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Localities

<u>Example</u>: G a finite group, $S \in Syl_p(G)$
$\mathcal{F} = \mathcal{F}_S(G)$; $S \in \Delta \subseteq Sub(S)$, $\mathcal{F}$-invariant, closed with respect to overgroups.

$$\mathcal{L} = \mathcal{L}_\Delta(G) = \text{"}\Delta\text{-locality of } G\text{"}$$
$$= \{ g \in G \mid S \cap {}^gS \in \Delta \}$$

$$\mathbb{D} = \{ (g_1, \dots, g_n) \in \underset{\text{free monoid on } G}{W(G)} \mid \exists (P_0, \dots, P_n) \in W(\Delta),\ P_0 \xrightarrow{c_{g_1}} \cdots \xrightarrow{c_{g_n}} P_n \}$$

$\Pi : \mathbb{D} \to \mathcal{L}$ (multivariate product in G)
$(g_1, \dots, g_n) \mapsto g_1 \cdot g_2 \cdot \dots \cdot g_n$

This is an example of a "partial group" and, moreover, of an "objective partial group" (to be defined later).

If $\Gamma \subseteq \Delta$ (with some conditions) then we have the notion of $\mathcal{L}/\Gamma$

<u>Problem</u>: Given a "rigid automorphism" β of $\mathcal{L}$ (to be defined), when does β extend to an automorphism of G?

<u>Special example</u> $G = V \rtimes GL_3(2)$
(V vector space over $\mathbb{F}_2$) with faithful action $\Delta = \{ Q_1, Q_2, S \}$ where $Q_i/V \cong C_2 \times C_2$

$$\mathcal{L} = N_G(Q_1) \underset{S}{\cup} N_G(Q_2)$$

β a "rigid automorphism" means automorphism which centralizes S

All such "rigid automorphisms" are given by pairs c_{z_i} $(i=1,2)$ with $z_i \in Z(S) = C_V(S)^{z_i}$, acting on $N_G(Q_i)$

<u>Exercice</u>. If V = Steinberg module for $GL_3(2)$ and $\{z_1, z_2\} = C_V(S)$, then β has no extension to an automorphism of G. If V = natural module (2^3) then β has an extension.

<u>Def 2.1</u> (Partial group) $\mathcal{M} \neq \emptyset$, $\mathbb{W} = \mathbb{W}(\mathcal{M})$, $\mathbb{D} \subseteq \mathbb{W}$, $\Pi : \mathbb{D} \longrightarrow \mathcal{M}$ satisfying:

(1) $\mathcal{M} \subseteq \mathbb{D}$ and $u \circ v \in \mathbb{D} \Rightarrow u, v \in \mathbb{D}$

(2) $u \circ v \in \mathbb{D} \Rightarrow (\Pi(u), \Pi(v)) \in \mathbb{D}$ and $\Pi(\Pi(u), \Pi(v)) = \Pi(u, v)$; $\Pi|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$

(3) Write 1 for $\Pi(\emptyset)$. Then we have if $u \circ v \in \mathbb{D}$ then $u \circ 1 \circ v \in \mathbb{D}$ (<u>exercice</u>: $\Pi(u \circ v) = \Pi(u \circ 1 \circ v)$).

An involution is an involutory bijection $x \longmapsto x^{-1}$ on $\mathcal{M}$ together with the map on $\mathbb{W}$ given by $(x_1, x_2, \ldots, x_n)^{-1} = (x_n^{-1}, \ldots, x_1^{-1})$.

(4) $u \in \mathbb{D} \Rightarrow u^{-1} \circ u \in \mathbb{D}$ and $\Pi(u^{-1} \circ u) = 1$.

A <u>partial group</u> is a triple $(\mathbb{D}, \Pi, {}^{-1})$.

<u>Def</u> (Partial subgroup) $\mathcal{N} \subseteq \mathcal{M}$ with $1 \in \mathcal{N}$, $\mathcal{N}$ closed under inversion, $u \in \mathbb{W}(\mathcal{N}) \cap \mathbb{D} \Rightarrow \Pi(u) \in \mathcal{N}$

<u>Conjugation</u>: For each $f \in \mathcal{M}$ set $\mathbb{D}(f)$ be the set of all $x \in \mathcal{M}$ s.th $(f^{-1}, x, f) \in \mathbb{D}$

Then write x^f for $\Pi(f^{-1}, x, f)$

<u>Def^n 2.5</u> $\mathcal{M}$ a partial group, $\Delta \subseteq Sub(\mathcal{M})$ (the set of "objects"). Set $\mathbb{D}_\Delta$ = the set of all words $w = (f_1, \dots, f_n) \in \mathbb{W}(\mathcal{M})$ such that $\exists\ (X_0, \dots, X_n) \in \mathbb{W}(\Delta)$ with $(X_{i-1})^{f_i} = X_i$ for all i, $1 \leq i \leq n$.
$(\mathcal{M}, \Delta)$ is an <u>objective partial group</u> if
(O1) $\mathbb{D} = \mathbb{D}_\Delta$
(O2) Given $X, Z \in \Delta$ and $f \in \mathcal{M}$ with $X^f \leq Z$ then $N_Y(X^f) \in \Delta$ for every subgroup Y of Z containing X^f.
(In particular $X^f \in \Delta$)
$N_{\mathcal{M}}(X)$ is a subgroup $\mathcal{M}$, if $X \in \Delta$

$$X \xrightarrow{f_1} X \xrightarrow{f_2} X \longrightarrow \cdots \xrightarrow{f_n} X$$

<u>Example</u> $M_{13} \subseteq Alt(13)$
Conway, Elkies, Martin (2005)
12 tiles + 1 hole on $\mathbb{P}(3)$.

<u>Def^n 2.8</u> Let p be a prime, S a finite p-subgroup of $\mathcal{L}$, $\mathcal{L}$ a partial group. Then $(\mathcal{L}, S)$ is a locality if $\mathcal{L}$ is finite and the exists $\Delta \subseteq Sub(S)$, $S \in \Delta$ such that
(L1) $(\mathcal{L}, \Delta)$ is objective
(L2) S is maximal in the poset of finite p-subgroup of $\mathcal{L}$.
Let $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ = fusion system on S generated

by all $\mathcal{L}$ conjugation maps between subgroups of S. $\mathcal{L}$ is a $\underline{\Delta\text{-linking system}}$ if $\Delta \subseteq \mathcal{F}^c$ and $O^p(C_{\mathcal{L}}(P)) = 1$ for all $P \in \Delta$

centric linking system $\overset{\text{def.}}{\Longleftrightarrow}$ $\mathcal{F}^c$ – centric.

Let $\Omega(\mathcal{L}, S) = \{\Delta \mid \Delta$ works in def^n 2.8$\}$.

<u>Def^n</u> $\mathcal{L}$ is <u>complete</u> if for each $f \in \mathcal{L}$ and each $\Delta \in \Omega(\mathcal{L}, S)$ the set
$$S_f = \{x \in S \mid x^f \in S\} \in \Delta$$

<u>Prop 2.9</u> Every locality is complete

<u>Proof</u>: Given $(\mathcal{L}, \Delta, S)$ a locality and $f \in \mathcal{L}$. Then $(f) \in \mathbb{D} = \mathbb{D}_\Delta$ so $P^f = Q$ $(P, Q \in \Delta)$

<u>Step 1</u> Let $a \in S_f$. Then $P^a \leq S_f$.

Set $b = a^f$, then

$(*)$ $(a^{-1}, f, b) \in \mathbb{D}$ $\quad P^a \xrightarrow{a^{-1}} P \xrightarrow{f} Q \xrightarrow{b} Q^b$.

So $(f, b) \in \mathbb{D}$. But also $(a, f) \in \mathbb{D}$ via $P^{a^{-1}}$

Given that $f^{-1} a f = b$ have $af = fb$
(uncancellation rule)

Then $a^{-1} f b = a^{-1}(af) = f$. so $(P^a)^f \leq S$ and $P^a \subseteq S_f$, $P^a \in \Delta$.

<u>Step 2</u> Let $x, y \in S_f$. Step 1 $\Rightarrow P^x \leq S_f, (P^x)^y \leq S_f$ (in Δ)

$w = ((f^{-1}, x), (f, f^{-1}), (y, f)) \in \mathbb{D}$ via P^f.

$\Pi(w) = x^f y^f = (xy)^f$. This shows that S_f is closed under product (closure under inverses)

$\square$

Corollary 2.19 (a) Subgroups of localities are local subgroups ($H \leq \mathcal{L} \Rightarrow H \leq N_{\mathcal{L}}(P)$, for some object P)

(b) Every p-subgroup of a locality $(\mathcal{L}, S)$ is conjugate into S.

Def^n 3.1 Let $\mathcal{M}, \mathcal{M}'$ be partial groups. A mapping $\beta : \mathcal{M} \longrightarrow \mathcal{M}'$ is a homomorphism of partial groups if $\mathbb{D}\beta_* \subseteq \mathbb{D}'$ (where β_* is the induced map $W \longrightarrow W'$ of free monoids) and $\Pi'(w\beta_*) = \Pi(w)\beta$ for all $w \in \mathbb{D}$.

$Ker(\beta) = \{f \in \mathcal{M} \mid f\beta = 1'\}$. Here $f^{-1}Ker(\beta)f \subseteq Ker$ $\forall f \in \mathcal{M}$

Def^n Let $(\mathcal{M}, \Delta)$, $(\mathcal{M}', \Delta')$ be objective partial groups. A homomorphism $(\mathcal{M}, \Delta) \longrightarrow (\mathcal{M}', \Delta')$ (of objective partial groups) consists of $\beta : \mathcal{M} \longrightarrow \mathcal{M}'$ (homomorphism of partial groups) s.th there exists $\gamma : \Delta \longrightarrow \Delta'$ with $w \in \mathbb{D}$ via $P \Rightarrow w\beta_* \in \mathbb{D}'$ via $P\gamma$.

This makes a category of partial groups and objective partial groups.

Isomorphism = invertible homomorphism.

Rigid isomorphism between $(\mathcal{L}, S, \Delta)$ and $(\mathcal{L}, S, \Delta)$ = isomorphism with $\gamma = id_\Delta$ and $\beta|_S = id_S$.

The method of descent

<u>Def^n</u> A fusion system $\mathcal{F}$ on S is saturated if:

(1) $\forall P \leq S$ has a fully normalized $\mathcal{F}$-conjugate Q. i.e. $\forall R \sim_{\mathcal{F}} Q$. $\exists\, \varphi \in \mathrm{Iso}(R, Q)$ that extends to $N_S(R)$

$$\begin{array}{ccc} N_S(R) & \xrightarrow{\tilde{\varphi}} & N_S(Q) \\ | & & | \\ R & \xrightarrow{\varphi} & Q \end{array}$$

(2) For every fully normalized $Q \in \mathcal{F}^c$ there exists a finite group M with Sylow p-subgroup $N_S(Q)$ and with $\mathcal{F}_{N_S(Q)}(M) = N_{\mathcal{F}}(Q)$

(3) $\mathcal{F}$ is generated by $\bigcup_{Q \in \mathcal{F}^{fc}} N_{\mathcal{F}}(Q)$.

<u>Special case of</u> existence and uniqueness concerns "constrained" fusion systems.
$\mathcal{F} = \mathcal{F}_S(M)$ where M is finite, $S \in Syl_p(M)$ and $\exists\, Q \in \mathcal{F}^c$ with $Q \trianglelefteq M$ and $O_{p'}(M) = 1$.

<u>6.1 Def^n</u> Let $\mathcal{L} = (\mathcal{L}, \Delta, S)$ be a locality and let $\mathcal{F}$ be a fusion system on S. Then $\mathcal{L}$ is $\mathcal{F}$-natural if $\mathrm{Hom}_{\mathcal{L}}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for all P, Q in Δ

<u>6.3 Hypothesis</u>: We are given a fusion system $\mathcal{F}$ on S, an $\mathcal{F}$-natural locality,

and a subgroup $T \leq S$, fully normalized in $\mathcal{F}$, $T \notin \Delta$, but with the properties that $\langle U, V \rangle \in \Delta$ for any ~~two~~ $U, V \in T$ with $U \neq V$.

Set $\Delta^+ = \Delta \cup \{ P \leq S \mid U \leq P$ for some $U \in T^{\mathcal{F}} \}$

<u>6.4 Lemma</u> Let $U \in T^{\mathcal{F}}$

(a) If $U \leq P \in \Delta$ then $N_P(U) \in \Delta$ (In particular $N_S(U) \in \Delta$).

(b) $\exists\, x \in \mathcal{L}$ such that $T^x = U$ and with $(N_{S_x}(T))^x = N_S(U)$
($f \in \mathcal{L}$, $S_f = \{ y \in S \mid y^f \in S \}$)

<u>Proof</u>

(a) (exercice) distinguish the case $U \trianglelefteq P$ and $U \ntrianglelefteq P$. The second case by induction on $|P:U|$.

(b) T fully normalized $\Rightarrow \exists\, \varphi \in \mathcal{F}$, $U\varphi = T$ and $N_S(U)\varphi = N_S(T)$. Here $N_S(U) \in \Delta$ by (a). As $\mathcal{L}$ is $\mathcal{F}$-natural there exists $y \in \mathcal{L}$ with $\varphi = c_y$. Take $x = y^{-1}$. □

<u>Assume</u> (with 6.3) a finite group M with $N_S(T) \in Syl_p(M)$ and with $\mathcal{F}_{N_S(T)}(M) = N_{\mathcal{F}}(T)$.

<u>Exercice</u> (see 2.17) $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is a locality ($\Delta_T = \{ P \in \Delta \mid T \trianglelefteq P \}$)

<u>Assume also</u>: we have a rigid isomorphism
$\lambda : N_{\mathcal{L}}(T) \longrightarrow \mathcal{L}_{\Delta_T}(M)$.

Thm 6.5 With above hypotheses (6.3. and $\exists M, \lambda$).

(a) There exists an $\mathcal{F}$-natural locality
$\mathcal{L}^+ = (\mathcal{L}^+, \Delta^+, S) = \mathcal{L}^+(\lambda)$ s.th.
$\mathcal{L}^+|_\Delta = \mathcal{L}$ and λ extends to an isomorphism (in a canonical way) to $\lambda^+ : N_{\mathcal{L}^+}(T) \longrightarrow M$.

(b) All $\mathcal{F}$-natural localities with set Δ^+ of objects arise in this way. (for some λ)

(c) $\mathcal{L}^+(\lambda) \overset{\text{rigid}}{\cong} \mathcal{L}^+(\lambda')$ (via the same M for λ and λ') iff $\lambda^{-1} \circ \lambda'$ extends to an automorphism of M

(d) (Existence + uniqueness of $\mathcal{L}^+$ ...).

What is $\mathcal{L}^+$?

Let Φ be the set of all triples $\varphi = (x^{-1}, g, y) \in \mathcal{L} \times M \times \mathcal{L}$ such that

(1) $T \le S_x \cap S_y$ $\quad (U = T^x,\ V = T^y)$

$N_{S_x}(T) = N_S(U)$, $N_{S_y}(T) = N_S(V)$

Define a relation $\sim_0$ on Φ by
$(x^{-1}, g, y) \sim_0 (\bar{x}^{-1}, \bar{g}, \bar{y})$ if $(\bar{x}x^{-1})\lambda \cdot g = \bar{g}\,(\bar{y}y^{-1})\lambda$.

$$
\begin{tikzcd}
U \arrow[r, "x^{-1}"] \arrow[d, equal] & T \arrow[r, "g"] & T \arrow[r, "y"] & V \arrow[d, equal] \\
U \arrow[r, "\bar{x}^{-1}"'] & T \arrow[u, "(\bar{x}x^{-1})\lambda"] \arrow[r, "\bar{g}"'] & T \arrow[u, "(\bar{y}y^{-1})\lambda"'] \arrow[r, "\bar{y}"'] & V
\end{tikzcd}
$$

Then $\sim_0$ is an equivalence relation.

Define also $\sim_1$ by $f \sim_1 (x^{-1}, (xfy^{-1})\lambda, y)$

$$
\begin{tikzcd}
U \arrow[r, "x^{-1}"] \arrow[rrr, bend right, "f"'] & T \arrow[r, "xfy^{-1}", "g = (xfy^{-1})\lambda"'] & T \arrow[r, "y"] & V
\end{tikzcd}
\quad = (x^{-1}, g, y).
$$

Let $\sim$ be the equivalence relation generated by $\sim_0$ and $\sim_1$

Set $\mathcal{L}_0^+$ = the set of such equivalence classes $(\Phi \sqcup \mathcal{L}_0)/\sim$
where $\mathcal{L}_0 = \{ f \in \mathcal{L} \mid U^f = V \text{ for some } U, V \in T^{\mathcal{F}} \}$

Set $\mathcal{L}_1^+ = \mathcal{L} \setminus \mathcal{L}_0$ and $\mathcal{L}^+ = \mathcal{L}_0^+ \sqcup \mathcal{L}_1^+$
Let $\mathbb{D}_0^+$ be the set of all $w = (c_1, \dots, c_n) \in W(\mathcal{L}_0^+)$ such that there are representatives $\varphi_i = (x_i^{-1}, g_i, y_i) \in C_i$ with
$w_0 = (g_1, (y_1 x_2^{-1})\lambda, g_2, \dots, g_{n-1}, (y_{n-1} x_n^{-1}), g_n)$
making sense.
$\Pi_0^+(w) = [x_1^{-1}, \Pi_M(w_0), y_n]$ (equiv class for $\sim$)

Show that the product is well defined
Show that Π_0^+ and Π agree on $\mathbb{D}_0^+ \cap \mathbb{D}$
$\mathbb{D}_0^+ \cap \mathbb{D} = \{ (c_1, \dots, c_n) \in \mathbb{D}_0^+$ s.th. each c_i has a representative $f_i \in \mathcal{L}$ (which is unique) and such that $(f_1, \dots, f_n) \in \mathbb{D} \}$.
Take $\mathbb{D}^+ = \mathbb{D}_0^+ \sqcup \mathbb{D}$, $\Pi_0^+ \sqcup \Pi$ and in this way obtain a partial group.
where $1^+ = [1, 1_M, 1]$ and $[x^{-1}, g, y]^{-1} = [y^{-1}, g^{-1}, x]$.

To show that $(\mathcal{L}^+, \Delta^+, S)$ is a locality need to prove that $(\mathcal{L}^+, \Delta^+)$ is objective.
<u>This is difficult!</u>
<u>At issue</u>: Given $C = [x^{-1}, g, y]$ and $U, V \in T^{\mathcal{F}}$, with $U^C = V$ (i.e. $\forall u \in U$, $\Pi^+(C^{-1}, u, C)$ is defined and is in V), one needs to show that a representative (x^{-1}, g, y) can be

chosen with $U \xrightarrow{x^{-1}} T \xrightarrow{g} T \xrightarrow{y} V$ as in condition (1) in definition of Φ.

<u>Scheme</u> for $\mathcal{F}$ and uniqueness.

- <u>Start</u> with $(\mathcal{L}_0, \Delta_0, S)$ where
  - $\mathcal{L}_0$ is a model for $N_{\mathcal{F}}(J(S))$, here $J(S)$ is generated by <u>abelian</u> subgroups of S of maximal order;
  - $\Delta_0 = \{P \leq S \mid J(S) \leq P\}$.

- <u>Choice</u> of T_1, λ_1 and then construct $(\mathcal{L}_1, \Delta_1, S)$ with $\mathcal{L}_1 = \mathcal{L}_0^+(\lambda_1)$ and $\Delta_1 = \Delta_0^+$, and how to iterate.