Bob Oliver

The obstruction theory for linking systems

Let $\mathcal{F}$ be a saturated fusion system,
$\mathcal{O}(\mathcal{F}^c)$, the orbit category of $\mathcal{F}^c$

Objects: $P \leq S$, P $\mathcal{F}$-centric

Morphisms: $\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q) \backslash \mathrm{Hom}_{\mathcal{F}}(P,Q)$

$$\mathcal{Z}_{\mathcal{F}} : \mathcal{O}(\mathcal{F}^c) \longrightarrow \mathcal{A}b$$
$$P \longmapsto Z(P)$$
$$(P \xrightarrow{\varphi} Q) \longmapsto (Z(P) \xleftarrow{\varphi^{-1}} Z(Q))$$

Obstruction to existence and uniqueness of a centric linking system associated to $\mathcal{F}$ lie in $H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}})$ $(i = 2,3)$

<u>Def</u> An [$\mathcal{F}$-invariant] interval is a subset $\mathcal{R} \subseteq \mathcal{F}^c$ s.th

1) $P \in \mathcal{R} \Longrightarrow P^{\mathcal{F}} \subseteq \mathcal{R}$
2) $P < Q < R$ and $P, R \in \mathcal{R} \Rightarrow Q \in \mathcal{R}$

If $\mathcal{R} \subseteq \mathcal{F}^c$ is an interval, then define
$$\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}} : \mathcal{O}(\mathcal{F}^c) \longrightarrow \mathcal{A}b$$
$$\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}(P) = \begin{cases} Z(P), & P \in \mathcal{R} \\ 1, & P \in \mathcal{R} \end{cases}$$
with induced morphisms as for $\mathcal{Z}_{\mathcal{F}}$

For all interval $\mathcal{R}$, we have $\mathcal{R} = \mathcal{Q} \setminus \mathcal{Q}_0$ where (if $S \notin \mathcal{R}$) $S \in \mathcal{Q}_0 \subseteq \mathcal{Q}$ are intervals

$(\mathcal{Q} = \{P \in \mathcal{F}^c \text{ s.th } \exists\, Q \mid P \geq Q \in \mathcal{R}\})$

<u>Remark</u> For all interval $\mathcal{R}$, $S \in \mathcal{R} \Rightarrow$
$\mathcal{R}$ is closed under overgroups.

<u>Lemma</u> In this situation ($\mathcal{R} = \mathcal{Q} \setminus \mathcal{Q}_0$) we have a short exact sequence of functors

$$0 \longrightarrow Z_{\mathcal{F}}^{\mathcal{R}} \longrightarrow Z_{\mathcal{F}}^{\mathcal{Q}} \longrightarrow Z^{\mathcal{Q}_0} \longrightarrow 0$$

[short exact sequence for each $P \in \mathcal{F}^c$]
<u>proof</u>: exercice.

In general, $\forall$ category $\mathcal{C}$, $\forall$ short exact sequence of funcors from $\mathcal{C}^{op}$ to Ab, ~~0~~$\to$ ~~F~~ $F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ we have a long exact sequence:

$$0 \to H^0(\mathcal{C}; F') \longrightarrow H^0(\mathcal{C}; F) \to H^0(\mathcal{C}; F'') \longrightarrow$$
$$\longrightarrow H^1(\mathcal{C}; F') \longrightarrow \cdots$$

[one has a short exact sequence of chain complexes: $0 \to C^*(\mathcal{C}; F') \to C^*(\mathcal{C}; F) \to \ldots$]

<u>Notation</u>: $L^i(\mathcal{F}; \mathcal{R}) := H^i(\mathcal{O}(\mathcal{F}^c); Z_{\mathcal{F}}^{\mathcal{R}})$
Thus we have (when $\mathcal{R} = \mathcal{Q} \setminus \mathcal{Q}_0$) an exact sequence:

$$L^{i-1}(\mathcal{F}; \mathcal{Q}_0) \to L^i(\mathcal{F}; \mathcal{R}) \to L^i(\mathcal{F}, \mathcal{Q}) \to L^i(\mathcal{F}, \mathcal{Q}_0)$$

Now, $\mathcal{Q}_0 \subseteq \mathcal{Q}$ are closed under overgroups, $\mathcal{R} = \mathcal{Q} \setminus \mathcal{Q}_0$ and
$L^3(\mathcal{F}; \mathcal{Q})$ contains the obstruction to existence of a $\mathcal{Q}$-linking system associated to $\mathcal{F}$.

$L^3(\mathcal{F};\mathcal{R})$ contains the obstruction to the existence of a $\mathcal{Q}$-linking system containing a given $\mathcal{Q}_0$-linking system

$L^2(\mathcal{F};\mathcal{Q})$ measures the difference of 2 $\mathcal{Q}$-linking systems.

Given $\mathcal{L}^1_{\mathcal{Q}}$ and $\mathcal{L}^2_{\mathcal{Q}}$, $L^2(\mathcal{F};\mathcal{R})$ contains the obstruction to extending an isomorphism between the $\mathcal{Q}_0$-linking subsystems.

$L^1(\mathcal{F};\mathcal{Q})$: $\{$rigid automorphisms of $\mathcal{L}_{\mathcal{Q}}\}/_{\mathrm{Inn}}$.

<u>Prop</u> $\mathcal{F}$ a saturated fusion system on S, $Y \trianglelefteq \mathcal{F}$ a normal central subgroup, $\mathcal{Q} = \mathrm{Sub}(S)_{\geq Y}$. Then

$$L^i(\mathcal{F},\mathcal{Q}) = \begin{cases} Z(\mathcal{F}) & i=0 \\ 0 & i \neq 0 \end{cases}$$

<u>proof</u> uses models for constrained fusion systems thus uses a result of Jackowski - McClure.

<u>Prop</u> (Change of fusion systems) Let $\mathcal{F}$ be a saturated fusion system on S, $Q \leq S$ fully normalized in $\mathcal{F}$, $\mathcal{E} = N_{\mathcal{F}}(Q)$. $\mathcal{R} \subseteq \mathcal{F}^c$ an interval such that $\forall R \in \mathcal{R}$ $\exists!$ $Q_* \in Q^{\mathcal{F}}$, $Q_* \trianglelefteq R$.

Set $\mathcal{R}_0 = \{ R \in \mathcal{R} \mid Q \trianglelefteq R \}$.
Then $L^i(\mathcal{F}; \mathcal{R}) = L^i(\mathcal{E}, \mathcal{R}_0)$, for all i.

<u>Special case</u>: $\mathcal{R} = Q^{\mathcal{F}}$, then
$L^i(\mathcal{F}; \mathcal{R}) \cong L^i(N_{\mathcal{F}}(Q); \{Q\})$
depends only on $\mathcal{Z}_{\mathcal{F}}^{\{Q\}}(Q) = Z(Q)$
and $\mathrm{Aut}_{\mathcal{O}(\ )}(Q) = \mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}(Q)/\mathrm{Inn}(Q)$.

<u>Thm</u>: There exist functors $\Lambda^i(\Gamma; M)$, $\forall i \geq 0$
where Γ is a finite group and M is
a $\mathbb{Z}_{(p)}[\Gamma]$-mod such that
$L^i(\mathcal{F}; \mathcal{R}) \cong \Lambda^i(\mathrm{Out}_{\mathcal{F}}(Q); Z(Q))$

For all finite group G, for all prime p, let
$\mathcal{O}_p(G)$ be the category with
objects: p-subgroups
morphisms: $\mathrm{Mor}_{\mathcal{O}_p(G)}(P, Q) = \mathrm{Map}_G(G/P, G/Q)$
$= \cancel{T_G}(P, Q)/Q$.

$F_M : \mathcal{O}_p(G) \longrightarrow Ab$, M a $\mathbb{Z}_{(p)}G$-module.

$$F_M(P) = \begin{cases} M & P = 1 \\ 0 & P \neq 1 \end{cases} \qquad \mathrm{Aut}_{\mathcal{O}_p(G)}(1) = G.$$

$\Lambda^i(G; M) := H^i(\mathcal{O}_p(G); F_M)$

<u>Thm</u> $F : \mathcal{O}(\mathcal{F}^c)^{op} \longrightarrow \mathbb{Z}_{(p)}$-mod, $P \in \mathcal{F}^c$
such that $F(Q) = 0$, $\forall Q \notin P^{\mathcal{F}}$ then
$H^i(\mathcal{O}(\mathcal{F}^c); F) \cong \Lambda^i(\mathrm{Out}_{\mathcal{F}}(P); F(P))$

$$H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}) = L^i(\mathcal{F}, \mathcal{F}^c)$$

$$\phi = \mathcal{Q}_0 \subseteq \mathcal{Q}_1 \subseteq \cdots \subseteq \mathcal{Q}_n = \mathcal{F}^c$$

obstruction lies in $L^3(\mathcal{F}; \mathcal{Q}_{i+1} \setminus \mathcal{Q}_i)$

G a finite group, M a $\mathbb{Z}_{(p)}G$-module.

$$H = C_G(M) = \mathrm{Ker}\,[G \longrightarrow \mathrm{Aut}(M)]$$

- $O_p(G) \neq 1$ or $p \mid |H| \implies \Lambda^i(G;M) = 0 \ \forall i$
- $p \nmid |H| \implies \Lambda^i(G;M) \cong \Lambda^i(G/H;M)$
- $p \mid |G| \implies \Lambda^0(G;M) = 0$
- $p \nmid |G| \implies \Lambda^i(G;M) = \begin{cases} M^G = C_M(G) & i = 0 \\ 0 & i \neq 0. \end{cases}$