(1) Be careful with the relationship

\[ \text{Hom}(Bord_\infty, C) \]

\[ \downarrow \]

\[ \text{Hom}(\text{Bord}_n, C|\Delta^n) \]

(2) This construction of \( \text{Bord}_n : \Delta^n \to \text{Spaces} \) looks like the nerve of an \( n \)-Cat in \( \text{Top} \).

True when \( n = 1 \), but not for \( n > 1 \). Problem, \( (n > 2) \).

Would replace paths by Moore paths, which does not work in high dimensions.

(3) \( \Psi_1 \). As in Galois, Maclane, Tillman, etc.

Kan德尔-Uillians

Duality: Start with strict 2-categories.

**Def:** Let \( C \) be a strict 2-category. With \( X \xrightarrow{f} Y \), say a 2-morphism as \( \frac{(k)}{(u)} \) (as in previous pages).

Is the unit of an adjunction between \( f \) and \( g \) if \( \exists \) 2-morph \( \frac{(k)}{(v)} \) such that \( (xx) \) and \( (xxx) \)

Such \( v \) is a counit of an adjunction between \( f \) and \( g \) and either \( u \) or \( v \) exhibits \( \frac{f}{g} \) as a left adjoint of \( \frac{g}{f} \). Say \( C \) has adjoints if every 1-morphism has a left and right adjoint.
And this is the notion of adjoint as usual.

\[ \text{Cat does not have adjoint...} \]

(1) \( M \) a (strict) monoidal cat.

\[ \text{BM strict 2-category with} \]

\[ \text{1-mor = } M, \text{ composition } M \times M \to M \]

\[ \text{ex: } (M, \otimes) = (\text{Vect}, \otimes) \]

\[ \text{def: } v \in M \text{ is dualizable if } v \text{ is 1-mor (BM), it } \]

\[ \text{admits a left adjoint.} \]

\[ \text{case of } (M, \otimes) = (\text{Vect}, \otimes) \quad \forall \text{ left dualizable } v \]

\[ 1 \xrightarrow{\text{ev}} v \]

\[ \text{(*)} \]

\[ \text{(**)} \]

\[ \text{for } \]

\[ \text{V satisfying} \]

\[ \text{V = V} \]

\[ \text{id}_V \]

\[ \text{V} \]

\[ \text{Ve Vect d. dualizable } \Rightarrow \text{ V id } \]
Morally, realize everything from level 2
take to $\mathbb{E}$ get a strict $2$-category
we are back to the previous case.

**Problem:** We don't really have morphisms anymore.

1. **(Rezk)** Given $(\infty,n)$-cat $X \in \text{SPSh}(\Theta_n)$ and $x, y \in X[0]_{\Theta_n}$, define

   $$M_X(x, y) \rightarrow X[1](x, y)$$

   $$\downarrow (\partial_1, \partial_0) \downarrow$$

   $$\sim$$

   $$\in \text{SPSh}(\Theta_n)$$

Since $\Theta_n = \Delta_2$.

2. Inductively, given $x, y \in X[1](x, y)$, the $(n-1)$-morphisms of $X$

   $$M^n_X(x, y) := M_{M^{n-1}_X(x, y)}$$

   $\Rightarrow$ Kan complex.

3. For a Cartesian closed cat with initial and terminal $*$.
For \( n = 2 \), it is the big nerve as before.

For \( C(\infty,n) \)-categories, \( h_2 C \) defined by

\[
\text{SpSh}(\Theta_n) \longrightarrow \text{SpSh}(\Theta_2) \longrightarrow \text{Str-2-Cat} \rightarrow h_2 C
\]

**Def.** Let \( C \) be an \((\infty,n)\)-cat \((n \geq 2)\). Say \( C \) has adjoints for \( \ell \)-morphisms if \( h_2 C \) has adjoints.

For \( 1 \leq k \leq n \), say \( C \) has adjoints for \( k \)-morphisms if for any \( x,y \in C(\omega) \), \( M_C(x,y) \) has adjoints.

If for all \( x,y \in C(\omega) \), \( M_C(x,y) \) has adjoints for \((k-1)\)-morphisms, say \( C \) has adjoints if \( \forall 1 \leq k \leq n \).

**Note.** If we regard an \((\infty,n)\)-cat as \((\infty,n+1)\)-cat and it has adjoints, it is a groupoid! (Hence we don't require \( k \leq n \) having adjoints is very close to being a groupoid.)
Can we use the monoidal structure of an \((\infty, n)\)-cat to create an \((\infty, n+1)\)-cat?

Other possibility: \(A \otimes (\infty, n)\)-cat \(C \in \text{SPsh}(\Theta^n)\) has duals if \(\eta \in C\) has duals. (Since \(\eta \in C\) will be a functor \(\text{R}^{op} \to \text{Cat}\).)

Lurie: this is supposed to be equivalent to the above. Below:

\[ C : \text{R}^{op} \to \text{SPsh}(\Theta^n) \]

\[ \Delta \times \Theta^n \quad \cong \quad (\Delta \Theta^n)^{op} \]

\[ \text{Lan}(\Delta) \quad \text{(or fibrant replacement)} \equiv \]

\[ \text{Ann} \quad \text{(R, n+1) cat} := \quad \mathcal{B}C \]

\( \text{Sat } C \text{ has duals if } \mathcal{B}C \text{ has adjoints as an } \]

\( (\text{R}, n+1) \text{ category}. \) (Where note: \( \mathcal{B}C \) should be preferably is)

A \( \otimes \text{-}(\text{R}, n+1) \text{ category} \)

Have

\[ \text{SPsh}(\Delta) \xrightarrow{\eta} \text{SPsh}(\Theta^n) \]

\[ C^\sim \quad \cong \quad C \]

"Sub-\(n\)-groupoid"
"Largest groupoid in \(C\)"

Note: a map \(C[0] \to C^\sim \) by universality of \(\eta\).
Theorem: Notion of bordism isomorphism is preserved under counits of \((\infty,n)\)-cats.

\[\mathsf{1} \in (\infty,n)\text{-CAT with duals} \to \mathcal{C} \in \mathsf{E}(\infty,n)\text{-CAT} 3\]

\[\mathsf{E} \mathsf{P} \to \mathsf{E}\]

**Bordism Hypothesis**

For \(\mathcal{C}\) an \((\infty,n)\)-category with duals,

\[\text{Hom}_{(\infty,n)\text{-CAT}}(\text{Lan}_{\text{Bord}}^{fr} \mathcal{C}, \mathcal{E}) \to \mathcal{E}^{\mathcal{C}}\]

\[\mathcal{C} \to \mathcal{C}^{(+)}\]

\[\infty\text{-Groupoid}\]

**Note:** Part of the statement is that the \((\infty,n)\)-cat on the left is an \(\infty\)-groupoid.

**Example:** Can take \(\mathcal{C} = \text{Lan}_{\text{Bord}}^{fr}\)

**E:** \(n = \text{anything, especially} 1\) ...

**Suppose** \(\mathcal{C}\) is an \(\infty\)-groupoid, symmetric monoidal (e.g., \(\mathcal{C} = K\text{-spectrum}\))

\[(\infty,n)\text{-CAT} \text{ via } \mathcal{C} .

\[\Rightarrow \text{TFT}_{n}^{fr}(\mathcal{C}) \cong \mathcal{C}^{\infty} \text{ and group completion (left)}

\[\text{adjoint to } \mathcal{C} \text{ of } \text{Lan}_{\text{Bord}}^{fr} \text{ by } |\text{Bord}^{fr} n| \text{ it implies}

\[|\text{Bord}^{fr} n| \cong \mathcal{C}^{\infty} 0 :\]
\[ \text{Hom} \left( \text{lan Bord}_h^\infty, L_n(\Omega^\infty E) \right) \rightarrow L_n \Omega^\infty E \]

\[ \downarrow \text{TARGET GROUP COMPUTED} \]

\[ \text{Hom} \left( L_n \text{lan Bord}_n^\infty, L_n \Omega^\infty E \right) \]

\[ L_n \Omega^{n-1} \text{ should co-represent the same thing as } L_n \text{lan Bord}_n^\infty \]. \( L_n \) is faithful.

Recovering Grothendieck-Madsen-Tillmann-Weiss -- (with their techniques).