Exercises on $p$-local finite groups

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Let $\mathcal{F}$ be a fusion system on the $p$-group $S$. Recall that an abstract transporter system on $\mathcal{F}$ is a category $\mathcal{T}$ whose objects are a collection of subgroups of $S$, closed under $\mathcal{F}$-conjugacy and overgroups, together with a pair of functors

$$\mathcal{T}_{\text{Ob}(\mathcal{T})}(S) \xrightarrow{\delta} \mathcal{T} \xrightarrow{\pi} \mathcal{F}$$

that satisfy

(A) On objects, $\delta$ is the identity and $\pi$ is the inclusion. Moreover, for each $P, Q \in \text{Ob}(\mathcal{T})$, the group

$$E(P) := \ker[\pi_{P,P} : \mathcal{T}(P) \to \mathcal{F}(P)]$$

acts freely (i.e., with trivial stabilizers) on $\mathcal{T}(P, Q)$ by right composition, and $\pi_{P,Q} : \mathcal{T}(P, Q) \to \mathcal{F}(P, Q)$ is the orbit map of this action. In particular, $\pi$ is surjective on morphisms. Also, $E(Q)$ acts freely on $\mathcal{T}(P, Q)$ by left composition.

(B) The functor $\delta$ is injective on morphisms, and for any $g \in N_S(P, Q)$, the composite $\pi_{P,Q} \circ \delta_{P,Q}$ sends $g$ to $c_g \in \mathcal{F}(P, Q)$.

(C) For all $g \in \mathcal{T}(P, Q)$ and $a \in P$, the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{g} & Q \\
\downarrow{\delta_{P,P}(a)} & & \downarrow{\delta_{Q,Q}(\pi(g)(a))} \\
P & \xrightarrow{g} & Q
\end{array}
$$

commutes in $\mathcal{T}$.

(I) $\delta_{S,S}(S) \in \text{Syl}_p(\mathcal{T}(S))$.

(II) For any $g \in \mathcal{T}(P, Q)_\text{Iso}$ and normal overgroups $P \triangleleft P' \leq S$ and $Q \triangleleft Q' \leq S$ such that

$$g \circ \delta_{P,P}(P') \circ g^{-1} \leq \delta_{Q,Q}(Q')$$

there is an element ("extension of $g$") $g' \in \mathcal{T}(P', Q')$ such that

$$
\begin{array}{ccc}
P' & \xrightarrow{g'} & Q' \\
\downarrow{\delta_{P,Q}(1)} & & \downarrow{\delta_{Q,Q'}(1)} \\
P & \xrightarrow{g} & Q
\end{array}
$$

commutes in $\mathcal{T}$. 

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Exercise 1. Let \( T \) be an abstract transporter system (e.g., a centric linking system), and let \( i \) denote any “inclusion” morphism, namely, any morphism of the form \( \delta_{P,Q}(1) \) for \( P \leq Q \). Let \( \text{Iso}(T) \) denote the set of isomorphisms of \( T \), and let \( \sim \) denote the equivalence relation on \( \text{Iso}(T) \) generated by restriction. Explicitly, if \( P \leq Q \) and we have isomorphisms \( g_P \in T(P,P')_{\text{Iso}} \) and \( g_Q \in T(Q,Q')_{\text{Iso}} \), we have \( g_P \sim g_Q \) if the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{g_Q} & Q' \\
\downarrow & & \downarrow \\
P & \xrightarrow{g_P} & P'
\end{array}
\]

commutes in \( T \). In general, \( \sim \) is the symmetric, transitive closure of this relation.

(a) Suppose that \( P \leq Q,R \) are objects of \( T \), and set \( A = Q \cap R \). Given \( g_Q, g_R \in \text{Iso}(T) \) two isomorphisms with sources \( Q \) and \( R \), respectively, such that \( g_Q|_P = g_R|_P =: g_P \), show that \( g_Q|_A = g_R|_A \), and that this morphism is an extension of \( g_P \).

(b) Suppose that \( P \subseteq Q,R \) are objects of \( T \). Given \( g_Q, g_R \in \text{Iso}(T) \) two isomorphisms with sources \( Q \) and \( R \), respectively, such that \( g_Q|_P = g_R|_P =: g_P \), show that there is some \( g_U \in \text{Iso}(T) \) with source \( U := \langle Q,R \rangle \) whose restrictions to \( P, Q, \) and \( R \) are the respective isomorphisms.

Hint: The extension axioms for transporter systems and fusion systems should both be helpful.

(c) Let \( g_Q \in \text{Iso}(T) \) be an isomorphism in \( T \) with source \( Q \), and suppose that \( g_Q \) has no proper extensions in \( T \). Show that if \( g_R \in \text{Iso}(T) \) is another isomorphism (with source \( R \)) such that \( g_Q \sim g_R \), then in fact \( R \leq Q \) and \( g_R = g_Q|_R \).

Hint: Use that \( g_Q \) does not have any proper extensions to conclude that we must only consider the case where we have \( P \leq Q,R \) is an object of \( T \) and \( g_Q|_P = g_R|_P \). Induct downward on the order of such a \( P \).

Exercise 2. Let \( T \) be a transporter system.

(a) Define an interesting map \( \theta \) from the set \( \text{Mor}(T) \) of morphisms of \( T \) to \( \pi_1(|T|,S) \) (here we view the object \( S \) of \( T \) as a basepoint for the fundamental group) that sends inclusions to the identity and compositions to multiplication.

(b) Show that the map \( \theta \) from (a) is universal in the following sense: If \( F : T \to BG \) is a functor from \( T \) to the classifying category of a discrete group that sends inclusions to the identity, there is a unique homomorphism \( \varphi : \pi_1(|T|,S) \to G \) such that

\[
\begin{array}{ccc}
\text{Mor}(T) & \xrightarrow{F} & G \\
\downarrow \theta & & \downarrow \varphi \\
\pi_1(|T|,S) & \xrightarrow{\varphi} & 
\end{array}
\]

commutes. You may assume the result from topology that states that the image of \( \theta \) generates \( \pi_1(|T|,S) \).

(c) Conclude with a description of \( \pi_1(|T|,S) \) in terms of generators and relations.

Exercise 3. Use the results of Exercise 1 to construct a partial group whose elements are the maximal isomorphisms of \( T \). Describe \( \pi_1(|T|,S) \) in terms of this partial group.