Localities

Example: $G$ a finite group, $S \in \text{Syl}_p(G)$, $T = T^*_S(G)$, $S \Delta \leq \text{Sub}(G)$, $\Delta$-invariant, closed with respect to overgroups.

$L = L_G^{\Delta}(G) = \{ \Delta \text{-locality of } G \}$

\[ = \{ g \in G \mid S \Delta^{g}\Delta S \in \Delta \} \]

\[ \Delta = \{ (g_1, \ldots, g_n) \in W(G) \mid \prod_{i=1}^{k} P_{i} \in W(G) \} \]

\[ \Pi : \Delta \rightarrow L \text{ (multivariate product on } G) \]

\[ (g_1, \ldots, g_n) \mapsto g_1 g_2 \cdots g_n \]

This is an example of a "partial group" and, moreover, of an "objective partial group" (to be defined later).

If $\Gamma \in \Delta$ (with some conditions) then we have the notion of $L/\Gamma$.

Problem: Given a "rigid automorphism" $\beta$ of $L$ (to be defined), when does $\beta$ extend to an automorphism of $G$?

Special example $G = V \times \text{GL}_3(2)$ ($V$ vector space over $\mathbb{F}_2$) with faithful action $\Delta = \{ Q_1, Q_2, S \}$, $Q_1 \cap V = Q_2$,

$L = S \cap (Q_1)$, $\forall N_G(Q_i)$, $\beta$ a "rigid automorphism" means automorphism which centralizes $S$. 

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All such "rigid automorphisms" are given by pairs \( G_i^2 \) (\( i = 1, 2 \)) with \( z \in G_i(\mathbb{F}) = \mathbb{C}_V(S) \), acting on \( N_G(\mathbb{Q}) \).

**Exercise.** If \( V = \text{Steinberg module for } GL_3(2) \) and \( \{z_1, z_2\} \in \mathbb{C}_V(S) \), then \( \beta \) has no extension to an automorphism of \( G \). If \( V = \text{natural module (28)} \) then \( \beta \) has an extension.

**Def 2.1 (Partial group) \( \mathcal{M} \neq \emptyset, W = W(\mathcal{M}) \), \( \Delta \subseteq W \), \( \Pi : \Delta \rightarrow \mathcal{M} \) satisfying:

1. \( \mathcal{M} \subseteq \Delta \) and \( u \circ v \in \Delta \Rightarrow u, v \in \Delta \)
2. \( u \circ v \in \Delta \Rightarrow (\Pi(u), \Pi(v)) \in \Delta \) and \( \Pi((\Pi(u), \Pi(v))) = \Pi(u, v) ; \Pi(\Pi(u) = \text{id}_{\mathcal{M}} \)
3. Write \( 1 \) for \( \Pi(\emptyset) \). Then we have if \( u \circ v \in \Delta \), then \( u \circ 0 \circ v \in \Delta \)

(Exercise: \( \Pi(u \circ v) = \Pi(u \circ 0 \circ v) \)).

An involution is an involutory bijection \( x \mapsto x' \) on \( \Delta \) with the map on \( W \) given by \( (x_1, x_2, ..., x_n)^{-1} = (x_1^{-1}, ..., x_n^{-1}) \).

4. \( u \circ 1 = \Pi(u) \in \Delta \) and \( \Pi(u \circ 0(\mathcal{M}) = 1 \).

A partial group is a triple \((\mathcal{M}, \Pi, 1)\).

**Def 2.2 (Partial subgroup) \( \mathcal{N} \subseteq \mathcal{M} \) with \( \mathcal{N} \neq \emptyset \), \( \mathcal{N} \) closed under inversion, \( u \in W(\mathcal{N}) \).

\( \Pi(u) \in \mathcal{N} \)

**Conjugation:** For each \( \mathcal{F} \subseteq \mathcal{M} \), set \( \mathbf{1}(\mathcal{F}) \) be the set of all \( x \in \mathcal{M} \) s.t. \((\mathcal{F}, x, \mathcal{F}) \in \Delta \).
Then write $x^f$ for $\Pi (f^{-1}, x, f)$

**Def. 2.5** All a partial group, $\Delta \leq \text{Sub}(M)$ (the set of "objects"). Set $D_\Delta$ is the set of all words $w = (f_1, \ldots, f_n) \in W(M)$ such that $f_i(x_0, \ldots, x_n) \in W(D)$ with $f_i(x_i) = x_i$ for all $i$, $1 \leq i \leq n$.

$(M, \Delta)$ is an objective partial group if

1. $D_\Delta = 0_D$
2. Given $X, Z \in \Delta$ and $f \in M$ with $X^f \subseteq Z$ then $N_X (X^f) \in \Delta$ for every subgroup $Y$ of $Z$ containing $X^f$.

(In particular $X^f \in \Delta$)

$N_M (X)$ is a subgroup $M$, if $X \in \Delta$

$x \xrightarrow{f} x \xrightarrow{f} x \xrightarrow{f} x \ldots \xrightarrow{f} x$

**Example** $M_{13} \leq \text{Alt} (13)$

Conway, Elkies, Martin (2005)
12 tiles + 1 hole on $P(3)$

**Def. 2.8** Let $p$ be a prime, $S$ a finite $p$-subgroup of $L$, $\Delta$ a partial group. Then $(L, S)$ is a locality if $L$ is finite and the exists $\Delta \leq \text{Sub}(S)$, $S \in \Delta$ such that

1. $(L, \Delta)$ is objective
2. $S$ is maximal in the poset of finite $p$-subgroup of $L$

Let $S = F_q (L)$ = fusion system on $S$ generated
by all \( L \) conjugation maps between subgroups of \( S \). \( L \) is a \( \Delta \)-linking system if
\[
\Delta \subseteq T^\circ \text{ and } \Omega^\Delta(X(P)) = 1 \text{ for all } P \in \Delta
\]
effective linking system \( \iff T^\circ \text{ - effective.} \)

Let \( \Delta(L,S) = \frac{1}{2} |\Delta| / \Delta \) work in def 2.8.

**Def.** \( L \) is **complete** if for each \( f \in L \) and each \( \Delta \in \Delta(L,S) \) the set
\[
S_f = \{ x \in S \mid x f \in S \} \subseteq \Delta
\]

**Prop 2.9.** Every locality is complete.

**Proof:** Given \( (L, \Delta, S) \) a locality and \( f \\in L \).

Then \( (f) \in \Delta = \Delta \Delta \) so \( P_f = Q \) \( (P, Q \in \Delta) \).

**Step 1.** Let \( a \in S_f \). Then \( P^a \leq S_f \).

Set \( b = a f \), then
\[
(\ast) \quad (a^{-1}, f, b) \in \Delta \quad P \xrightarrow{\theta^{-1}} P^a \xrightarrow{f} Q \xrightarrow{b} Q^a.
\]

So \( (f, b) \in \Delta \). But also \( (a, f) \in \Delta \) via \( P^a \).

Given that \( f^{-1} a f = b \) we have \( a f = f b \) (uncancellation rule).

Then \( \psi \psi b = a^{-1} (a f) = f \cdot \psi(\psi a f) \leq S \)
and \( P^a \leq S_f \), \( P_f \in \Delta \).

**Step 2.** Let \( x, y \in S_f \). Step 1 \( \Rightarrow P^x \leq S_f \), \( (P^x) \psi \leq S_f \).

\[ w = (x, y, f), (x, f, y), (y, f) \in \Delta \text{ via } P^{-f}. \]

\[ \pi(w) = x y f = (xy)^{-1} f. \] This shows that \( S_f \) is closed under product (closure under inverses).
Corollary 2.19  (a) Subgroups of localities are local subgroups \( H \leq L \Rightarrow H \leq \mathbb{L}_P(P) \), for some object \( P \).

(b) Every \( p \)-subgroup of a locality \( (L, S) \) is conjugate into \( S \).

Definition 3.1. Let \( \mathcal{M}, \mathcal{M}' \) be partial groups. A mapping \( \beta: \mathcal{M} \rightarrow \mathcal{M}' \) is a homomorphism of partial groups if \( \beta\, B \leq \mathcal{M}' \) (where \( B \) is the induced map \( \mathbb{W} \rightarrow \mathbb{W}' \) of free monoids) and \( \Pi'(w B') = \Pi(w) \beta \) for all \( w \in \mathcal{M} \).

\[ \text{Ker}(\beta) = \{ f \in \mathcal{M}' : \beta\, f = 1 \} \] . Here \( \cdot \text{Ker}(\beta) \cdot \text{Lker} \).

Definition 4. Let \( (\mathcal{L}, \Delta) \), \( (\mathcal{L}', \Delta') \) be objective partial groups. A homomorphism \( (\mathcal{L}, \Delta) \rightarrow (\mathcal{L}', \Delta') \) of objective partial groups consists of \( \beta: \mathcal{L} \rightarrow \mathcal{L}' \) (homomorphism of partial groups) such there exists \( f: \Delta \rightarrow \Delta' \) with \( w \in \Delta \) via \( \beta = w B \in \Delta' \) via \( \beta \).

This makes a category of partial groups and objective partial groups.

Isomorphism = invertible homomorphism.

Rigid isomorphism between \( (L, S, \Delta) \) and \( (L', S', \Delta') \) = isomorphism with \( f = \text{id}_\Delta \) and \( \beta' = \text{id}_S \).
The method of descent.

**Def.** A fusion system $F$ on $S$ is saturated if:

1. **(1)** $\forall P \leq S$ has a fully normalized $F$-conjugate $Q$, i.e., $\forall R \approx_F Q$. If $\exists R \leq Q$ such that $\exists (R, Q)$ that extends to $N_S(R)$:
   \[
   N_S(R) \xrightarrow{\psi} N_S(Q)
   \]

2. **(2)** For every fully normalized $Q \leq F$, there exists a finite group $M$ with $Syl_p\sigma_p$ subgroup $N_S(Q)$ and with:
   \[
   F_{N_S(Q)}(M) = N_{\overline{F}}(Q)
   \]

3. **(3)** $F$ is generated by $\cup_{Q \leq F} N_{\overline{F}}(Q)$.

Special case of existence and uniqueness conditions "constrained" fusion systems.

$F = F_S(M)$ where $M$ is finite, $S \in Syl_p(M)$ and $\exists Q \leq F$ with $Q \leq M$ and $Q \cap (M) = 1$.

6.1 **Def.** Let $L = (L, A, S)$ be a locality and let $F$ be a fusion system on $S$.

Then $L$ is $F$-natural if $Hom_L(AQ) = Hom(F_AQ)$ for all $A, Q$ in $A$.

6.3 **Hypothesis:** We are given a fusion system $F$ on $S$, an $F$-natural locality.
and a subgroup $T \leq S$, fully normalized in $F$, $T \not \subseteq \Delta$, but with the properties that $\langle u, v \rangle \in \Delta$ for any two $u, v \in T$ with $u \neq v$.

Set $\Delta^+ = \Delta \cup \{ P \leq S \mid U \leq P \text{ for some } U \in T \}$

6.4 Lemma. Let $U \in \Delta^+$

(a) If $U \leq P \in \Delta$ then $N_P(U) \in \Delta$

(b) $x \in S^T$ such that $T^x = U$ and $N_T(U) = N_G(U)$

Proof.
(a) (exercise) distinguish the case $U \leq P$ and $U \not \subseteq P$. The second case by induction on $LP: U$.

(b) $T$ fully normalized $\Rightarrow \exists y \in F \setminus U \setminus \Delta$ and $N_T(U) = N_G(U)$. Here $N_G(U) \in \Delta$ by (a).

As $x$ is $F$-natural, there exists $y \in L$ such that $U = \langle y \rangle$. Take $x = y^{-1}$.

Assume (with 6.3) a finite group $M$ with $N_S(T) e \text{Sy}_{gp}(M)$ and with $F_{N_S(T)}(M) = N_S(T)$.

Exercise (see 2.17) $(N_S(T), \Delta_T, N_S(T))$ is a locality $(\Delta_T = \{ P \in \Delta \mid T \leq P \})$.

Assume also: we have a rigid isomorphism $\lambda : N_S(T) \to \mathbb{L}_{\Delta_T}(M)$. 
Thm 6.5 With above hypotheses (6.3. and 7.12).
(a) There exists an $F$-natural $L^+$-locality
\[ L^+ = (L^+, \Delta^+, \varepsilon) = L^+(\lambda), \text{ so that} \\
L^+|_{\Delta} = L \text{ and } \lambda \text{ extends to an isomorphism} \\
(\text{in a canonical way}) \text{ to } \lambda^*: N^+_F(\tau) \rightarrow M. \\
(b) All $F$-natural localities with set $\Delta^+$ of objects arise in this way. (for some) \\
(c) $L^+(\lambda) \cong L^+(\lambda')$ (use the same $M$ for $\lambda$ and $\lambda'$) \\
iff $\lambda^{-1}\lambda'$ extends to an automorphism of $M$.
(d) (Existence and uniqueness of $L^+$ ...).

What is $L^+$?
Let $\Phi$ be the set of all triples $\tau = (x^-, g, y) \in L^+ \times M \times L$ such that

1. $T \leq S_x \cap S_y$ \( \Upsilon = T \times, \psi = T \psi \)
2. $N^x_\psi(T) = N^y_\psi(U), N^y_\psi(T) = N^x_\psi(U) \)

Define a relation $\sim$ on $\Phi$ by
\[(x^-, g, y) \sim (x^-, \tilde{g}, \tilde{y}), \text{ if } (\tilde{x}^-, \tilde{g}, \tilde{y}).\]

\[ U \xrightarrow{x^-} T \xrightarrow{g} T \xrightarrow{y} V \]
\[ \Upsilon \xrightarrow{(x^-, \tilde{g}, \tilde{y})} \Upsilon \]
\[ U \xrightarrow{x^-} T \xrightarrow{\tilde{g}} T \xrightarrow{\tilde{y}} V \]

Then $\sim$ is an equivalence relation.

Define also $\sim$ by $f \sim (x^-, (xy)^{-1}y, y)$
\[ U \xrightarrow{x^-} T \xrightarrow{g^{-1}(xy)^{-1}y} T \xrightarrow{y} V \]

Let $\sim$ be the equivalence relation generated by $\sim$ and $\sim$. 
Set $L_0^+ = \{ u L_0 \} / \sim$ where $L_0 = \{ \frac{1}{n} \in \mathbb{Z}^+ \mid u \neq \emptyset \}$ for some $u \in \mathbb{Z}^+$.

Set $L_1^+ = L \setminus L_0$ and $L^+ = L_0 + u L_1^+$.

Let $\Delta^+ = \{ (c_1, \ldots, c_n) \in W(L_0^+) \mid \exists \text{ representatives } y_i = (x_i, g_i, y_i) \in \mathcal{C}_i \text{ with } w_0 = (g_1, (y_1, x_1^{-1}), g_2, \ldots, (y_n-x_n), g_n) \}$ making sense.

Show that the product is well defined.

Show that $\Pi_0^+$ and $\Pi$ agree on $\Delta^+ \cap \Pi_0$.

$\Pi_0^+(\Delta^+) = \{ (c_1, \ldots, c_n) \in \Delta^+ \mid \text{each } c_i \text{ has a representative } f_i \in \mathcal{L} \text{ which is unique} \}$.

Take $\Delta^+ = \Delta_0^+ \cup \Delta$, $\Pi_0^+ \cup \Pi$ and in this way obtain a partial group $\Pi^+$.

To show that $(L^+, \Delta^+)$ is a locally need to prove that $(L^+, \Delta^+)$ is objective.

This is difficult.

At issue: Given $C = [x^{-1}, g, y]$ and $u, v \in \mathbb{F}$, with $u C = v$ (i.e. $\forall u \in \mathcal{C}$, $\Pi^+(C^{-1}, u, C)$ is defined and is in $V$), one needs to show that a representative $[x^{-1}, g, y]$ can be
chosen with \( U \xrightarrow{T} T' \xrightarrow{T} T'' \xrightarrow{Y} V \) as in condition (1) in definition of \( D \).

Scheme for \( T \) and uniqueness.

- **Start with** \((L_0, \Delta_0, S)\) where
  - \(L_0\) is a model for \( \text{N}_{\text{tf}}(f(S)) \), here
  - \( f(S) \) is generated by abelian subgroups of \( S \) of maximal order;
  - \( \Delta_0 = \{ P \leq S \mid f(S) \leq P \} \).

- **Choice of** \( T_1, \lambda_1 \) and then
  - construct \((L_1, \Delta_1, S)\) with \( L_1 = L_0 \circ (1) \) and
  - \( \Delta_1 = \Delta_0 \), and how to iterate.