Carles Broto: Equivalences between fusion systems of finite groups of Lie typ[e]

$$\mathcal{F}_p(G) \rightsquigarrow H^*(G;\mathbb{F}_p) \cong H^*(BG;\mathbb{F}_p) \leftarrow BG^\wedge_p$$

← How much does this say about $BG^\wedge_p$?

Thm [Broto-Levi]: $H^*(X;\mathbb{F}_p) \underset{\uparrow}{\cong} H^*(PSL_2(q);\mathbb{F}_p) \Rightarrow X^\wedge_p \simeq BPSL_2(q)^\wedge_p$

incl. operations

In particular, $H^*(BPSL_2(q')^\wedge_p;\mathbb{F}_p) \cong H^*(BPSL_2(q);\mathbb{F}_p)$
$\Rightarrow BPSL_2(q')^\wedge_p \quad BPSL_2(q)^\wedge_p$

Martino-Priddy: $(BM_{11})^\wedge_2 \simeq BSL_3(3)^\wedge_2$

Quillen: $H^*(BGL_n(q);\mathbb{F}_p)$ $(p,q)=1$ same algebra (+ operations)
if p odd, $q \equiv q' \pmod p$, $\nu_p(q^s-1) = \nu_p(q'^s-1)$ $\quad s = \mathrm{ord}(q)$ in $\mathbb{Z}/p$

Friedlander: ...

Thm: (jt. w/ Møller and Oliver): $\mathbb{G}$ connected reductive group scheme over $\mathbb{Z}$.
Fix p a prime, q, q' both prime to p.

a) $\mathcal{F}_p(\mathbb{G}(q)) \cong \mathcal{F}_p(\mathbb{G}(q'))$ if $\overline{\langle q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times$

b) $\mathbb{G} = A_n, D_n, E_6$, τ graph auto: $\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p({}^\tau\mathbb{G}(q'))$ if $\overline{\langle q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times$

c) If the Weyl group of ~~order~~ $\mathbb{G}$ contains ψ^{-1}: an element that inverts all elements in max torus, then $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$; $\mathcal{F}_p({}^\tau\mathbb{G}(q)) = \mathcal{F}_p({}^\tau\mathbb{G}(q'))$ if $\overline{\langle -1,q\rangle} = \overline{\langle -1,q'\rangle} \le \mathbb{Z}_p^\times$

d) $\mathbb{G}$ of type A_n, D_n, n odd, E_6, τ graph auto of order 2.
$\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ if $\overline{\langle -q\rangle} = \overline{\langle q'\rangle} \le \mathbb{Z}_p^\times$

Other cases: $\mathcal{F}_p(G_2(q)) \simeq \mathcal{F}_p({}^3D_4(q))$, $p \neq 3$, $q \equiv 1 \pmod p$

$\mathcal{F}_p(F_4(q)) \simeq \mathcal{F}_p({}^2E_6(q))$, $p \neq 2$, $q \equiv 1 \pmod p$

[Friedlander] + [Jackowski-McClure-Oliver]

∃ fibre square:

$$\begin{tikzcd}
(B^\tau\mathbb{G}(q))^\wedge_p \arrow[r] \arrow[d] & B\mathbb{G}(\mathbb{C})^\wedge_p \simeq (B\mathbb{G})^\wedge_p \arrow[d, "\Delta"] \\
B\mathbb{G}(\mathbb{C})^\wedge_p \arrow[r, "(1,\tau\psi^q)"] & B\mathbb{G}(\mathbb{C})^\wedge_p \times B\mathbb{G}(\mathbb{C})^\wedge_p
\end{tikzcd}$$

Homotopy pullback

$$\begin{tikzcd}
X^{h\alpha} \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \arrow[r, "(1,\alpha)"] & X \times X
\end{tikzcd}$$

$I \longrightarrow X$

$X^{h\alpha} = \{\omega \in map(I,X) \mid \omega(1) = \alpha(\omega(0))\}$

$X_{h\alpha} := X \times I/(x,0) \sim (\alpha(x),1)$ (mapping torus)

Have

$$\begin{tikzcd} X \arrow[d] & \\ X_{h\alpha} = X \times I/\sim \arrow[d] \arrow[d, shift left=8ex, "\pi"] & \\ S^1 = I/\sim & \end{tikzcd}$$

If α is homeo, then π is also a fibre bundle with fibre X.
Sections
$$\Gamma(X_{h\alpha} \downarrow S^1) \simeq X^{h\alpha}$$

① If α was not a homeomorphism, we can subst. (X, α) by the double mapping telescope $(Tel(X), \hat{\alpha})$

$$X^{h\alpha} \simeq Tel(X)^{h\hat{\alpha}} \quad \Leftarrow \text{h. fixed pts are invariant}$$

In some cases, p-completion functor preserves fibrations

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\simeq"'] & X_{h\alpha} \arrow[r] \arrow[d] & S^1 = B\mathbb{Z} \\ X^\wedge_p \arrow[r] & (X_{h\alpha})^\wedge_p \arrow[r] & (S^1)^\wedge_p = B\mathbb{Z}^\wedge_p \\ X^\wedge_p \arrow[r] \arrow[u] & (X_{h\beta})^\wedge_p \arrow[r, dash] \arrow[u] & S^1 \arrow[u] \end{tikzcd}$$

$, \pi_1(S^1) = \mathbb{Z}$

$, \pi_1(S^{1\wedge}_p) = \mathbb{Z}_p$

again a fibration if $\pi_1(S^1) = \mathbb{Z}$ acts nilpotently on $H^*(X; \mathbb{F}_p)$

$$X^{h\alpha} = \Gamma(X_{h\alpha} \downarrow S^1) \simeq \Gamma((X_{h\alpha})^\wedge_p \downarrow (S^1)^\wedge_p)$$
$$\simeq \Gamma(X_{h\beta} \downarrow S^1) \simeq X^{h\beta}$$