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Linking systems and classifying spaces for finite groups

Fusion system: category $\mathcal{F}$

$Ob(\mathcal{F}) = \{P \leq S\}$ S a p-group

$Hom_{\mathcal{S}}(P,Q) \subseteq Hom_{\mathcal{F}}(P,Q) \subseteq Inj(P,Q)$

$\forall \varphi \in Hom_{\mathcal{F}}(P,Q)$: $\varphi = (incl) \circ (\mathcal{F}\text{-isom})$

[Roberts - Specterov] (definition of saturation)

$\forall P \leq S$

- P is fully automized in $\mathcal{F}$ if
  $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$
- P is receptive in $\mathcal{F}$ is $\forall \varphi \in Iso_{\mathcal{F}}(Q,P)$
  φ extends to $\overline{\varphi} \in Hom_{\mathcal{F}}(N_\varphi, S)$
  where $N_\varphi = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in Aut_S(P)\}$.

$\mathcal{F}$ is saturated if each $P \leq S$ is $\mathcal{F}$-conjugate to a subgroup which is fully automised + receptive. in $\mathcal{F}$

A subgroup $P \leq S$ is $\mathcal{F}$-centric if
$\forall Q \leq P^{\mathcal{F}}$, $C_S(Q) \leq Q$ $(\Leftrightarrow C_S(Q) = Z(Q))$.

A centric linking system $\mathcal{L}$ associated to $\mathcal{F}$, is a category
$Ob(\mathcal{L}) = \mathcal{F}^c := \{P \leq S \mid P \text{ is } \mathcal{F}\text{-centric}\}$.

with $\mathcal{T}^c_S(S) \xrightarrow{\delta} \mathcal{L} \twoheadrightarrow \mathcal{F}^c$

such that

$\forall P, Q$:

$Z(P)$ acts freely on $Mor_{\mathcal{L}}(P,Q)$
via $\delta_P : N_S(P) \longrightarrow Aut_{\mathcal{L}}(P)$

$$N_S(P) = Aut_{\mathcal{F}_S(S)}(P)$$

and $\pi_P : Aut_{\mathcal{L}}(P) \xrightarrow{\text{onto}} Aut_{\mathcal{L}}(P)$ orbit map.

+ compatibility conditions.

<u>Geometric realization of a category $\mathcal{C}$:</u>

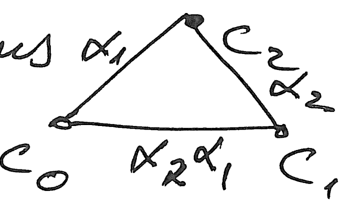
- $\mathcal{C}$ is a small category ($Ob(\mathcal{C})$ is a set)

Let $|\mathcal{C}|$ be the cell complex with

- 1 vertex for each object
- 1 edge for each morphism (attached to its source and target)

<u>but:</u> id_c is collapsed to the vertex $\bullet_c$

- a 2-simplex for each $c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} c_2$ composable pair of morphisms



<u>but</u> $c_0 \xrightarrow{\alpha} c_1 \xrightarrow{Id_{c_1}} c_1$ collapses to $c_0 \xrightarrow{\alpha} c_1$

etc ...

- an n-simplex for every composable sequence of n-morphisms. with natral identifications.

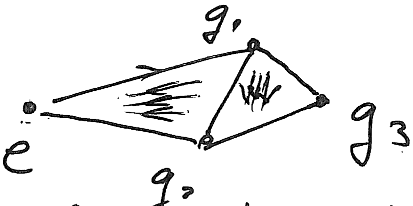
Examples

- $\mathcal{C}$: 1 object, 1 morphism (Id). then $|\mathcal{C}| = pt$.

- $\forall$ group G, $\mathcal{E}(G)$: $Ob(\mathcal{E}(G)) = G$
  $\forall g, h \ \exists! \ g \longrightarrow h$

  $B(G)$: $Ob(B(G)) = \{*\}$
  $Mor_{B(G)}(*, *) = G$.
  composition = group multiplication.

  $|\mathcal{E}(G)|$ is contractible:



G acts freely on $|\mathcal{E}(G)|$ via the action on $\mathcal{E}(G)$ (permutes objects by translation)

and $|\mathcal{E}(G)|/G \xrightarrow{\cong} |B(G)|$

<u>Thm</u> $|\mathcal{E}(G)| \longrightarrow |B(G)|$ is a covering space with covering group G.

$BG = |B(G)|$ is the classifying space for G.
$EG = |\mathcal{E}(G)|$ is its universal cover.

<u>Def</u>: Let $X \underset{p}{\sim} Y$ (X, Y are spaces) is an equivalence relation generated by. $X \sim_p Y$ if $\exists f: X \to Y$ s.th.

$$f_\#: H_*(X;\mathbb{F}_p) \xrightarrow{\simeq} H_*(Y;\mathbb{F}_p) \text{ is a isomorphism}$$

in fact:

$X \sim_p Y \iff \exists Z$ with $X \xrightarrow{f} Z \xleftarrow{g} Y$ such that $f_\#$ and $g_\#$ are isomorphism.

<u>Conjecture</u> [Martino - Priddy]
$\forall G, H$ finite groups, $\forall p$ prime $\left(\begin{matrix} S \in Syl_p(G) \\ T \in Syl_p(H) \end{matrix}\right)$

$$BG \sim_p BH \iff \underbrace{\mathcal{F}_S(G) \simeq \mathcal{F}_T(H)}_{\text{induced by some } \varphi: S \xrightarrow{\simeq} T.}$$

<u>Thm [BLO]</u> $\forall$ finite group G, $S \leq G$ $S \in Syl_p(G)$ we have

$$BG \sim_p |\mathcal{T}_S(G)| \sim_p |\mathcal{T}_S^c(G)| \sim_p |\mathcal{L}_S^c(G)|$$

$$\mathrm{Mor}_{\mathcal{T}_S(G)}(P, Q) = T_G(P, Q) = \{ g \in G \mid {}^gP \leq Q \}.$$

$$\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P, Q) = T_G(P, Q) / O^p(C_G(P))$$

<u>Hence</u>: $\mathcal{L}_S^c(G) \simeq \mathcal{L}_S^c(H)$ ($\underline{\text{any}}$ equivalence of categories)

$$\Rightarrow BG \sim_p |\mathcal{L}_S^c(G)| \simeq |\mathcal{L}_T^c(H)| \sim_p BH$$

Moreover : $BG \simeq_p BH \iff \mathcal{L}^c_S(G) \simeq \mathcal{L}^c_T(H)$.

<u>Left with the problem</u>:

Does $\mathcal{F}_S(G) \simeq \mathcal{F}_T(H) \Longrightarrow \mathcal{L}^c_S(G) \simeq \mathcal{L}^c_T(H)$

<u>Special case of</u>:
$\forall$ saturated fusion system $\mathcal{F}$ is there an associated centric linking system an if so, is it unique?

[BLO]: Want to associate a classifying space to each fusion system.

$\mathcal{F}$: fusion system
$\mathcal{L}$: centric linking system associated to $\mathcal{F}$.

$|\mathcal{L}|^\wedge_p$ = a classifying space for $\mathcal{F}$ (universal space for $\simeq_p$ equivalence class of $|\mathcal{L}|$).

Hence Chermak's thm $\Rightarrow \exists !$ classifying space associated to each saturated fusion system.

($\forall$ G: $BG^\wedge_p \simeq |\mathcal{L}^c_S(G)|^\wedge_p$ is the classifying space for $\mathcal{F}_S(G)$)

$\mathcal{F}$ a s.f.s. over S, $\mathcal{L}$ an associated centric linking system.

$$1 \to \{Z(P)\}_{P \leq S} \longrightarrow \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c \longrightarrow 1 \qquad \mathcal{F}^c \subseteq \mathcal{F}$$

$$\hat{Z} : (\mathcal{F}^c)^{op} \longrightarrow Ab.$$
$$P \longmapsto Z(P) = C_S(P)$$
$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

Need a copy $P \leq \mathrm{Aut}_{\mathcal{L}}(P)$

$$1 \to Z(P) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow 1.$$

<u>~~Obit~~ Orbit category</u> $\mathcal{O}(\mathcal{F}^c)$

$$Ob(\mathcal{O}(\mathcal{F}^c)) = Ob(\mathcal{F}^c)$$
$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P,Q)$$

We have:

$$1 \to \{P\}_{P \leq S} \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}(\mathcal{F}^c) \longrightarrow 1.$$

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{L}}(P,Q) \arrow[rr] \arrow[dr] & & \mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P,Q) \\ & \mathrm{Hom}_{\mathcal{F}}(P,Q) \arrow[ur] & \end{tikzcd}$$

So we can think of the kernel as a functor:

$$\mathcal{O}(\mathcal{F}^c) \xrightarrow{\text{incl}} Gps/Inn.$$

$$1 \to P \to Aut_{\mathcal{L}}(P) \to Aut_{\mathcal{O}(\mathcal{F}^c)}(P) \to 1.$$

We deal with extensions of non-abelian kern

$$1 \to H \to G \to \Gamma \to 1.$$

$$\begin{array}{ccc} H & \longrightarrow & Inn(H) \\ \downarrow & & \downarrow \\ G & \longrightarrow & Aut(H) \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & Out(H) \end{array}$$

Extension problem: Given H, Γ and $\varphi: \Gamma \to Out(H)$ describe all extension. Existence? Uniqueness?

Th [McLane?]

- The obstruction to existence of an extension lies in $H^3(\Gamma; Z(H))$
- If there are any, then $H^2(\Gamma; Z(H))$ acts fully and transitively on the set of all extensions.

Procedure:

- choose a lifting $\hat{\phi}: \Gamma \to Aut(H)$ (map of sets).

- $\forall \gamma, \delta \in \Gamma$ choose $t(\gamma,\delta) \in H$ s.th.
$$\hat\phi(\gamma) \circ \hat\phi(\delta) = c_{t(\gamma,\delta)}\, \hat\phi(\gamma\delta) \in \mathrm{Aut}(H)$$

Set $G = H \times \Gamma$ (as a set)
an define the product by
$$(h,\gamma) * (k,\delta) = (h\cdot\hat\phi(\gamma)(k)\cdot t(\gamma,\delta), \gamma\delta)$$

<u>Note (Exercise)</u>
$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(H) \\ (h,\gamma) &\longmapsto c_h \circ \hat\phi(h) \end{aligned}$$
is a homo (when G is a group)

We have
$$\begin{array}{ccccc} \Gamma & \xhookrightarrow{\text{set}} & G & \hookleftarrow & H \\ \gamma & \longmapsto & (1,\gamma) & & \\ & & (h,1) & \longleftarrow\!\shortmid & h \end{array}$$

$\forall \gamma, \delta, \varepsilon$: $\exists!$ $u(\gamma,\delta,\varepsilon) \in Z(H)$ such that
$$(\gamma\delta)\varepsilon = u(\gamma,\delta,\varepsilon)\, \gamma(\delta\varepsilon).$$

$$u \in C^3(\Gamma; Z(H)) = \mathrm{Map}(\Gamma^3, Z(H)).$$

Now
$$H^n(\Gamma; Z(H)) = H^n\left(C^{n-1} \xrightarrow{d} C^n(\Gamma; Z(H)) \xrightarrow{d} C^{n+1}\right)$$
where
$\forall \xi \in C^n$
$$d\xi(g_1, \dots, g_{n+1}) = g_1(\xi(g_2, \dots, g_{n+1})) + \\ + \sum_i (-1)^i \xi(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1}\xi(g_1, \dots, g_n).$$

$du = 0$

$u = 0 \Rightarrow G$ associative.

$u \in \mathrm{Im}(d) \Rightarrow$ can modify to make is associative

hence $[u] \in H^3(\Gamma; Z(H))$ is $0 \Leftrightarrow \exists$ such extension.

Generalize all that to categories:

$\mathcal{C}$ a category $F : \mathcal{C}^{op} \longrightarrow Ab$.

$$C^n(\mathcal{C}; F) = \prod_{c_0 \to \cdots \to c_n} F(c_0)$$

$$d : C^n \longrightarrow C^{n+1}$$

$$d\xi(c_0 \xrightarrow{\alpha} c_1 \to \cdots \to c_{n+1}) = \alpha^*(\xi(c_1 \to \cdots \to c_n)) + \\ + \sum (-1)^i \xi(c_0 \to \cdots \xrightarrow{\hat{c_i}} \to c_n).$$

$$H^n(\mathcal{C}; F) = \mathrm{Ker}[C^n \longrightarrow C^{n+1}] \Big/ \mathrm{Im}[C^n \longrightarrow C^{n+1}]$$

Thm For a given saturated fusion system

- $\exists\ [u] \in H^3(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}})$ is the obstruction to the existence of a linking system. associated to $\mathcal{F}$
- If $\exists\, \mathcal{L}$: $H^2(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}})$ acts freely on the set of all linking systems associated to $\mathcal{F}$.

where $\mathcal{Z}_{\mathcal{F}} : \mathcal{O}(\mathcal{F}^c) \longrightarrow Ab$

$$P \longmapsto Z(P) = C_S(P)$$

$$(P \xrightarrow{\varphi} Q) \longmapsto (Z(P) \xleftarrow{\varphi^{-1}} Z(Q))$$

Thm (Chermak) $\forall$ saturated fusion system $\mathcal{F}$, $H^n(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}) = 0$, $\forall\ n \geq 2$ and also for $n = 1$ if $p > 0$.

<u>Note</u>: $\forall$ saturated fusion system $\mathcal{F}$, assoc to $\mathcal{L}$. $\exists$ sequence

$$1 \to H^1(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}) \longrightarrow \operatorname{Out}_{typ}(\mathcal{L}) \longrightarrow \operatorname{Out}(S, \mathcal{F}) \longrightarrow H^2(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}})$$

where

$\operatorname{Aut}(S, \mathcal{F}) = \{ \alpha \in \operatorname{Aut}(S) \mid {}^{\alpha}\mathcal{F} = \mathcal{F} \}$ (α is fusion preserving)

$$\operatorname{Out}(S, \mathcal{F}) = \operatorname{Aut}(S, \mathcal{F}) / \operatorname{Aut}_{\mathcal{F}}(S).$$