Gernot Stroth : F-modules

H finite gp.

$F^*(H) = O_p(H)$, i.e. $C_H(O_p(H)) \leq O_p(H)$ "H is a gp. of characteristic p"

$P \in Syl_p(H)$, $J(P)$

Case 1: $\Omega_1(Z(P)) \trianglelefteq H$ ⇝ not so interesting

Case 2: $V = \langle \Omega Z(P)^H \rangle \leq Z(O_p(H))$

$H_1 = C_H(V)$

Case 2.1: $J(P) \leq H_1$

$$P \cap H_1 \in Syl_p(H_1)$$
$$\text{\rotatebox{90}{$\leq$}}$$
$$J(P) = J(P \cap H_1)$$

Frattini argument: $H = H_1 N_H(P \cap H_1)$
$= H_1 N_H(J(P))$
$= N_H(\Omega Z(P)) N_H(J(P))$ "Thompson factorization"

Case 2.2: $J(P) \not\leq H_1$

Let $A \leq P$, $A \not\leq H_1$, A el. abelian of maximal order

$V(A \cap H_1)$ el. ab.

$$\Rightarrow |V(A \cap H_1)| \leq |A|$$
$$\| $$
$$\frac{|V|\,|A \cap H_1|}{|V \cap A|}$$

$$\Rightarrow |V/C_V(A)| \leq |V/V \cap A| \leq |A/A \cap H_1| = |A/C_A(V)|$$

"Failure of factorization" = "FF-module" = "F-module"

H_1, H_2 gps s.t. $F^*(H_i) = O_p(H_i)$, $i = 1,2$.

$V_i \leq Z(O_p(H_i))$, $V_i \trianglelefteq H_i$, $i=1,2$.

$V_1 \leq H_2$, $V_2 \leq H_1$

$[V_1, V_2] = 1 \rightsquigarrow$ no way to say anything

$[V_1, V_2] \neq 1$

$\Rightarrow 1 \neq |V_1/C_{V_1}(V_2)| \leq |V_2/C_{V_2}(V_1)| \neq 1 \Rightarrow V_1$ is F-module

without loss of generality

$A \in \mathcal{P}(G,V)$ (called "best offender")

$\Rightarrow$ A offender on V.

Proof: $|A|\,|C_V(A)| \geq 1 \cdot |V|$ $(B := 1)$

Proof of Lemma 1.2:

$B := C_A(V_1)$

$|A|\,|C_V(A)| \geq |B|\,|C_V(B)| \geq |B|\,|V_1 C_V(A)| = |B| \frac{|V_1|\,|\cancel{C_V(A)}|}{|C_{V_1}(A)|} =$

$\Rightarrow |A| \geq |B| \cdot |V_1/C_{V_1}(A)| \Rightarrow |V_1/C_{V_1}(A)| \leq |A/C_A(V_1)|$.

$F(G)$ = maximal nilpotent normal sgp.

$E(G)$ Product of all $\overbrace{\text{subnormal quasisimple}}^{\text{component}}$ sgps.

(K quasisimple $:\Leftrightarrow [K,K] = K$ & $K/Z(K)$ simple)

$[F(G), E(G)] = 1$

$F^*(G) := F(G)E(G)$ "generalized Fitting sgp.". Most important property: $C_G(F^*(G)) \leq F^*(G)$.

Timmesfeld replacement :

Can replace a bestoffender by a quadratically acting best offender

$$([V, C_A(M), C_A(M)] \leq [M, C_A(M)] = 1)$$

Case $[K, A] = K$ in Lemma 1.5 :

$[V,K] =: V_1$, $[K,A] = K$

$\Rightarrow$ A offender on V_1 .

Set $K_1 = [F(G), A]$ and $K_2 = [E(G), A]$

$[K_1, V]$, $[K_2, V]$

F-module for $A/C_A([K_1,V])$

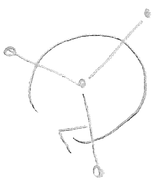
Assume $B := C_A([K_1,V])$ is not F-module offender on $[K_2,V]$

$\Rightarrow$ $|[K_2, V] : C_{[K_2,V]}(B)| > |B|$

$[K_1,V][K_2,V] = V$

$|[K_1,V] : C_{[K_1,V]}(A)| = |A/C_A([K_1,V])|$

$Spin_8(p^f)$



Thm 1.7 (6):

$3A_6 \subseteq SL_3(4)$, $V(6,2)$

$3A_6$ acts irreducibly on $V(6,2)$

have transvections in $3A_6$ on $V(6,2)$

Thm 1.7 (7)

$A_7 \leq A_8 = SL_4(2)$ $\qquad V(4,2)$

$\langle (12)(34), (13)(24) \rangle$ offender on $V(4,2)$

$A \in \mathcal{P}(G,V)$, $X \leq G$

$Y = \langle X, A \rangle$

$$|Y| = |\langle X,A\rangle| \geq |XA| = \frac{|X|\,|A|}{|X\cap A|}$$

$$|C_V(Y)| = |C_V(\langle X,A\rangle)| = |C_V(X) \cap C_V(A)|$$

$$|C_V(X)\, C_V(A)| = \frac{|C_V(X)|\,|C_V(A)|}{|C_V(X)\cap C_V(A)|}$$

$$\Rightarrow |Y|\,|C_V(Y)| = \frac{|X|\,|A|}{|X\cap A|} \cdot |C_V(X)\cap C_V(A)|$$

$$= \frac{|X|\,|A|}{|X\cap A|} \cdot \frac{|C_V(X)|\,|C_V(A)|}{|C_V(X)\, C_V(A)|} \geq \frac{|X|\,|C_V(X)| \cdot |A|\,|C_V(A)|}{|X\cap A|\,|C_V(X\cap A)|} \qquad \frac{|A|\,|C_V(A)|}{|X\cap A|\,|C_V(X\cap A)|} \geq 1$$

$$\boxed{\Rightarrow |Y|\,|C_V(Y)| \geq |X|\,|C_V(X)|}$$

$Y := \langle A^g, A \rangle$

$$|Y|\,|C_V(Y)| \geq |A^g|\,|C_V(A^g)| = |A|\,|C_V(A)|$$

$Y = \langle A^{g_1}, \dots, A^{g_n}, A \rangle$, $\quad |Y|\,|C_V(Y)| \geq |A|\,|C_V(A)|$

$X := \langle A^{g_1}, \dots, A^{g_{n-1}}, A \rangle$

$$\Rightarrow |Y|\,|C_V(Y)| \geq |X|\,|C_V(X)| \geq |A|\,|C_V(A)|$$

$\langle A^G \rangle$

$|\langle A^G \rangle| \, |C_V(\langle A^G \rangle)| \geq |A| \, |C_V(A)|$

$\quad \parallel$

$|G| \, |C_V(G)| \geq |A| \, |C_V(A)|$

If $|C_V(G)| = 1$ then

$|G| \geq |A| \, |C_V(A)| \geq |V|$

$\Rightarrow \boxed{|V| \leq |G|}$

Proof of Lemma 2.1:

$a, b \in A$, $v \in V$

$v^a = v v_a \qquad (v_a := [v, a])$

$v^b = v v_b$

$v^{ab} = (v v_a)^b = v v_b v_a^b = v v_b v_a$

$v^{ba} = (v v_b)^a = v v_a v_b^a = v v_a v_b$

$v^{a^p} = (v v_a)^a = v v_a v_a^a = v v_a v_a^a = v v_a^p$

$\Rightarrow a^p = 1$

$i \in G$, i involution, $p = 2$

$\Rightarrow v^i = v v_i$, $v = v^{i^2} = v \underbrace{v_i v_i^i}_{=1}$

$\Rightarrow$ i acts quadratically

So as involutions always acts quadratically, assume always $|A| > 2$.

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groups H_1, H_2, $F^*(H_i) = O_p(H_i)$, $P \in Syl_p(H_1) \cap Syl_p(H_2)$

$V_i = \langle \Omega Z(P)^{H_i} \rangle$, $i = 1, 2$.

$K_1 := \bigcap_{g \in H_1} (H_1 \cap H_2)^g \qquad K_2 := \bigcap_{g \in H_2} (H_1 \cap H_2)^g$

$\boxed{\text{Assume } C_{H_i}(V_i) \leq K_i \text{ if } V_i \neq \Omega_1(Z(P)).}$

$G = \langle H_1, H_2 \rangle$

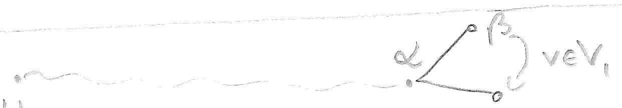
$\Gamma = \Gamma(H_1, H_2)$ coset graph

vertices: cosets of H_i in G

gH_1, kH_2 joined by an edge iff $gH_i \cap hH_j \neq \emptyset$.

$x g H_1 = g H_1$

$\Leftrightarrow g^{-1} x g \in H_1 \Leftrightarrow x \in g H_1 g^{-1}$



$V_1 \leq H_1$

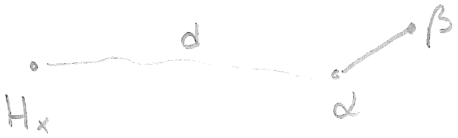
$V_1 \leq H_\alpha \leftarrow$ stabilizer of α in G

α of minimal distance s.t. $V_1 \leq H_\alpha$, but there is $\beta \in \Delta(\alpha)$ s.t. $V_1 \not\leq H_\beta$

$d_1 := d(H_1, \alpha)$

$d_2 := \min(H_2, \delta)$

$d := \min(d_1, d_2)$



$x \in \{1, 2\}$

$V_x \leq H_\alpha$

$V_\alpha \leq H_1$

$\boxed{\text{May assume } V_1 \neq \Omega_1(Z(P)) \quad V_1 = \Omega_1(Z(P))}$

$[V_x, V_\alpha] \leq V_x \cap V_\alpha$, so $[V_x, V_\alpha, V_\alpha] = 1$

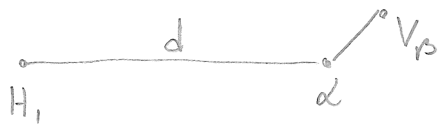
If $[V_x, V_\alpha] = 1$ then

$V_x \leq C_{H_\alpha}(V_\alpha)$

$W_\alpha = \langle V_\beta \mid \beta \in \Delta(\alpha) \rangle$

Assume $d \geq 3$. $\Rightarrow$ W_α elementary abelian.

$[V_\beta, V_{\tilde\beta}] = 1$ for $\beta \in \Delta(\alpha)$



$\delta \in \Delta(H_1)$

W_δ

W_α

$V_\beta \le H_\delta \implies W_\alpha \le H_\delta$

$V_1 \le W_\delta \quad , \quad [W_\delta, W_\alpha] \le W_\delta$

$[W_\alpha, V_1, V_1] \le [W_\alpha, V_1] = 1$

<u>$d=1$</u>

$V_1 \not\le O_p(H_2)$

$[O_p(H_2), V_1, V_1] \le [V_1, V_1] = 1$

So get always quadratic action!

$H = SL_2(9)$

$|V| = 3^4$

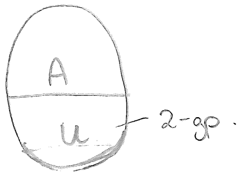
$\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$ acts quadratically

$L_2(9) \cong A_6 \ge A_5 \cong SL_2(5)$

$H \ge G \cong SL_2(5)$

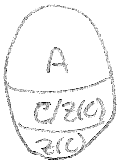
V is quadratic module for G

"Proof" of Lemma 1.6



$C \leq U$
critical

$[A, Z(C)] = 1$



$$C/Z(C) = \prod_{i=1}^{n} C_{C/Z(C)}(B_i)$$

$|C_i / C_i \cap Z(C)| = 4 \quad , \quad [A, C_i] = C_i$

$\langle A/B_i, C_i \rangle \cong SL_2(3)$

$[C_i, C_j] = 1 \quad i \neq j$

$C_1 \dots C_n = SL_2(3) \times \dots \times SL_2(3)$

$V = V_1 \times \dots \times V_n \qquad [V_i, y_i] = V_i \quad , \quad [V_i, y_j] = 1 \quad i \neq j$

$|A| = 3^n$

$|V_i : C_{V_i}(A)| \geq 3$

$3^n = |A| \geq |V : C_V(A)| \geq 3^n$

Consequence of 2.16

$e \in E$, $|C_V(e)|$ maximal, $f \in E$, $f \neq e$.

$[C_V(e), f] \neq 1 \Rightarrow [V,e,f] \cap [V,f] \neq 1$

We may assume $C_V(e) = C_V(f) \quad \forall e, f \in E$

$$\begin{array}{c} C_V(e) \\ \| \\ C_V(E) \end{array}$$

$C_V(E)$ is normalized by $\langle C_G(e), C_G(f) \rangle = G$

$[V,e] \leq C_V(e)$

$\Rightarrow [V,E] \leq C_V(E)$

$[V, \langle E^G \rangle] \leq C_V(E)^G = C_V(E^G)$

$\Rightarrow \langle E^G \rangle$ acts quadratically.

$\Rightarrow \quad \langle E^G \rangle \leq O_2(G)$

<u>Lemma</u>: M_{11} possesses no quadratic fours gp.

<u>Proof</u>: facts: (i) M_{11} has only one conjugacy class of involutions

(ii) If $e \in M_{11}$ is an involution then $C_G(e) \simeq Gl_2(3)$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

and $C_G(e)$ is maximal in M_{11}.

E acts quadratically.

$C_G(e) \simeq Gl_2(3)$, $|Z(Gl_2(3))| = 2$.

$\xRightarrow{2.16}$ $\exists\, e, f \in E^{\#}$ s.t. $e \neq f$ and $[V,e] \cap [V,f] \neq 1$.

$R_e = Gl_2(3) = C_G(e)$

$\langle R_e, R_f \rangle = G$. $\quad 2.17 \xRightarrow{\#} [V,e] \cap [V,f] \leq C_V(G) = 1 \quad$ ↯

ad 2.19

$3A_7 \leq 3M_{22}$

has quadratic module which is 12-dimensional

2F-modules

H_1 , H_2

$\overset{\bullet}{1}$ —— $\overset{\bullet}{2}$ $\qquad\qquad$ $\overset{\bullet}{\alpha}$

$V_1 \ntrianglelefteq O_p(H_2)$ $\qquad\qquad$ W_α

In the situation of Lemma 3.2:

$$[y, y] \leq [A \cap O_p(X), A^\gamma \cap O_p(X)] \leq A \cap A^\gamma$$

$$[y, y, y] = 1$$

$$[A, y] \leq y$$

$$[A, y, y, y] = 1 \qquad \text{cubic action}$$

Set $q = p^f$ or $q = 2$ if $X/O_2(X) \cong D_{2n}$.

$$|y/A \cap A^\gamma| = q^{2x}$$

$$\underbrace{|A^\gamma \cap O_p(X)/A \cap A^\gamma|}_{\bar{y}} = q^x$$

$$|A : C_A(\bar{y})| \leq q^x q \leq |\bar{y}|^2$$

$$\boxed{\text{(1) 2F module with cubic offender}}$$

If $\bar{y}$ is quadratic then $A \cap O_p(X) \leq C_A(\bar{y})$

$\Rightarrow |A : C_A(\bar{y})| \leq q \leq |\bar{y}| \rightsquigarrow$ have F-module ✓

V natural module for $SL_2(q)$, $q = p^f$

Sylow p-sgp. y

$[V, \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}]$ 1-space

$a \in A \setminus O_p(X)$

$$\boxed{[a, y]\, C_y(A) = [A, y]\, C_y(A)} \quad (*)$$

(2) Modules with cubic group $\mathcal{Y}$ acting s.t. $(*)$ holds are called nearly quadratic.