Nakahara: [String Topology & BG's], Chateau-Renishan, Westerland

Fix a field $F$ and let $X = BG$, $G$ a finite group or $X$ a 1-connected space with $H^*(X; F)$ finitely generated such as $BG$, $G$ connected compact Lie groups $p$-compact groups for $F = F_p$.

**Theorem**

For $X$ as above, $H^*(LX; F)$ is an HCFT (positive boundary) where $LX = \text{map}(S^1, X)$ (up to (degree))

HCFT: Homological Conformal Field Theory.

Let $C$ be the topological category with $\text{Obj} C = N = 1$ closed 1-dim. manifolds?

$\text{Mor}_C(n, m) =$ moduli space of Riemann cobordisms from $n$ to $m$ circles such as in $\text{Mor}_C(3, 3)$

![Diagram of a cobordism]

i.e. the space of Riemann structures on such cobd.

$$\cong \bigotimes_{k=0}^{n+m} \text{BT} \times \cdots \times \text{BT}$$

Where $\text{BT} \cong D_0 \text{Diff}(S^{2g}, \text{rel} D)$

$C$ is a symmetric monoidal category under disjoint union

**Def.** A CFT is a monoidal functor

$$\Phi: C \to \text{Hilb}$$

s.t. $\Phi(n+m) = \Phi(n) \otimes \Phi(m)$
Let $C \times C$ be the linear category with the same objects as $C$ but $\text{Mor}_{C \times C}(n, m) = C \times \text{Mor}_C(n, m)$. 

**Def.** A TCFT is a monoidal functor 
$$\Phi : C \times C \rightarrow \text{Chain complexes} / \mathbb{F}$$ 
$s.t.$ 
$$\Phi(n+m) = \Phi(n) \otimes \Phi(m).$$

Let $H \times C$ be the linear category with the same objects as $C$ but $\text{Mor}_{H \times C}(n, m) = H \times \text{Mor}_C(n, m)$. 

**Def.** An HCFT is a monoidal functor 
$$\Phi : H \times C \rightarrow \text{Gr. Vect} / \mathbb{F}$$ 
$s.t.$ 
$$\Phi(n+m) = \Phi(n) \otimes \Phi(m).$$ 
$(\text{Obs: } \Phi(0) = \mathbb{F})$ 

Let $C^+ \subset C$ be the subcategory with the same objects and morphisms are such that every component of its cobordism have non-empty incoming & outgoing boundary cycle (i.e. $n_i, m_i > 0$). 

**Rephrased Theorem** 
The map $n \mapsto H_\ast(LX; \mathbb{F}) \otimes^n$ can be extended to a monoidal functor from $H_\ast C^+$ to graded $\mathbb{F}$-vector spaces. 

In particular we have maps, $\forall g, n, m$

$$H_\ast(T_g, n+m; \mathbb{F}) \otimes H_\ast(LX; \mathbb{F}) \otimes^n \rightarrow H_\ast(LX; \mathbb{F})$$

**Classical String Topology**

Clan Sullivan. $H_{\ast+d}(LM)$ is a BV-algebra, i.e. there is a commutative product of degree zero and a
Degree one operator

Observe: The BV-structure is exactly the part of an NCFT coming from $\otimes$ giving the product and $\otimes_0$ of degree 1 giving $\Delta$.

Theorem (Godin)
The Chea-Sullivan BV-structure extends to an NCFT on $H_*(LM)$.

Back to $BG = X$.

Consider the diagrams

1. $$(LX)^n \xrightarrow{\text{in}} \text{Rep}(Sg_{n+m}, X) \xrightarrow{\text{out}} (LX)^m$$

These maps are Diff$(Sg_{n+m})$ - equivariant where Diff acts trivially on $LX$. Take this Borel construction.

2. $\text{BDiff} \times (LX)^n \xrightarrow{\pi} E \text{Diff} \times \text{Diff} \text{Rep}(Sg_{n+m}, X) \xrightarrow{} \text{BDiff} \times X^n$.

We want now transfer maps in homology. If so, we set the NCFT by taking the composition. We need to understand the fiber of $\pi$.

Proposition

$\text{Rep}(S, X)^m \times (LX)^m$ is a fibration with fiber $(S^m X \times \ast)$ if $X$ is $(n-1)$-connected.
proof: It is a fibration because it is induced by a cofibration. Since \( LX \) is connected (\( X \) is 1-connected), we look at the fiber over the trivial loops \( \{1\} \subseteq X \). It is

\[ \tau_{\ast}( S/\{1\}, X ) \text{, but } \Sigma_{n} = \{1\} \]

Since we assume \( H_{\ast}( LX; \mathbb{F} ) \) is finite, the Dwyer transfer exists.

Corollary: The fiber of \( \pi_{\ast} \) is \((S\times X) - \{1\} \). 

Note: If \( x \in F \), since \( \Sigma_{n} = \{1\} \), we have \( \text{map}( \{1\}, B\mathbb{F} ) \) easy to describe...

Integration along fibers

If \( H_{\ast}( F; \mathbb{F} ) = \# \) is kbp dim., \( \pi_{\ast} B \) acts

basically on \( H_{\ast}( F; \mathbb{F} ) \). Be \( SS \rightarrow E \rightarrow B \)

gives

\[ H_{p}( B ) \cong E_{p,d} = H_{p}( B; H_{\ast}( F; \mathbb{F} ) ) \]

In this case, we get \( d = \text{dim}( S \times X ) \)

depends on \( g, n, m \).

The orientability condition holds as well since we have a pull-back

\[ \tau_{\ast}( S, X ) \rightarrow (LX)^{n} \text{ orientable} \]

\[ \tau_{\ast}( S, X ) \rightarrow X^{n} \text{ orientable} \]