TITLES AND ABSTRACTS

MASTERCLASS ON FUSION SYSTEMS AND \( p \)-LOCAL GROUP THEORY

1. Lecture series by Andrew Chermak

First Lecture: Localities (Objective partial groups, localities, homomorphisms, partial normal subgroups, completeness.)

Second Lecture: The method of descent (Quotient localities, Frattini lemma, associativity, the method of descent.)

Third Lecture: The argument with FF-pairs and automorphisms

Centric Linking Systems are reviewed as “group-like” structures called “localities”. Some properties of localities in general will be discussed, lending to a proof of existence and uniqueness of centric linking systems associated with any saturated fusion system.

2. Lecture series by Bob Oliver

First lecture: Linking systems and classifying spaces for fusion systems

In this first talk, we recall the definition by Broto, Levi, and Oliver of linking systems associated to a saturated fusion system, and how they can be used to construct a “classifying space” for the fusion system. We then discuss some of the reasons for wanting to prove that there is a unique linking system associated to any given saturated fusion system. For example, one consequence of this (to the uniqueness) is the Martino-Priddy conjecture. This says that two groups \( G \) and \( H \) have isomorphic fusion at \( p \) if and only if their \( p \)-completed classifying spaces are homotopy equivalent. This last condition means that the mod \( p \) cohomology rings of the classifying spaces \( BG \) and \( BH \) are not only isomorphic, but via a sequence of isomorphisms realized by continuous maps of spaces.

Second lecture: The obstruction theory for linking systems

For a given fusion system \( \mathcal{F} \), one can think of an associated linking system \( \mathcal{L} \) as an extension of the subcategory \( \mathcal{F}^c \subseteq \mathcal{F} \) or of its “orbit category”, in a way which is analogous to extensions of groups. Using this analogy, one can construct certain “cohomology” groups (cohomology groups of a category with coefficients in a functor) which contain the obstructions to the existence and uniqueness of linking systems associated to \( \mathcal{F} \). In other words, to prove the existence and uniqueness of an associated linking system, it suffices to prove that these groups vanish. After discussing the problem of extensions, we will describe briefly some of the techniques which can be used to compute these cohomology groups.
Third lecture: Existence and uniqueness of linking systems: an outline of the proof

Andy’s proof of the existence and uniqueness of linking systems was written in terms of direct manipulation of the linking systems; or rather, of the localities which he showed are equivalent to linking systems. This proof can also be carried out in terms of the obstruction theory described in my second lecture, while following essentially the same outline. We will describe how, following this procedure, the problem can first be reduced to a question involving quadratic best offenders, a question which is answered using the classification of such offenders by Meierfrankenfeld and Stellmacher.

3. Lecture series by Gernot Stroth

First Lecture: F-modules

Second Lecture: Quadratic modules

Third Lecture: 2F-modules, nearly quadratic modules

In this series of three lectures we will investigate the three most important kinds of modules, which are related to local group theory. These are F-modules, quadratic modules and 2F-modules (nearly quadratic modules). For the investigation of F-modules it is important to consider them as special quadratic modules or 2F-modules. In each lecture we will start with some motivation for the particular kind of module, which will come from amalgams. Then we will prove some basic properties, with the aim to show that it is enough to classify the irreducible ones belonging to almost simple groups (the automorphism group of a quasisimple group). Every lecture then ends with some ideas of the classification of the quasisimple groups, which admit the modules in question.

In the third lecture we will relate this to the current revision of the classification of the finite simple groups and to some open questions about saturated fusion systems.

4. Carles Broto: Equivalences between fusion systems of finite groups of Lie type

This is a talk about the interplay between algebra and homotopy theory. We use homotopy theoretic methods in order to show that the p-fusion system of certain finite groups of Lie type are equivalent.

5. Ran Levi: p-local compact groups and unstable Adams operations

The concept of a p-local compact group is to p-local finite groups what a compact Lie groups is to finite groups. I will recall the concept and some of the basic properties, and then move on to discuss certain self maps on these objects. These maps, which were constructed by F. Junod in his thesis, are analogs of unstable Adams operations, as defined on compact Lie groups and p-compact groups, but their construction uses intrinsically the algebraic structure of a p-local compact group. I will discuss some recent progress in the study of these operations, and in particular some progress towards their classification up to homotopy.
Block algebras are finite-dimensional algebras which in addition to their invariants as algebras give rise to fusion systems, and have indeed prompted the introduction of this concept by Puig in the early 1990’s. The interplay between invariants of the module categories of block algebras and of their fusion systems is far from understood - on one hand, finiteness conjectures such as Donovan’s, suggest that there should be only finitely many Morita equivalence classes of block algebras with a given fusion system. On the other hand, it is not known whether Morita equivalent block algebras should have isomorphic fusion systems. Broue’s abelian defect conjecture can be interpreted as saying that the derived bounded category of a block with an abelian defect group should be an invariant of the associated centric linking system, prompting the question as to what would be a reasonable generalisation of this conjecture to nonabelian defect groups. A key technique to relate the module category of a block algebra to its fusion system is via cohomology - there is a canonical map from the cohomology of a fusion system of a block to the Hochschild cohomology of the block algebra. In conjunction with Symonds’ recent proof of Benson’s regularity conjecture, this can be exploited to show that the Hilbert series of the Hochschild cohomology of a block algebra and its fusion system determine each other up to finitely many possibilities.

This talk is a report on the work of my Ph.D. thesis, which makes a contribution to Aschbacher’s program for the classification of simple 2-fusion systems by investigating fusion systems \( F \) on \( S \) with a centralizer \( C \) on \( T \) of an involution \( x \in S \) having a ”standard component” fusion system \( K \) on a nonabelian dihedral 2-group. Under the ”Baumann hypothesis” of Aschbacher (i.e. that \( C_T(x) \) contains the Baumann subgroup of \( S \) and supposing that \( C_C(K) \) is a cyclic 2-group, it is shown that \( F \) is the fusion system of \( L_4(q) \) for some odd \( q \).

A key ingredient in the proof is a generalization of the classical Thompson transfer lemma to fusion systems, which extends known results in the group case as well. We also make use of the recent work of Andersen, Oliver, and Ventura on reduced and tame fusion systems.

When \( G \) is a finite group with Sylow \( p \)-subgroup \( S \), the \( (S, S) \)-biset \( G \) plays an important role for transfer theory and Mackey functors defined on subgroups of \( G \) (e.g. group cohomology). This lecture will cover the fusion system analogue, which comes with a surprising twist, allowing us to identify saturated fusion systems on \( S \) with certain idempotents in the double Burnside ring of \( S \).

I’ll define the class of fusion systems mentioned in the title. Then I’ll give an overview on our approach to its classification: First we consider fusion systems over ”small” groups, then we extend the methods used via certain types of amalgams of groups. This is joint work with Kasper Andersen and Bob Oliver.