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The argument with FF-pairs and automorphisms.

Correction

$(\mathcal{M}, \Delta)$ objective partial group (definition)

Replace (2)+(3) by $u \circ v \circ w \in \mathbb{D} \Rightarrow u \circ \Pi(v) \circ w \in \mathbb{D}$ and $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$.
there is an example by M W Jacobsen with a partial group $\mathcal{M}$ such that $N_{\mathcal{M}}(X)$ is not a subgroup.

Associativity

$\mathcal{M}$ a partial group, $\mathbb{W} = \mathbb{W}(\mathcal{M})$
$w, w' \in \mathbb{W}$, write $w \downarrow w'$ if $w = u \circ v \circ u'$ with $v \in \mathbb{D}$ and $w' = u \circ (\Pi(v)) \circ u'$. Let $\Downarrow$ be the transitive closure of $\downarrow$
$\mathcal{M}$ is associative if, whenever $w \in \mathbb{W}$ and $f, f' \in \mathcal{M}$ with $w \Downarrow f$ and $w \Downarrow f'$ then $f = f'$. Define $\sim$ on $\mathbb{W}$ by $u \sim v$ if $\exists w$ $w \Downarrow u$ and $w \Downarrow v$. Then $\sim$ is an equiv. relation. There is a group $\mathbb{W}/\sim$ given by $[u][v] = [u \circ v]$. This is in fact $\pi_1(|\mathcal{L}|)$ in case that $\mathcal{M}$ is a linking system $\mathcal{L}$.

Associativity of $\mathcal{L}$ $\equiv$

$$\mathcal{L} \hookrightarrow \mathbb{W}/\sim$$
$$f \longmapsto [f]$$

But we know examples where $\pi_1(|\mathcal{L}|) = 1$

It is not true in general that the localities are associative.

Def^n: A *general setup* consists of a (M, S, X) where M is a finite group. $S \in Syl_p(M)$, X a normal p-subgroup of M, such that $C_M(X) \leq X$. A *reduced setup* is a general setup (H, S, Y) where $Y = C_S(Z(Y)) = O_p(H)$ and $O_p(H/C_H(Z(Y))) =$

Lemma Let (M, S, X) be a general setup. Then there exists a maximal reduced setup. (H, S, Y), with Δ a unique largest subgroup of $Z(X)$ such that $Z(S) \leq \Delta \trianglelefteq M$ and s. th. $O_p(M/C_M(\Delta)) = 1$; Set $Y = C_S(\Delta)$ and $H = N_M(\Delta)$.

* (H, S, Y) is called the *reduced core* of (M, S, X).

Notation For reduced (H, S, Y) with $\Delta = Z(Y)$ let $J(S, \Delta)$ be the preimage in S of the subgroup of S/Y generated by offenders in S/Y on Δ.

Prop 7.11 Let (H, S, Y) be a reduced setup, $\Delta = Z(Y)$, $\Gamma = \{$ all overgroups Q of Y in S such that $J(Q, \Delta) \not\leq Y \}$

Assume $S \in \Gamma$; Set $\mathcal{L} = \mathcal{L}_\Gamma(H)$ and let $\beta \in Aut_0(\mathcal{L})$. Then β extends to an rigid automorphism of H.

The proof uses general FF-module theorem, and thus uses CFSG.

Prop 7.12. Let (M, S, X) be a general setup and (H, S, Y) be its reduced core. Set $\mathcal{F} = \mathcal{F}_S(M)$, let Γ be an $\mathcal{F}$-interval containing S such that $\forall (Y \leq) Q \in \Gamma$, we have $J(Q, D) \in \Gamma$ (where $D = Z(Y)$). Then each $\beta \in Aut_0(\mathcal{L}_\Gamma(M))$ extends to an automorphism of M

Lemma 7.13 (Uniqueness of $\mathcal{L}_\Gamma(M)$ in 7.12).
Suppose $\mathcal{L} = \mathcal{L}_\Gamma(M)$ and $\mathcal{L}'$ is any other $\mathcal{F}$-natural Γ- linking system, $\exists$ a rigid isomorphism $\mathcal{L} \xrightarrow[0]{\cong} \mathcal{L}'$
(Sketch of the proof): Involves a descent through $\mathcal{L}_0 \cong \mathcal{L}_0'$, $N_\mathcal{L}(S) \cong N_{\mathcal{L}'}(S)$.
Come as far as $\mathcal{L}_n \xrightarrow[\Delta_n]{\cong} \mathcal{L}_n'$ with some set Δ_n of objects. $\Delta_n \subseteq \Gamma$
$\mathcal{K} = \mathcal{L}_n$, $\mathcal{K}' \subseteq \mathcal{L}_n'$
Choose some T with which to advance to $\mathcal{K}^+$, $(\mathcal{K}')^+$, $\mathcal{L}_{n+1} = \mathcal{K}^+ = \mathcal{K}^+(\iota)$ where ι is the identity onto $N_\mathcal{K}(T)$.
$\mathcal{L}_{n+1}' = \mathcal{K}^+(\lambda)$ for some λ.
Show that all rigid automorphisms of $N_\mathcal{K}(T)$ extend to $N_\mathcal{L}(T)$. $\square$

<u>Hypothesis</u>: $\mathcal{F}$ a saturated fusion system on S, but no $\mathcal{L}$, or more than one $\mathcal{L}$.

"Filtration" of $\mathcal{F}^c$, $\mathcal{H} = \{(X_i, \Delta_i)\}_{i=0}^N$ where $X_i \leq S$ and Δ_i is an $\mathcal{F}$-invariant interval.

Let $X_0 = J(S)$, $\Delta_0 =$ all overgroups of $J(S)$ in Δ_i

Assuming we have X_n, Δ_n proceed to choose X_{n+1}

(1) so that $d(X_{n+1})$ is as large as possible. ($d(-)$ is the order of largest abelian subgroup) then

(2) so that $|J(X_{n+1})|$ is as large as possible, then

(3) if possible, so that $J(X_{n+1}) \in \mathcal{F}^c$ and then

(4) so that $|X_{n+1}|$ is minimal if $J(X_{n+1}) \in \mathcal{F}^c$ (so $X_{n+1} = J(X_{n+1})$), but with $|X_{n+1}|$ maximal if $J(X_{n+1}) \notin \mathcal{F}^c$.

(Also X_{n+1} is fully normalized in $\mathcal{F}$, and $J(X_{n+1})$ is fully normalized in $\mathcal{F}$)

$\Delta_{n+1} = \Delta_n \cup \{$subgroups of S containing a conjugate of $X_{n+1}\}$.

<u>Lemma</u> $\mathcal{H}$ has the "filtration property": ~~two~~ distinct $\mathcal{F}$-conjugate X_i generate a member of Δ_{i-1}.

<u>Proof</u> $n =$ least index for which lemma fails. Then $X = X_n$ has $\mathcal{F}$-conjugates $U \neq V$ with $\langle U, V\rangle \notin \Delta_{n-1}$. Set $Q = \langle U, V\rangle$. Then $d(Q) = d(X)$ and $J(Q) = J(U) = J(V)$.

If $J(X) \in \mathcal{F}^c$ then $X = J(X)$, $U = V \Rightarrow\Leftarrow$
So $J(X) \notin \mathcal{F}^c$ and maximality of $|X|$ yields $U = V = Q$. Again $\Rightarrow\Leftarrow$ $\square$.

Lemma 7.16 Let n be an index with $1 \leq n \leq r$. Suppose $J(X_n) \in \mathcal{F}^c$. Let $Q \in \Delta_{n-1}$ with $X_n \leq Q$. Then $J(Q) \in \Delta_{n-1}$.

Notation Assume existence and uniqueness (in the strong sense) of an $\mathcal{F}$-natural Δ_n-linking system (some $n \geq 0$). Set $X = X_{n+1}$, $R = N_S(X)$, M a model for $N_{\mathcal{F}}(X)$, Let (H, R, Y) be the reduced core of (M, R, X), $D = Z(Y)$, $\Delta = \Delta_n$. We have $Y \in \Delta \Rightarrow H \leq \mathcal{L}_{\Delta_X}(M)$

8.17 Suppose $Y \notin \Delta$

(a) $X = J(X) = J(Y)$

(b) $\Delta_X =$ the set of all subgroups $Q \leq R$ over X, such that $J(Q) \neq X$.
($\rightsquigarrow$ hypothesis of 7.12, $Q \in \Delta_X \Rightarrow J(Q, D) \in \Delta_X$).

Proof: Set $B = C_S(J(X))$. Then B is normalized by X, so BX is a p-group and $N_{BX}(X) = N_B(X)X$. [The aim is to show that $N_B(X) \leq X$, whence $B \leq X$]
$N_B(X) = C_R(J(X)) \leq C_R(Z(X)) \leq C_R(D) = Y$.
$N_B(X) = C_Y(J(X))$. But $J(X) = J(Y)$ as

$Y \notin \Delta$, so $C_Y(J(X)) = C_X(J(Y)) \leq J(Y) \leq X$

Thus $N_B(X) \leq X$, $X = J(X)$, so (a) holds.

(b) Let $Y \leq Q \leq R$, $Q \notin \Delta$. Then $J(X) = J(Q)$.

Conversely if $Q \in \Delta$, then, by 7.16 $J(Q) \in \Delta$.

This implies that $J(Q) = J(X)$ and so $J(Q) \leq X$.

Finish the case $Y \notin \Delta$. That is, advance to the existence and uniqueness of an $\mathcal{L}_{n+1}$.

7.13| $\lambda : N_{\mathcal{L}}(X) \xrightarrow[o]{\cong} \mathcal{L}_{\Delta_X}(M)$. Then 6.5(a) says There's an $\mathcal{L}^+(\lambda) = \mathcal{L}_{n+1}$ (existence)

7.12| Any $\lambda^{-1}\lambda'$ extends to an automorphism of M, 6.5(c) says $\mathcal{L}^+(\lambda) \overset{\sim}{=}_{o} \mathcal{L}^+(\lambda')$.

Existence and uniqueness of $\mathcal{L}$ holds.