Bob Oliver

Existence and uniqueness of centric linking systems

Def: G a finite group, $\Delta \trianglelefteq G$ normal abelian subgroup. Let $C_G(\Delta) \leq \mathcal{J}(G,\Delta) \leq G$ be the subgroup satisfying

$$\mathcal{J}(G,\Delta)/C_G(\Delta) = \langle \underbrace{\mathcal{P}(G/C_G(\Delta), \Delta)}_{\text{set of best offenders}} \rangle$$

Def A general setup is (Γ, S, Y) such that Γ is a finite group, $S \in Syl_p(\Gamma)$, $Y \trianglelefteq \Gamma$ normal centric p-group $(C_\Gamma(Y) \leq Y)$.

A reduced setup is a general setup (Γ, S, Y) such that $Y = O_p(\Gamma)$, $Y = C_S(Z(Y))$, $O_p(\Gamma/C_\Gamma(Z(Y))) = 1$.

Prop A (7.11) Let (Γ, S, Y) be a reduced setup $\Delta = Z(Y)$, $\mathcal{R} = \{R \in Sub(S)_{\geq Y} \mid \mathcal{J}(R,\Delta) = Y\}$. Assume $\Gamma/C_\Gamma(\Delta)$ is generated by quadratic best offenders on Δ. Let $\mathcal{F} = \mathcal{F}_S(\Gamma)$. Then $L^k(\mathcal{F}, \mathcal{R}) = 0$ for $k=1$ $(p>2)$ and $k=2$ $(p=2)$.

Prop B (7.12 + 7.13) (Γ, S, Y) a general setup $\mathcal{F} = \mathcal{F}_S(\Gamma)$, $\Delta = Z(Y)$, $\mathcal{R} \subseteq Sub(S)_{\geq Y}$ is an interval, $\mathcal{F}$-invariant such that

$Q \in Sub(S)_{\geq Y}$, $Q \in \mathcal{R} \iff \mathcal{J}(Q,\Delta) \in \mathcal{R}$

Then $L^k(\mathcal{F}, \mathcal{R}) = 0$, $\forall k \geq 2$; $\forall k \geq 1$ if p odd.

Note on Δ in Andy's talk:

$\Delta = \mathrm{Sub}(S)_{\geq Y} \setminus \mathcal{R}$

Exact sequence $\Rightarrow L^k(\mathcal{F};\mathcal{R}) \cong L^{k-1}(\mathcal{F},\Delta)$.

Example (Prop A in case $p=2$, $k=1$).

$S = D_8$, $\Gamma = \Sigma_4$, $Y = O_2(\Gamma)$ then (Γ, S, Y) is a reduced setup and Γ/Y is generated best offenders on $Y = D$

$\mathcal{R} = \{Y\}$. Then $L^1(\mathcal{F},\mathcal{R}) = H^1(\mathcal{O}(\mathcal{F}^c); Z_{\mathcal{F}}^{\mathcal{R}})$

$\cong \Lambda^1(\underbrace{\Sigma_3}_{\mathrm{Out}_\Gamma(Y)}; C_2^2) \cong \mathbb{Z}/2$

Hence $L^1(\mathcal{F},\mathcal{R}) \neq 0$.

Proof of proposition B Assume false and let $(\Gamma, S, Y, \mathcal{R}, k)$ be a couter-example where $(k, |\Gamma|, |\Gamma/Y|, |\mathcal{R}|)$ is minimal in lexicographic order.

Step 1: $\mathcal{R} = \{P \in \mathrm{Sub}(S)_{\geq Y} \mid J(P,D) = Y\}$

Step 2: $k=2$ $(p=2)$ or $k=1$ $(p>2)$

Step 3: (Γ, S, Y) reduced

Step 4: $\Gamma/C_\Gamma(D)$ generated by quadratic offenders

Then the contradiction comes from prop A.

Step 2 We have $Y \in \mathcal{R} \subseteq \mathrm{Sub}(S)_{\geq Y}$

$\mathcal{Q} = \mathrm{Sub}(S)_{\geq Y} \setminus \mathcal{R}$ (closed under overgroups)

If $k \geq 2$: $L^i(\mathcal{F}; \mathrm{Sub}(S)_{\geq Y}) = 0$ $(\forall i > 0)$

$\Rightarrow L^k(\mathcal{F};\mathcal{R}) \cong L^{k-1}(\mathcal{F},\mathcal{Q}) \Rightarrow$

$(\Gamma, S, Y, \mathcal{Q}, k-1)$ smaller counterexample

Hence if $k > 2$ $(p = 2)$ or $k > 1$ $(p > 2)$ then $(\Gamma, S, Y, Q, k-1)$ give a contraction.

...

Proof of Thm: $\forall \mathcal{F}$: $H^i(\mathcal{O}(\mathcal{F}_c); \mathcal{Z}_{\mathcal{F}}) = L^i(\mathcal{F}, \mathcal{F}^c) = 0$ $\forall i \geq 2$; $\forall i \geq 1$ if $p > 2$.

using Prop B.

Choose $X_0, X_1, \ldots, X_N$

$$\phi = Q_{-1} \subseteq Q_0 \subseteq \cdots \subseteq Q_N = \mathcal{F}^c$$

intervals closed under overgroups.

For $n > 0$, assume Q_{n-1}, X_{n-1} defined.

Let $\mathcal{U}_1 = \{ P \in \mathcal{F}^c \setminus Q_{n-1} \mid \underbrace{d(P)}_{\text{order of largest abelian subgroup of } P} \text{ maximal} \}$

$\mathcal{U}_2 = \{ P \in \mathcal{U}_1 \mid |J(P)| \text{ maximal} \}$

$\mathcal{U}_3 = \{ P \in \mathcal{U}_2 \mid J(P) \in \mathcal{F}^c \}$

$$\mathcal{U}_4 = \begin{cases} \{ P \in \mathcal{U}_3 \mid |P| \text{ minimal} \} & \text{if } \mathcal{U}_3 \neq \phi \\ \{ P \in \mathcal{U}_2 \mid |P| \text{ maximal} \} & \text{if } \mathcal{U}_3 = \phi \end{cases}$$

Choose $X_n \in \mathcal{U}_4$ which is fully normalized in $\mathcal{F}$.

Set $Q_n = Q_{n-1} \cup \{ Q \leq \underline{\underline{P}} \leq S \mid Q \in X_n^{\mathcal{F}} \}$

$\mathcal{R}_n = Q_n \setminus Q_{n-1}$ $(n \geq 0)$

<u>Note</u>: $X_0 = J(S)$, $Q_0 = \mathrm{Sub}(S) \geq J(S)$

Must show: $L^k(\mathcal{F}; \mathcal{R}_n) = 0$, $\forall\, 0 \leq n \leq N$

We have an exact sequence.

$$0 = L^k(\mathcal{F}; \mathcal{R}_n) \longrightarrow L^k(\mathcal{F}; Q_n) \longrightarrow L^k(\mathcal{F}; Q_{n-1})$$

so $L^k(\mathcal{F}, Q_{n-1}) = 0 \Rightarrow L^k(\mathcal{F}, Q_n) = 0$. Hence $L^k(\mathcal{F}, Q_N) = 0$

Case 1

If $\mathcal{U}_3 = \phi$, then $J(X_u) \notin \mathcal{F}^c$ and $\mathcal{R}_u - X_u^{\mathcal{F}}$

$\Rightarrow L^k(\mathcal{F}; \mathcal{R}_u) = \Lambda^k(N_\Gamma(X_u)/X_u ; \underbrace{Z(X_u)}_{Z_{\mathcal{F}}(X_u)})$.

$J(X_u)$ not centric in S but centric in X_u.

$C_S(J(X_u)) \leq X_n \cdots \exists g \in N_\Gamma(X_u) \setminus X_u$.

$$[g, Z(X_u)] \leq [g, J(X_u)] = 1$$

$\Rightarrow 1 \neq [g] \in N_\Gamma(X_u)/X_n$ p-element act trivialy on $Z(X_u)$.

$\Rightarrow \Lambda^k(\cdots) = 0 \Rightarrow L^k(\mathcal{F}; \mathcal{R}_u) = 0$.

Case 2 $J(X_u) \in \mathcal{F}^c$ $(\mathcal{U}_3 \neq \phi)$

$X_n = J(X_u)$ and $|X_u|$ is minimal in $\mathcal{U}_3$

$\forall X_u \leq P \in \mathcal{R}_n = \mathcal{Q}_n \setminus \mathcal{Q}_{u-1}$, $J(P) = X_u$.

Let $\mathcal{E} = N_{\mathcal{F}}(X_u)$, let Γ be a model for $\mathcal{E}$, hence $\mathcal{E} = \mathcal{F}_{N_S(X_u)}(\Gamma)$

Let $\mathcal{R} = \{R \in \mathcal{R}_u \mid R \geq X_u\} = \{R \in \mathcal{R}_u \mid J(R) = X_u\}$

$\Rightarrow L^k(\mathcal{F}; \mathcal{R}_n) \cong L^k(\mathcal{E}; \mathcal{R})$ by the change of fusion system proposition.

Remains to check that $L^k(\mathcal{E}; \mathcal{R}) = 0$ by Prop B. (applied to $(\Gamma, N_S(X_u), X_u)$ general setup. $\square$

Def In a finite group G a radical p-chain is a sequence $P_0 < P_1 < \cdots < P_u$ in G where P_i is a p-subgroup of G, $\forall i$, $P_0 = O_p(N_G(P_0))$ (P_0 is radical)

$$P_i = O_p\big(\underbrace{N_G(P_0, P_1, \ldots, P_i)}\big) \; \forall i$$
$$= N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_i)$$
$$P_k \in Syl_p(N_G(P_0, \ldots, P_{k-1})).$$

<u>Prop</u> $\forall G$, M a $\mathbb{F}_p G$-module if $\Lambda^k(G;M) \neq 0$ then there exists a radical p-chain $1 = P_0 < P_1 < \cdots < P_k < G$ such that $M \supseteq \mathbb{F}_p P_k$

<u>Prop C</u> $G \neq 1$ finite group, $O_p(G) = 1$, V a faithful $\mathbb{F}_p G$-module. $\mathcal{U}$ a set of non-trivial quadratic best offenders in G on V, invariant under G-conjugacy, such that $G = \langle \mathcal{U} \rangle$
Set $G_0 = O^p(G)$, $W = C_{[G_0, V]}(G_0)$ $[G_0, V]/C_V(G_0)$
Set $k = 2$ (if $p = 2$) or $k = 1$ (if $p > 2$)
Let $P_0 < P_1 < \cdots < P_k$ be a radical of length k such that P_0 does not contain any $U \in \mathcal{U}$.
Then $C_W(P_0) = W^{P_0}$ does not contain a copy of $\mathbb{F}_p(P_k/P_0)$.

<u>Proof of Prop C $\Rightarrow$ Prop A.</u>
Let (Γ, S, Y) be a reduced setup, $\Delta = Z(Y)$, $V = \Gamma_1(\Delta)$
$C_\Gamma(\Delta) = C_\Gamma(V)$ since $O_p(\Gamma/C_\Gamma(\Delta)) = 1$.
Then $G = \Gamma/C_\Gamma(\Delta)$ acts faithfully on Δ, V.
For all $H \leq \Gamma$, $\overline{H} := HC_\Gamma(\Delta)/C_\Gamma(\Delta) = \mathrm{Im}(H \hookrightarrow \Gamma \twoheadrightarrow G)$

$N_G(\overline{P}) = \overline{N_\Gamma(P)}$, $\forall\, Y \leq P \leq S$
$\mathcal{U} = \{$ quadratic best offenders in G on $\Delta \}$
$\subseteq \{$ —— " —— in G on $V \}$

$\mathcal{R} = \{ P \in \mathcal{F}^c \mid \mathcal{J}(P, D) = Y \} =$
$= \{ P \in \mathcal{F}^c_{\geq Y} \mid \overline{P}$ contains no best offenders on $D \}$

$\mathcal{Q} = \mathrm{Sub}(S)_{\geq Y} \setminus \mathcal{R} = \{ P \mid \overline{P}$ contains best offenders on $D \}$

Must show: $L^k(\mathcal{F}; \mathcal{R}) = 0$
for that it is enough to show that
$\Lambda^k(\underbrace{\mathrm{Out}_{\mathcal{F}}(R)}_{N_\Gamma(R)/R}; \underbrace{Z(R)}_{C_D(\overline{R})}) = 0 \ \forall R \in \mathcal{R}.$

[Assume $D = V = \Omega_1(D)$].
Assume $\Lambda^k \neq 0 \Rightarrow O_p(N_\Gamma(R)/R) = 0$
and there exist a radical p-chain
$R = P_0 < P_1 < \dots < P_k$ in Γ such that
$\mathbb{F}_p(P_k/P_0) \subseteq C_V(\overline{R})$
can arrange so $\mathbb{F}_p[P_k/P_0] \subseteq C_W(\overline{R})$
$P_0 = \underline{R} \in \underline{\mathcal{R}} \Rightarrow \overline{P_0}$ does not contain any $U \in \mathcal{U}$
then $\overline{P_0} < \overline{P_1} \dots < \overline{P_k}$ is a radical p-chain in G
which contradicts conclusion of Prop C.

Proof of special case of Prop C

Assume G is quasisimple of Lie type in char p.
We know the radical subgroups of G. They are O_p(parabolic).
• $\forall$ radical p-chain $P_0 < P_1 < \dots < P_k$ in G
we have $P_k \in \mathrm{Syl}_p(G) \Rightarrow \exists U \in \mathcal{U}, U \leq P_k$.
By assumption $U \not\leq P_0$ set $U_0 = U \cap P_0 \lneq U$.
Assume $C_W(P_0) \supseteq \mathbb{F}[P_k/P_0]$. Hence
$C_W(P_0) \supseteq \mathbb{F}[U/U_0]$.

Hence, $rk(C_W(U_0)) - rk(C_W(U)) \geq p^a - 1$,
where $p^a = |U/U_0|$.
Now, U is a best offender implies
$$|U||C_W(U)| \geq |U_0||C_W(U_0)|.$$
Thus $rk(C_W(U_0)) - rk(C_W(U)) \leq \log_p(|U/U_0|) =$
$\Rightarrow a \geq p^a - 1 \Rightarrow p = 2$.
Now $p = 2 \Rightarrow k = 2$ and implies that
$|P_k/P_0| \geq 4$ ... $\square$