Problem Session

Chaired by Jesper Grodal

J. Grodal

$$\begin{tikzcd}[column sep=large, row sep=huge]
\left\{\begin{array}{c}\text{p-fusion structure}\\ \mathcal{F}\end{array}\right\} \arrow[r, bend left, leftarrow] \arrow[r, dashed, "?"'] & \left\{\begin{array}{c}\text{p-local structure}\\ \mathcal{L}\end{array}\right\} \arrow[d, dotted, shift left, "\text{existence and uniqueness thms}"] \arrow[dr, dotted, bend left, "?"] & \\
 & \{\text{Finite groups}\} \arrow[u] \arrow[r, hook] & \{\text{All groups}\}
\end{tikzcd}$$

Martino-Priddy conjecture for group case

$\lim^3, \lim^2$ vanish?

$\lim^1 = 0$ for p odd?

Problem: Prove existence & uniqueness of finite simple groups using p-local finite groups

Subproblem (Quillen) Odd order theorem this way?

Approach: Study $\pi_1(|\mathcal{L}|)$ finite in many cases groups of Lie type in defining characteristic, Symmetric groups $p=2$ p-solvable groups. What about $\pi_*(|\mathcal{L}|)$?

Problem: Alperins conjecture:

value depending on {p-local structure} = value depending on {finite groups}

C. Wilkerson

X finite simply conn CW-complex. $BAut(X)$ $\pi_1 = Out(X)$

2) Sullivan/Wilkerson $Out(X)$ ~~is arithmetic group~~, has properties like arithmetic groups

$X = S^n$ $BAut_1(S^n) \underset{\text{F-isomorphism}}{\sim} \begin{cases} p=2 & BSO(n+1) \\ p>2 & BX(p-1,p-1,n) \text{ p-compact groups.} \end{cases}$

identity component

J. Smith + C. Wilkerson $\left\{ \begin{array}{c} X \\ \downarrow \\ E \\ \downarrow \\ B\mathbb{Z}/2 \end{array} \right\}$ Fiber homotopy equivalences $\sim \Leftrightarrow$

$[B\mathbb{Z}/2, BAut_1(X)]$

roughly speaking = elements of order 2 i $Aut_1(X)$

Question: Is 2-rank of $Aut_1(X)$ bounded? Interpretations

$H^*(BAut_1(X))$ (Krull dimension) Transcendence degree

Conjecture: Yes. Motivation: If X is PDA uses 60's result from 60's Mann. OK in this case

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J. Grodal: Followup P finite p-group

$Rep(P, O(n+1)) / \underset{\text{of or reps}}{\text{Gal conjugation}} \longrightarrow [BP, BAut((S^n)_p^\wedge)]$

Question: If $BAut((S^n)_p^\wedge)$ is like a Lie group, is the Weyl group $\mathbb{Z}_p^* \wr \Sigma_n$?

Question (Wilkerson) What is Rank of $Diff(M)$.

③ Marcus Linckelmann k finite field of char p.

B block of kG gives rise to fusion system $\mathcal{F}_P(B)$, P defect group.

B' —— kG' ———— $\mathcal{F}_{P'}(B')$, P' ——

Question: Does $\mathrm{Mod}(B) \cong \mathrm{mod}(B') \implies \begin{cases} P \cong P' \\ \mathcal{F} \cong \mathcal{F}' \end{cases}$?

Modular isomorphism problem: ~~Does~~ $kP \cong kP' \Rightarrow P \cong P'$?
(subproblem)

$\mathcal{O}$ complete DVR of char 0, $\mathcal{O}/J(\mathcal{O})$ char p

Thm $\mathcal{O}P \cong \mathcal{O}P' \Rightarrow P \cong P'$ (....-Scott)
Roggenkamp?

J. Carlsson: $\left\{\begin{array}{c}\text{fusion systems} \\ \text{of blocks}\end{array}\right\} \overset{?}{=} \left\{\begin{array}{c}\text{fusion systems} \\ \text{of groups}\end{array}\right\}$ Q1

M. Linckelmann Defect group D of block comes from fusion system of a block with D Sylow? Q2

Q1: Ok for Σ_n, ft gps of Lie type non defining char.

J. Grodal: Does $Qd(p) = (\mathbb{Z}/p)^2 \rtimes SL_2(p)$ act freely on a finite complex $\simeq S^n \times S^m$, p odd?

Known: Action cannot have equivariant projection, $n \neq m$

(4) Question (Ian Hambleton): What is the analogue of Milnor's condition (every elt of order 2 is central) in the case of products of spheres?

Thm (~~Smith~~ Adem-Unlu-Davis) Every rank 2 p-group acts freely and smoothly on a product of two spheres, $p>2$.

Unlu: work for $p=2$ OK for $|G| \leq 256$ (except perhaps one).

J. Davis

(1) k field, G torsion free.

- Are there nontrivial idempotents in kG or $\mathbb{Z}G$? Conj (Bass): No
- ~~Every~~ f.g projective $/kG$ or $\mathbb{Z}G \Rightarrow$ stably free?

$\tilde{K}_0(\mathbb{Z}G)=0$?

Free: wrong (Dunwoody)

(2) Borel conjecture

Closed aspherical manifolds M, N. Conj $M \cong N$

~~[illegible]~~

(3) Finite group actions on S^3 are linear?

Perelman: free actions
Smith conj: nonfree actions
} May be solved

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4. **Hilbert-Smith conj**: Any locally compact topological group acting effectively on a manifold is a Lie group?

Question: Does $\mathbb{Z}_p$ act effectively on a manifold?

Positive results: ask Bob. Smooth version is OK.

5. **Conjecture**: If $(\mathbb{Z}/p)^r$ acts freely $S^{n_1} \times \ldots \times S^{n_k}$ then $r \leq k$?

True for $k=1, k=2$

True for $n_1 = \ldots = n_k$ if $p > 2$

Question: Does every finite group act freely and homologically trivially on some product of spheres?

Tiny question: non abelian group $G = p_+^{2+1}$ (order p^3, exp p)
For which (m,n) does G act freely on $S^m \times S^n$?

6. Find a simply connected manifold with no effective action of a nontrivial finite group?

Might be solved: Kreck, Puppe

Jon Alperin:

Thm (Jordan) $\exists J: \mathbb{N} \to \mathbb{N}$, G complex linear group of dim n
Then G has an abelian normal subgroup of index $J(n)$.

⑥ What about nonlinear effective actions?

J. Grodal Direct proofs of Lie group results for pcgs ~~directly~~:

Botts thm ~~[illegible]~~ on X/T?

ΩX torsion free? (Lins result)

Borel-Steinbergs theorems?

Homotopy Kač-Moody groups: Theory ..... (Kitchloo)

ΛBX ~~[illegible]~~ p-compact Kač-Moody groups

free loop space

Weyl group $L \rtimes W$

Bob Oliver:

Exotic p-local finite groups

(1) extensions $1 \to \mathcal{F}_0 \to \mathcal{F} \to Q \to 1$ $\mathcal{F}_0, \mathcal{F}$ saturated fusion system

Q p-group or p'-group. Classification in [BCGLO?]

2. Examples with one exotic and the other nonexotic? (4 cases)

Most interesting case: Fix $\mathcal{F}_0/S_0$. Assume $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$

Does $\mathrm{Aut}_{fus}(S_0) / \mathrm{Aut}_{\mathcal{F}_0}(S_0) = \mathrm{Out}_{fus}(S_0)$

have automorphisms not ~~from~~ in image $\mathrm{Out}(G_0)$? Phrased this way

answer is no.

(7) [illegible] Does there exist $\alpha \in Out_{fus}(S_0)$ not in the image of $Out(G_0')$ for any G_0' with $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0')$??

If α exists can build example: If $|\alpha| = p^k$ then $\exists$

$$\exists\ 1 \to \mathcal{F}_0 \to \underset{\substack{\uparrow \\ \text{exotic}}}{\mathcal{F}} \to \langle\alpha\rangle \to 1$$

<u>Classification of fusion systems</u>: Is there another exotic fusion system at $p=2$ besides $\mathcal{F}_{Sol(q)}$?

<u>Larry Smith</u>

<u>Steenrod algebra</u> $\mathbb{F}_q$, $V = \mathbb{F}_q^n$, $\mathbb{F}_q[V] \xrightarrow{\mathcal{P}} \mathbb{F}_q[V][X]$ (X: degree $-(q-1)$)

$\mathcal{P}$ map of algebras$/\mathbb{F}_q$
$z \in \mathbb{F}_q[V]_1 : \mathcal{P}(z) = z + z^q X$ } determines $\mathcal{P}$.

For $f \in \mathbb{F}_q[V]$ $\mathcal{P}(f) = \sum_{i+j=\deg(f)} \mathcal{P}^i(f) \cdot X^j$ ($\mathcal{P}^i$: Steenrod operations)

Let $\varphi : U = \mathbb{F}_q^m \to V = \mathbb{F}_q^n$

$\deg(\mathcal{P}^i f) = i(q-1) + \deg(f)$

$$\begin{array}{ccc} \mathbb{F}_q[V] & \xrightarrow{\varphi^*} & \mathbb{F}_q[U] \\ \downarrow \mathcal{P} & & \downarrow \mathcal{P} \\ \mathbb{F}_q[V][X] & \xrightarrow{\varphi^*} & \mathbb{F}_q[U][X] \end{array}$$

$\mathcal{P}^i \in End_{\mathbb{F}_q}(\mathbb{F}_q[-])$.

⑧ ~~𝔉~~ P^i's generate an algebra $\mathcal{P}^*$ (Steenrod algebra without Bockstein)

$$P^iP^j = \sum (\ ) P^{i+j-k}P^k$$

use Bullett–Macdonald identity.

$\mathbb{F}_q[V]$ module over $\mathcal{P}^*$

<u>Hit problem</u>: What is the minimal number of module generators?

<u>Known</u>: $\dim V = 1 \to$ exercise

$\dim V = 2 \longrightarrow q=2$ Peterson

$\dim V = 3, q=2 \to$ Kameko

? in general

Motivation: Immersion problem: Given manifold M. What is the lowest codimension of immersion $N \looparrowright \mathbb{R}^?$ with M & N cobordant.
Combinatorial school. Other school:

$\mathbb{F}_q[V]/I$ PDA, I $\mathcal{P}^*$-invariant. $f \in I$, $P^i f \in I$

explains some results of Hubbuck/Kameko

Hit problem for $\mathcal{D}I(n) = \mathbb{F}_q[V]^{GL(V)}$.