IMPORTANT MODULES

1. F-modules

First we will give a definition of F-modules and then say why they are important.

Definition 1.1. *(F-module) Let G be a finite group and V be a faithful $\mathbb{F}_pG$ - module. We say that V is an F-module for G if there is a nontrivial elementary abelian p-subgroup A of G such that*

$$|V : C_V(A)| \leq |A|.$$

$$\textit{(or } \dim V/C_V(A) \leq \log_p(|A|)\textit{)}$$

In this case we will call A an offender.

The first occurrence of F-modules goes back to John Thompson.

Let P be a p-group. Define

$$J(P) = \langle A \mid A \leq P, A \textit{ elementary abelian, with } |A| \textit{ maximal } \rangle$$

Then $J(P)$ is called the Thompson subgroup of P.

Let H be some group with $P \in \mathrm{Syl}_p(H)$ and $C_H(O_p(H)) \leq O_p(H)$ $(O_p(H) = F^*(H))$.

Set $V = \langle \Omega_1(Z(P))^H \rangle \leq Z(O_p(H))$ and

$$H_1 = C_H(V).$$

If $J(P) \leq H_1$, then $J(P) = J(P \cap H_1)$ and so by Frattini

$$H = N_H(J(P))H_1 = N_H(J(P))C_H(\Omega_1(Z(P))).$$

So assume that $J(P) \not\leq H_1$. In particular

$$V \neq \Omega_1(Z(P)).$$

Now there is some $A \leq P$, A elementary abelian of maximal order such that $A \not\leq H_1$.

As $V(A \cap H_1)$ is elementary abelian, we have that

$$|V(A \cap H_1)| \leq |A|.$$

In particular

$$|V||A \cap H_1|/|V \cap A| \leq |A|.$$

So

$$|V : V \cap A| \leq |A/A \cap H_1|.$$

As $V \cap A \leq C_V(A)$, we get

$$|V : C_V(A)| \leq |A/A \cap H_1|$$

and so V is an F-module with offender $A/A \cap H_1$.

Here is another occurrence:

Let G be a group with subgroups H_1, H_2 and $F^*(H_i) = O_p(H_i)$, $i = 1, 2$.

Let $V_i \leq \Omega_1(Z(O_p(H_i)))$ be a normal subgroup of H_i, $i = 1, 2$.

Now suppose $V_{3-i} \leq H_i$, $i = 1, 2$.

One possibility of course is $[V_1, V_2] = 1$. The other is $[V_1, V_2] \neq 1$. We will investigate this possibility.

We may choose notation such that

$$|V_1 : C_{V_1}(V_2)| \leq |V_2 : C_{V_2}(V_1)|.$$

Then $V_2/V_2 \cap C_{H_1}(V_1)$ induces an F-module offender on V_1.

Concerning F-modules the following notation is helpful:

Denote by $\mathcal{P}(G,V)$ the set of all elementary abelian subgroups A of G such that

$$|A||C_V(A)| \geq |B||C_V(B)| \text{ for all}$$

$$B \leq A \text{ including } B = 1.$$

In particular A is an F-module offender.

Furthermore if A is a minimal offender, then

$$A \in \mathcal{P}(G,V).$$

Lemma 1.2. *Let $A \in \mathcal{P}(G, V)$ and V_1 a submodule of V, then either $[V_1, A] = 1$ or $|V_1 : C_{V_1}(A)| \leq |A/C_A(V_1)|$.*

So we may assume that V is irreducible.

We will consider the structure of G.

We may assume that A act faithfully on

$$F^*(G) = F(G)E(G).$$

What can we say about this group?

Proof. Set $B = C_A(V_1)$ and assume that $B \neq A$. Then $V_1 C_V(A) \leq C_V(B)$. In particular

$$|A||C_V(A)| \geq |B||C_V(B)| \geq |B||V_1 C_V(A)| =$$

$$|B||V_1||C_V(A)|/|C_{V_1}(A)|.$$

Hence

$$|A/B| \geq |V_1 : C_{V_1}(A)|$$

$\square$

Lemma 1.3. *(Timmesfeld replacement):* *Let $A \in \mathcal{P}(G, V)$. Set $M = [V, A]$. Then we have that $|A||C_V(A)| = |C_A(M)||C_V(C_A(M))|$ and $C_A(M) \neq 1$.*

So we have that

$$|V| \leq |A||C_V(A)| = |C_A(M)||C_V(C_A(M))|$$

and then

$$|V : C_V(C_A(M))| \leq |C_A(M)|.$$

In particular $C_A(M)$ is also an offender.

We further have

$$[V, C_A(M), C_A(M)] \leq [M, C_A(M)] = 1.$$

Definition 1.4. *(quadratic). Let G be a group and V be a faithful $\mathbb{F}_p$-module. If there is some p-group $A \leq G$ with $[V, A] \neq 1 = [V, A, A]$, we say that V is a quadratic module and A is an offender.*

We just have shown that any F-module admits a quadratic offender. We will consider quadratic modules in more details in the next lecture.

There is an important result, which goes back to Thompson, was also proved by Aschbacher (using the classification) and a very nice proof was given by Chermak. (We will comment on this next lecture)

Lemma 1.5. *Let $A \in \mathcal{P}(G, V)$. If K is a component with $[K, A] \neq 1$, then $[K, A] = K$.*

Assume first that A acts faithfully on $E(G)$.

Maybe V is not irreducible for $E(G)$ but we may assume that $V = [V, E(G)]$.

We are going to show that there is $B \leq A$ and a component K with $[K, B] = B$, such that B induces an F-module offender on $[V, K]$.

Let K_1 be a component with $[K_1, A] \neq 1$. Then we may assume that $[K_1, A] = K_1$.

If A acts faithfully on K_1, we have that $[V, K_1]$ is an F-module with offender A.

So assume that $B_1 = C_A(K_1) \neq 1$.

Let further $K_2 = [E(G), B_1]$.

Then also $[K_2, A] = K_2$.

If A acts faithfully on K_2, then by induction we get the assertion.

Hence we may assume that $B_2 = C_A(K_2) \neq 1$.

For $i = 1, 2$ we have $B_i = C_A(K_i)$ and

$$K_i = [K_i, B_{3-i}].$$

As A acts quadratically we have

$$[V, B_{3-i}, B_i] \leq [V, A, A] = 1.$$

Then

$$1 = [V^{K_i}, B_{3-i}^{K_i}, B_i^{K_i}] = [V, K_i, B_i]$$

and then also

$$1 = [V^{K_{3-i}}, K_i^{K_{3-i}}, B_i^{K_{3-i}}] = [V, K_i, K_{3-i}].$$

This holds for $i = 1, 2$.

Further we have that

$$[K_i, B_i, V] = 1.$$

As $[V, B_i, A] = 1$, we also have

$$[B_i, V, K_i] = [B_i, V, A^{K_i}] = 1.$$

By the three subgroup lemma we get that

$$[V, K_i, B_i] = 1.$$

We have that

$$[V, K_1 \times K_2] = [V, K_1][V, K_2].$$

and

$$[V, K_1] \cap [V, K_2] \leq C_V(K_1 K_2) = C_V(E(G)).$$

We now assume that neither $[V, K_1]$ nor $[V, K_2]$ is an F-module with offender $\hat{B}_i = A/B_i$, $i = 1, 2$. Then

$$|[V, K_1] : C_{[V,K_1]}(A)| > |\hat{B}_1|.$$

As B_1 centralizes $[V, K_1] \cap [V, K_2]$, we get that

$$|[V, K_2] : C_{[V,K_2]}(B_1)| < |B_1|,$$

and so B_1 induces an F-module offender on $[V, K_2]$.

Hence by induction we get some subgroup $A_0 \leq A$ and a component K of $E(G)$ such that

$$[K, A_0] = K, C_{A_0}(K) = 1$$

and A_0 induces an F-module offender on $[V, K]$.

This reduces the study of F-modules to those for automorphism groups of quasisimple groups (not quite).

Suppose now that G is solvable, what then?

We have that A acts faithfully on $F(G)$.

In this case we have a lemma, which is more related to quadratic modules, so we will sketch a proof maybe the next lecture

Lemma 1.6. *Let $G = UA$, where A is an elementary abelian p-group and U is some p' group. Let V be some irreducible faithful $\mathbb{F}_p$-module for G on which A induces an F-module offender. Then $p = 2$ or 3 and $|V : C_V(A)| = |A|$.*

In particular we can have an F-module offender in a solvable groups just for $p = 2$ or $p = 3$.

Furthermore we will see that A has to induce a transvection on V.

Now we consider $F^*(G)$.

Set $K_1 = [F(G), A]$ and $K_2 = [E(G), A]$.

We can mimic the proof above to get that there is subgroup $B \leq A$, which either induces an F-module offender on $[V, K_2]$ or on $[V, K_1]$.

Let us assume that there is no such offender on $[V, E(G)]$. Then B_2 in the notation above induces an F-module offender on $[V, K_1]$.

Hence by the lemma above we have

$$|[V,K_1] : C_{[V,K_1]}(B_2)| = |B_2|.$$

But then as $[V,K_2]$ is not an F-module we must have $\hat{B}_2 = 1$ and so $B_2 = A$ acts faithfully on K_1 and centralizes K_2.

In particular the problem is reduced to the automorphism groups of quasisimple groups.

Here is the result which treats this case at least for the known finite simple groups.

This is due to M. Aschbacher, B. Cooperstein, Th. Meixner in the first generation.

R. Guralnick, G. Malle, B. Lawther and U. Meierfrankenfeld, G. Stroth in the second generation.

Theorem 1.7. *Let $H = F^*(G)$ be a known finite quasisimple group and V be some irreducible $\mathbb{F}_pG$-module, which is an F-module. Then one of the following holds*

(1) H *is classical and* V *is the natural module.*

(2) $p = 2$, $H = G_2(2^f)$ *and* V *is the natural 6-dimensional module.*

(3) $H/Z(H) \cong PSL_n(p^f)$, $n \geq 5$, *and* V *is the exterior square of the natural module.*

(4) $H \cong \mathrm{Spin}_7(p^f)$ *and* V *is a spin module of dimension* 8.

(5) $H \cong \mathrm{Spin}^+_{10}(p^f)$ *and* V *is a half-spin module of dimension* 16.

(6) $H \cong 3.\mathrm{Alt}(6)$, $p = 2$, $|V| = 2^6$.

(7) $H \cong \mathrm{Alt}(7)$, $p = 2$, $|V| = 2^4$.

(8) $G \cong \mathrm{Sym}(n)$, $p = 2$, $n \equiv 1(2)$, V *is a natural* $\mathrm{Sym}(n)$-*module.*

(9) $G \cong \mathrm{Alt}(n)$ *or* $\mathrm{Sym}(n)$, $p = 2$, n *is even,* $n \geq 6$, V *is a corresponding natural module.*

In fact much more is known.

We know the offender in most cases. (Meierfrankenfeld/Stellmacher)

We just have restricted ourself to irreducible modules. So we have to check which have (nonsplit) extension by trivial modules or other F-modules. (Meierfrankenfeld/Stellmacher)

It is an open problem, and a solution would be very important for the revision of the classification of the finite simple groups, if we could drop the word known in this theorem.

In fact if $p > 3$ this can be done. But this is another story, we will tell in the second lecture.

The theorem can be obtained as a corollary of a more general theorem, which we will see in the third lecture.

There is a nice property which can be used to show that some groups do not have F-modules or to determine the F-modules for groups, which are not of Lie type in characteristic p.

Choose $A \in \mathcal{P}(G, V)$. Let X be any group. Set $Y = \langle X, A \rangle$.

Then we have $|Y| \geq |XA| = |X||A|/|X \cap A|$.

Furthermore we have that

$$\begin{aligned} |C_V(A)C_V(X)| &= |C_V(A)||C_V(X)|/|C_V(A) \cap C_V(X)| \\ &= |C_V(A)||C_V(X)|/|C_V(Y)|. \end{aligned}$$

Hence

$$|C_V(Y)| = |C_V(A)||C_V(X)|/|C_V(A)C_V(X)|$$
$$\geq |C_V(A)||C_V(X)|/|C_V(A \cap X)|.$$

This implies

$$|Y||C_V(Y)|$$
$$\geq |A||X||C_V(A)||C_V(X)|/|A \cap X||C_V(A \cap X)|.$$

As $A \in \mathcal{P}(G, V)$ we have that

$$|Y||C_V(Y)| \geq |X||C_V(X)|.$$

Now set $X = A^g$. Then we have

$$|Y||C_V(Y)| \geq |A^g||C_V(A^g)| = |A||C_V(A)|.$$

Now we have $A^G = \langle A^{g_1}, \ldots, A^{g_n} \rangle$.

Lets assume that we have shown that for

$$Y = \langle A^{g_1}, \dots, A^{g_{n-1}}, A \rangle$$

we have that $|Y||C_V(Y)| \geq |A||C_V(A)|$.

Then we get for $Y_1 = \langle Y, A^{g_n} \rangle$ that

$$|Y_1||C_V(Y_1)| \geq |Y||C_V(Y)| \geq |A||C_V(A)|.$$

Now if $G = \langle A^G \rangle$ and $C_V(G) = 1$ this implies

$$|G| \geq |A||C_V(A)| \geq |V|.$$

This is the reason why these modules and also those we will see in the next lectures are also called small modules.