p-Noetherian groups

Def: A p-Noetherian group is a triple \((X, BX, e)\)
where \(H^*(X; \mathbb{F}_p)\) is Noetherian, \(BX\) is \(p\)-complete.
\[\Omega BX = X\]

Ex: 1. \((K(2p, 2), K(2p, 3), e)\) is a p-Noetherian
with \(H^*(K(2, 3); \mathbb{F}_p) = \mathbb{F}_p[\pi_3, \mathbb{S}^2\pi_3, \mathbb{S}^4\pi_3, \ldots]\), is not.

\(\mathbb{Q} H^*(K(2, 3))\) is almost \(\Sigma F(1)\)

More precisely, \(\mathbb{Q} H^*(K(2, 3)) \cong \Sigma F(1)\)

The class in degree 2 is missing \("3\pi_3\)."

Example 2 is the only example with \(BX\)
finite by Hopkins' theorem.

2. \((S^3 < 3>_p, (BS^3 < 4>_p, e)\)

For \(p = 2\), \(H^*(S^3 < 3>_p) = \mathbb{F}_2[\pi_4] \otimes \Lambda(\nu_5)\)

\[H^*(B(S^3))\]
\[ K(2,3) \to B(S^3 \wedge S^7) \to B S^3 \to K(2,4) \]

\[ K(2,3) \to \quad B(\Sigma^3 S^7) \]

2. For many 1-connected compact Lie groups, they computed \( H^\ast(BG) \) as an algebra. It always look like:

\[ A \cong \mathbb{L} \quad \text{for} \quad A = H^\ast(K(2,3)) \]

\[ B \cong H^\ast(BG) \]
Part I: $H^*(B_X)$ is finitely generated as an algebra $/A_p$.

We study fibrations

$$K(A,n) \rightarrow E \rightarrow X$$

where

1) $E, X$ are 1-connected.
2) $A$ is an abelian gp of fin. type.
3) $H^*(X)$ is noetherian.

Thm: $H^*(E)$ is fg. as an algebra $/A_p$.

Notes:

- Not true without assumption 3).
  - By example
    $$K(2,2) \rightarrow \Sigma K(2,2) \rightarrow K(2,2) \vee K(2,3) \rightarrow K(2,3)$$

- Could be true replacing $K(A,n)$ by any fin. postnikov system.

- Might even be true replacing $K(A,n)$ by any space with $H^*( )$ fg. as a ring over $A_p$. 

Step 1: The semi s.s. collapses at some fit stage $H^X$ is Noetherian, the ascending chain of id of these elements hit by a differential $d_k$ is stabilize.

So by Koda there is a subalgebra $G^* \leq H^X(K(A,n))$ generated by "large" Steenrod operations on the fundamental classes.

$N = 0 \Rightarrow \quad d^N = 0$.

$$H^*(K(A,n)) = G^* \otimes F^* \text{ as algebras}$$

Here $G^*$ but not $F^*$ is stable under Steenrod operations.

In fact $G^* \leq QH^*(K(A,n))$ f.g. in $\mathbb{N}$.

So $G^*$ is f.g. in $\mathbb{N}$ since locally Noetherian.

Consider the quotient s.s.

$$E_2 = H^X(Y) \otimes F^*.$$

By the Evans-Drage-Williams argument this s.s. (collapse).

Moreover $E_r = G^* \otimes E_r$ and therefore collapses also on the same page.
By induction $a \otimes g \in E_{r+1} \otimes G$. Then $d_r(a \otimes g) = d_r(a) \otimes g$

$\Rightarrow E_{r+1} = E_{r+1} \otimes G^*$

**Step 2:** The cohomology $H^*(E)$ is f.g.
as a module over $G^*$.

From Step 1, we know $E_{\infty}$ is f.g. as a module on $H^*(X) \otimes \mathbb{F}_p$, where $\mathbb{F}_p$ is a field and $H^*(X)$ is a graded algebra.

$E_{\infty}$ is f.g. as a module over $H^*(X) \otimes \mathbb{F}_p^*, \mathbb{F}_p^*$.

$\Rightarrow H^*(E) = \mathbb{I}$

$\Rightarrow H^*E$ is finitely generated as an algebra over $A_p$.

and generators for $H^*(X)$ and $\mathbb{Z}_p^*$.

**Step 3:** $H^*E$ is an algebra.

Warning: $G^* = H^*(K(2,3))$, $B^* = H^*(K(2,3))$ with

trivial action of $A_p$. $B^*$ is f.g. as modules over $G^*$.

The problematic need compatibility between action of

$A_p$ on $B^*$ and on $G^*$.
Here $\mathbb{G}^\ast$ acts on $H^\ast \mathcal{E}$ as follows.

$$H^\ast \mathcal{E} \xrightarrow{\ast \ast} H^\ast (K(A,\mathcal{R}))$$

$G^\ast$ free alg.

$g$ act via this section.

$$g \cdot x = s(g) x.$$  

for $\Theta \in G^\ast$ confusion

$$s(\Theta x) = s(\Theta g) \cdot x + \Theta s(x)$$

But we have map $\varphi: H^\ast \mathcal{E} \xrightarrow{\ast \ast} H^\ast (K(A,\mathcal{R}))$ of algebras.

Choose generators $1 = b_1, b_2, \ldots, b_m$ of $H^\ast \mathcal{E}$ as a $G^\ast$-algebra.

Claim: $b_1, \ldots, b_m, s(g_1), \ldots, s(g_k)$ generate $H^\ast \mathcal{E}/A^\ast$ if $g_i$ gen $G^\ast,A^\ast$. 

$H^\ast \mathcal{E} \cong H^\ast (K(A,\mathcal{R}))$.
The only problem is to write $s(\Theta g)$ in terms of $s(g'_1)$'s.

If $1, c, \ldots, c$ are module generators for $H^*E$ over $H^X \otimes \mathbb{F}_p[z]$, we can assume $c \in ker p$.

Consider

$$s(\Theta g) - \Theta s(g)$$

so

$$p(s(\Theta g) - \Theta s(g)) = \Theta g - \Theta g = 0.$$ 

Reduced to looking at things in the kernel of $p$.

$$\exists = \lambda_0 + \lambda_1 c_1 + \ldots + \lambda_k c_k, \quad \lambda_i \in \mathbb{F}_p$$

By induction can handle $\lambda_1, \ldots, \lambda_k$, $p(3) = 0$

$$\lambda_0 + \lambda_1 p(c_1) + \ldots + \lambda_k p(c_k)$$

so

$$\lambda_0 = - (\lambda_1 p(c_1) + \ldots + \lambda_k p(c_k)).$$