Madsen-Weiss, Galatius Theorems, etc.

0-dimensional case: Tim of Barrett-Prid09-Quillen
\[ H^*(\Sigma_n) \cong H^*_+(\mathbb{R}^\infty \mathcal{S}^n) \]
\[ H^*_+ (\Sigma_n) \]

2-dimensional case: Madsen-Weiss

1-dim: Galatius
\[ H^*_+ (\text{Aut}(F_n)) \cong H^*_+(\mathbb{R}^\infty \mathcal{S}^n) \cong H^*_+ (\Sigma_n) \]
\[ H^*_+ (\text{BAut}(F_n)) \quad \text{space of graphs} \]

3-dimensional: special classes of 3-MFDs
- \( V_g = \text{genus } g \) handlebody
\[ H^*_+ (\text{BDiff}(V_g)) \cong H^*_+(\mathbb{R}^\infty \mathcal{S}^n (\text{BSO}(3)_+)) \]
\[ H^*_+ (\text{BDiff}(V_g)) \quad \text{(same as one would get for dim 3-MFDs with bordism)} \]

- \( H^*_+ (\text{BDiff}(S^1 \times S^2)) \cong H^*_+(\mathbb{R}^\infty \mathcal{S}^n (\text{BSO}(4)_+)) \)
\[ \text{bordism of a 4-dim handlebody} \]
$\Sigma_k = \text{Sym. Group}$

$\Sigma_n \subseteq \Sigma_{n+1}$, $\Sigma_k = \bigcup_{n} \Sigma_n$

$L^0 S^n = \text{Space of Maps } (S^m,*) \rightarrow (S^n,*)$

$L^0 S^n \subset L^{n+1} S^{n+1}$ by suspension

$L^{\infty} S^n = \bigcup_{n} L^n S^n$

$\pi_i (L^{\infty} S^n) = \pi_i S^n = \pi_{i+n} (S^n)$

$\pi_0 (L^{\infty} S^n) = \pi_n S^n = \mathbb{Z}$

All components are highly connected

$H_\ast (\Sigma_k) = H_\ast (\text{0.S.}^k)$

Model for 0.S.E.: Space of unordered distinct $\ast$ tuples of points in $\mathbb{R}^\infty = C_\ast (\mathbb{R}^\infty) = \bigcup_{n} C_\ast (\mathbb{R}^n)$

Scanning: Take a configuration $C \in C_\ast (\mathbb{R}^n)$

Powerful magnifying lens with limited view.

See at most one point at a time.

Point in a ball $B^n$ disappears at $\partial B^n$.

Get a point in $S^n$, for each position of the lens.

Here $\mathbb{R}^n \rightarrow S^n$ taking $P$ to $dP$.

I.e. $\mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$

$C_\ast (\mathbb{R}^n) \rightarrow L^n S^n$.
$h \to \infty \quad \implies \quad C_k(\mathbb{R}^\infty) \to \mathbb{R}^{\infty-\infty}$

Connected leaves in $k^{\text{th}} (-k?)$ component.

$k \to \infty : B \times \mathbb{R} \to \mathbb{R}^{\infty-\infty}$

This induces $H_r^\text{iso}$

SCANNING MANIFOLDS: Given a smooth, closed, orient.
MFD $M$ of dim $n$.

MODEL FOR $BLiff(M)$: Space of smooth oriented submanifolds
of $\mathbb{R}^\infty$ diffeo to $M = C(M)$

$C_n(M)$ = submanifolds in $\mathbb{R}^n \subset \mathbb{R}^\infty$

Scan a config in $C_n(M)$:

$AG_{m,n} = \text{affine Grassmannian of } M \text{-plane in } \mathbb{R}^n$

$\approx \text{almost flat plane}$

SEE A SMALL PIECE OF $M$, ALMOST FLAT $M$-PLANE
IN A BALL $B^n$.  

CENTER OF MASS
\[ C_n(M) \rightarrow L^n AG_{m,n} \]

Let \( n \rightarrow \infty \) \[ C(M) \rightarrow L^\infty AG_{m,\infty} \]

\[ \text{BDiff}(M) \rightarrow SC^\infty AG_{\infty,0} \]

compactly supported

\[ g_0: \text{This is an example on } H_k. \]

\[ \text{BAR-CUNIE: Diffe}_g(S^\infty) \text{ has contractible components} \]

Can replace by \( T_0 \text{Diffe}_g(S^\infty) \).

\[ \text{HARER STABILITY: } H_j(T_0 \text{Diffe}_g(S^\infty)) \text{ is inde} \text{ep of } g \text{ for } g > 0. \]

\[ \text{Art}(F_n): \text{MODEL FOR } BA_k(F_n): \text{SPACE OF FINITE CONNECTED BASEPOINTED GRAPHS IN } \mathbb{R}^n \text{ WITH } T_1 \subset F_n. \]

\[ \text{ALLOW EDGES TO COLLAPSE.} \]

\[ \text{ALLOW VOLUME 1 VERTICES.} \]

[\text{WICKER-VOGTMANN, IGUSA}] \[ \text{SCAN GRAPHS IN } \mathbb{R}^n: \]

\[ \text{SPACE OF TREES IN } \mathbb{R}^n = \text{SPACE OF POINTS} = S^n. \]
\[ \text{BAut}(F_2) \to \mathbb{S}^\infty \]
\[ H_*(\text{BAut}(F_2)) \cong H_* \mathbb{S}^\infty \cong H_* \mathbb{E}_\infty \]

**Part II: Generous' Thm**

Space \( G^n \) of graphs in \( \mathbb{R}^n \): smooth edges, linear near vertices, valence 0, 1, 2 allowed, X of angled, not necessarily compact, connected. Edges can extend to \( \infty \).

Conical compactification/expansion of edges:

**Neto of a given graph \( X \in G^n \):**
All small conical expansions of vertices, and isotopeed.

Globally: "Compact-open topology." Take all graphs \( X' \) which in some large ball \( B^n \) (centered at 0) with \( X \cap B^n \) are close to \( X \).
\( \Pi_0 G^n = 0 \quad (n \geq 1) \):

Conical expansion is realized by

\( \text{Expand radially from a point in } \mathbb{R}^n - x \).

\[ \text{Genus filtration of } G^n : G^{n_0} \subset G^{n_1} \subset G^{n_2} \subset \ldots \subset G^n = G^n \]

\( G^{n_k} = \text{Graphs in } \mathbb{R}^n \times I^{n-k} \quad \text{infinite in } k \text{ directions} \)

4 main steps

1. \( \Pi_0 G^{n_0} = \text{Homotopy type of finite graphs (possibly 0)} \)

\[ \text{Let } G^{n_0} = \text{Compact convex graphs of rank } k, \text{ containing } 0 \text{ as a vertex.} \]

\[ \text{Prop: } G^{n_0} \text{ is a BANt F}_k \]

\[ \text{Invar: } \text{Classifying space of the set of finite rank } k \text{ graphs, with morphisms respecting connected and isomorphisms is a BANt F}_k \]
3. Have inside $G^n$ the graph with ≤ 1 point, forming an $S^n$.

**Prop:** $S^n \rightarrow G^n$ is a Homotopy-Equiv.

**Proof Sketch:** Show $\pi_i(G^n, S^n) = 0$ for $x_t \in G^n$, $t \in D^i$, $x_t \in S^n$, $t \in 0\ldots 1$.

Want $B^n_t \cap x_t = \emptyset$ or no.

**Difficulty:**

![Diagram](attachment:diagram.png)

Replace by

![Diagram](attachment:diagram.png)

(Modification of the trivalent part of $\pi_i(G^n, S^n)$)

Don't do anything on edge through $0$?

Then expand $B^n_t$ to $\mathbb{R}^n$ and shrink each tree to its center point.
(3) Map $G^n \times \rightarrow \mathbb{R}G^n_{\infty}$ by performing flows from $-\infty$ to $+\infty$ in $(\epsilon, t)$-coordinates.

Prop: $G^n \times \rightarrow \mathbb{R}G^n_{\infty}$ is a homotopy equiv.

Hence $G^n_{\infty} \times \mathbb{R}G^n_{\infty} = \mathbb{R}G^n_{\infty} \times \mathbb{R}G^n_{\infty}$.

Idea: Define $G^n_{\infty, \infty}$ as space of graphs $v$ in $G^n_{\infty, \infty}$ with slices $S(a_i) = \mathbb{R} \times \{a_i\} \times I^{n-2}$, object from $X$, with weights $w_i \geq 0$, $\sum w_i = 1$.

Expand radius $r$ from $a_i$.

For families, use a partition of unity. The weights are "damping factors."
Monoid version of $G^n$: By "stacking and compression" (or Moore loops - like construction)

\[ G^{n,\mathcal{E}} \cong M^{n,\mathcal{E}} \]

\[ B M^{n,\mathcal{E}} \cong G_{s}^{n,\mathcal{E}} \]

So $L G^{n,\mathcal{E}} \cong L B M^{n,\mathcal{E}} \cong M^{n,\mathcal{E}} \cong G^{n,\mathcal{E}}$

\[ \Pi_{0} M^{n,\mathcal{E}} = 0 \]

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3rd Talk

$G^n = \text{space of graphs in } \mathbb{R}^n$.

$G^{n,0} \subset G^{n,1} \subset \ldots \subset G^{n,n} = G^n$

- $G^{n,0}$ connected, rank $k$
- $G^{n,0} = \text{Bund}(F_k)$
- $G^n \cong \Sigma^n$
- $G^{n,2} = \Sigma G^{n,2+1}$ for $k \geq 1$

(Following G-R-W, the above)

\[ G^{n,0} \rightarrow L G^{n,n} \text{ not a homotopy equivalence} \]

\[ \Pi_{0} G^{n,0} = \text{homotopy types of finite graphs } (n \geq 4) \text{, monoid} \]

\[ \Pi_{0} (L G^{n,n}) = \Pi_{1} G^{n,n} \text{ group} \]

\[ \Pi_{n} G^{n,0} \text{ not abelian, } \Pi_{n} (L G^{n,n}) = \Pi_{2} G^{n,n} \text{ abelian} \]
Take $n = \infty$. Better $M^\infty, o_0 = M^\infty = \text{Connected graphs in } [0, a]\times \mathbb{R}$

Containing $[0, a] \times 0$ (base line), Meeting $0 \times \mathbb{R}^\infty$ and $a \times \mathbb{R}^\infty$ in one point.

(Allow $a = 0$).

$\Rightarrow$ Monoid.

**Prop:** $BM^\infty \cong G^{\infty, 1}$

$G^{\infty, 1}$ = Graphs containing $\mathbb{R} \times 0$ with slices, weights, ...

$G^\infty \xrightarrow{\text{Forgetting the infinite parts to left and right}} BM^\infty$

$G^{\infty, 1} \xrightarrow{\text{Homotopy Equiv.}}$

(a) Introduce baseline

(b) Connecting ties to baseline

(c) $\text{SC}(a) \xrightarrow{\text{Connect to } C^0}$

$\xrightarrow{\text{Back connected}}$
All Together:
\[ H_*(\text{BAut}\mathcal{F}_\mathbb{E}) \times H_*(\mathbb{L} M^{\mathbb{Z}}) \cong H_*(\mathbb{L} \mathbb{B}M^{\mathbb{Z}}) \]
by a variant of \((\ominus)\).

\[ \cong H_*(\mathbb{L} \mathbb{B}M^{\mathbb{Z}}) \]
by \((\ominus)\),

\[ \cong H_*(\mathbb{L} \mathbb{B}G^{\mathbb{Z}}) \]
by \((\ominus)\),

\[ \cong H_*(\mathbb{L} \mathbb{B}G^{\mathbb{Z}}) \]
by \((\ominus)\),

\[ \cong H_*(\mathbb{L} \mathbb{B}S^{\mathbb{Z}}) \]
by \((\ominus)\).

\[ \overset{\text{Variant for } g\text{-bin morphisms}}{\rightarrow} H_*(E_g) \cong H_*(\mathbb{L} \mathbb{B}S^{\mathbb{Z}}) \]

\[ \overset{\text{Variant: Particles with } \mathbb{Z}\text{-vectors; counting } \rightarrow \text{ Dimension}}{\rightarrow} \]

\[ \overset{\text{Ex} - \text{Thm: Free abelian group}}{\cong} K(\mathbb{Z}, n) \]

\[ \overset{\text{Homology}}{\text{V}_g} \]
$H_4(BO\mathbf{F}(V_2))$ Stabilizer

Very similar proof to show $H_4(\text{Lie} \mathbf{T}_0, BO\mathbf{F}(V))$

Main difference in 1 and 2

2. $BO\mathbf{F}(V_2)$ = space of $V_2 \times \mathbb{R}^\infty$

with handle decomposition

Connection of disjoint discs cutting the handlebody into disjoint balls.

(The discs have weights and move around ...)

Thicken to form handle.

Make the 2-discs flat and round (using $\text{Diff}(\mathbb{D}^2) \cong O(2)$)

3-disc (0-handle) rounds ($-D^3 = \emptyset$)

3-plane graph

$x \times \mathbb{R}^\infty$

Field of 3-planes, including tangent under to edges.

Pointed 3-plane

($\text{plane}$ discrete when the point $\to \infty$

$\to$ Thom space $(\mathbb{T} + \mathbb{N} \to G_3, \mathbb{N})$

$\cong \Sigma^\infty(BO(3)^+, \mathbb{N})$)

$\cong \Sigma^\infty(BO(3)^+, \mathbb{N})$


For $S^1 \times S^2$ same with the complex of spheres instead of discs... (Stability not known.)