

On some analogies between minimal 2-surfaces and Einstein 4-manifolds

Antoine Song

UC Berkeley

April 2021

Goal

Einstein 4-manifolds and minimal surfaces in 3-manifolds share many common properties. Most of these analogies are well known in some corners of geometry.

Goal

Einstein 4-manifolds and minimal surfaces in 3-manifolds share many common properties. Most of these analogies are well known in some corners of geometry.

The proofs of those analogous properties are different, but certain principles remain the same and results for one kind of objects give intuition for the other kind.

Goal

Einstein 4-manifolds and minimal surfaces in 3-manifolds share many common properties. Most of these analogies are well known in some corners of geometry.

The proofs of those analogous properties are different, but certain principles remain the same and results for one kind of objects give intuition for the other kind.

We will review some important results illustrating these points, and present new results in that vein describing quantitatively Einstein 4-manifolds and minimal surfaces.

Definitions

Minimal surfaces:

Let Σ be a closed surface embedded into a closed 3-manifold (N^3, g) , Σ is minimal if the mean curvature vanishes, i.e.

$$H_{\Sigma} = 0.$$

Einstein 4-manifolds:

Let (M^4, g) be a closed 4-manifold, g is Einstein if the Ricci curvature is a constant multiple of the metric, i.e.

$$\text{Ric}_g = \lambda g$$

where $\lambda \in \{-1, 0, +1\}$.

First analogies

Minimal surfaces are critical points of the area functional $\mathcal{A}$ and are fixed points of the Mean Curvature Flow.

First analogies

Minimal surfaces are critical points of the area functional $\mathcal{A}$ and are fixed points of the Mean Curvature Flow.

At second order, a minimal surface Σ minimizes $\mathcal{A}$ modulo a finite dimensional space. This dimension is the Morse index of Σ .

First analogies

Minimal surfaces are critical points of the area functional $\mathcal{A}$ and are fixed points of the Mean Curvature Flow.

At second order, a minimal surface Σ minimizes $\mathcal{A}$ modulo a finite dimensional space. This dimension is the Morse index of Σ .

Simons inequality:

$$\Delta_{\Sigma}|A|^2 \geq -C(1 + |A|^2)^2.$$

First analogies

Einstein 4d metrics are critical points of the normalized Hilbert functional $\mathcal{H}$ and are fixed points of the Ricci flow.

First analogies

Einstein 4d metrics are critical points of the normalized Hilbert functional $\mathcal{H}$ and are fixed points of the Ricci flow.

At second order, an Einstein 4d metric (M, g) maximizes $\mathcal{H}$ along traceless-transverse variations modulo a finite dimensional space. This dimension is the index of (M, g) .

First analogies

Einstein 4d metrics are critical points of the normalized Hilbert functional $\mathcal{H}$ and are fixed points of the Ricci flow.

At second order, an Einstein 4d metric (M, g) maximizes $\mathcal{H}$ along traceless-transverse variations modulo a finite dimensional space. This dimension is the index of (M, g) .

Elliptic inequality:

$$\Delta_M |\text{Rm}| \geq -4 |\text{Rm}|^2.$$

Monotonicity

For Σ minimal surface in (N, g) , for any $p \in N$ and r small:

$$\frac{\text{Area}(\Sigma \cap B(p, r))}{r^2}$$

is almost non-decreasing.

Monotonicity

For Σ minimal surface in (N, g) , for any $p \in N$ and r small:

$$\frac{\text{Area}(\Sigma \cap B(p, r))}{r^2}$$

is almost non-decreasing.

For a Einstein 4-manifold (M, g) , for any $p \in M$ and r small:

$$\frac{\text{Vol}(B(p, r))}{r^4}$$

is almost non-increasing.

Monotonicity

For Σ minimal surface in (N, g) , for any $p \in N$ and r small:

$$\frac{\text{Area}(\Sigma \cap B(p, r))}{r^2}$$

is almost non-decreasing.

For a Einstein 4-manifold (M, g) , for any $p \in M$ and r small:

$$\frac{\text{Vol}(B(p, r))}{r^4}$$

is almost non-increasing.

These monotonicity properties lead to:

- ▶ control on area and on coverings by balls
- ▶ existence of tangent cones for limits,
- ▶ more generally, analysis of singularities of limits, stratification theories (Cheeger, Naber, Valtorta, Jiang...)

ϵ -regularity

Given a 3-manifold (N, g) ,

ϵ -regularity

Given a 3-manifold (N, g) ,

Choi-Schoen (Schoen-Simon-Yau):

There are $\epsilon > 0$ and $C > 0$ such that if $r > 0$ is small enough, then for any $p \in N$ and minimal surface Σ ,

$$\int_{\Sigma \cap B(p, r)} |A|^2 \leq \epsilon \Rightarrow \max_{\Sigma \cap B(p, r/2)} |A|^2 \leq Cr^{-2}.$$

ϵ -regularity

Given a 3-manifold (N, g) ,

Choi-Schoen (Schoen-Simon-Yau):

There are $\epsilon > 0$ and $C > 0$ such that if $r > 0$ is small enough, then for any $p \in N$ and minimal surface Σ ,

$$\int_{\Sigma \cap B(p, r)} |A|^2 \leq \epsilon \Rightarrow \max_{\Sigma \cap B(p, r/2)} |A|^2 \leq Cr^{-2}.$$

Schoen (Schoen-Simon):

There is $C > 0$ such that if $r > 0$ is small enough, then for any $p \in N$ and minimal surface Σ :

$$\Sigma \cap B(p, r) \text{ has Morse index } 0 \Rightarrow \max_{\Sigma \cap B(p, r/2)} |A|^2 \leq Cr^{-2}.$$

ϵ -regularity

Nakajima, Anderson, Gao, Cheeger-Tian:

There are $\epsilon > 0$ and $C > 0$ such that if r is small enough, then for any Einstein 4-manifold (M, g) and any $p \in M$,

$$\int_{B(p,r)} |\text{Rm}|^2 \leq \epsilon \Rightarrow \max_{B(p,r/2)} |\text{Rm}| \leq Cr^{-2}.$$

Compactness

Sharp, Chodosh-Ketover-Maximo:

If $\{\Sigma_i\}$ is a sequence of minimal surfaces in (N, g) such that

$$\text{Morse index}(\Sigma_i) \leq C, \text{ and } \text{Area}(\Sigma_i) \leq C$$

then Σ_i subsequentially “converges” to a smooth minimal surface $\Sigma_\infty \subset N$, and the genus of Σ_i is uniformly bounded.

Compactness

Anderson, Bando-Kasue-Nakajima, Gao, Anderson-Cheeger:

If $\{(M_i, g_i)\}$ is a sequence of Einstein 4-manifolds such that

$$\chi(M_i) \leq C \text{ and } \text{Vol}(M_i, g_i) \geq C^{-1} \text{ and } \text{Diam}(M_i, g_i) \leq C$$

then (M_i, g_i) subsequentially “converges” to a smooth Einstein 4-orbifold (M_∞, g_∞) , and the number of diffeomorphism types of M_i is finite.

Thick/thin decomposition for Einstein 4-manifolds

Idea of “large/small-decomposition”: divide a complicated object into two pieces, each piece being “simple” from a certain point of view. One of the first examples of thick-thin decomposition in geometry is given by Margulis’ lemma for hyperbolic manifolds.

Thick/thin decomposition for Einstein 4-manifolds

Idea of “large/small-decomposition”: divide a complicated object into two pieces, each piece being “simple” from a certain point of view. One of the first examples of thick-thin decomposition in geometry is given by Margulis’ lemma for hyperbolic manifolds.

Consider $\nu_0 > 0$ and $\epsilon > 0$ small constants. For an Einstein 4-manifold (M, g) , define at $p \in M$:

$$r_\epsilon(p) := \sup\{r \in (0, 1]; \quad \int_{B(p,r)} |\text{Rm}|^2 \leq \epsilon\},$$

Thick/thin decomposition for Einstein 4-manifolds

Idea of “large/small-decomposition”: divide a complicated object into two pieces, each piece being “simple” from a certain point of view. One of the first examples of thick-thin decomposition in geometry is given by Margulis’ lemma for hyperbolic manifolds.

Consider $v_0 > 0$ and $\epsilon > 0$ small constants. For an Einstein 4-manifold (M, g) , define at $p \in M$:

$$r_\epsilon(p) := \sup\{r \in (0, 1]; \quad \int_{B(p,r)} |\text{Rm}|^2 \leq \epsilon\},$$

and set

$$M_{>v_0} := \{x \in M; \quad \text{Vol}(B(x, r_\epsilon(x))) > v_0 r_\epsilon(x)^4\},$$

$$M_{\leq v_0} := \{x \in M; \quad \text{Vol}(B(x, r_\epsilon(x))) \leq v_0 r_\epsilon(x)^4\}.$$

Sheeted/non-sheeted decomposition for minimal surfaces

Let (N, g) be a 3-manifold. Consider $n_0 > 0$ a large constant and $\bar{r} > 0$ a small constant. For a minimal surface $\Sigma \subset N$, define at $p \in M$:

$$s(p) := \sup\{r \leq \bar{r}; \quad \Sigma \cap B(p, r) \text{ is stable}\},$$

Sheeted/non-sheeted decomposition for minimal surfaces

Let (N, g) be a 3-manifold. Consider $n_0 > 0$ a large constant and $\bar{r} > 0$ a small constant. For a minimal surface $\Sigma \subset N$, define at $p \in M$:

$$s(p) := \sup\{r \leq \bar{r}; \quad \Sigma \cap B(p, r) \text{ is stable}\},$$

and set

$$\Sigma_{>n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area larger than } n_0 s(x)^2\},$$

$$\Sigma_{\leq n_0} := \{x \in \Sigma; \quad \Sigma \cap B(x, s(x)) \text{ has area at most } n_0 s(x)^2\}.$$

Unbounded families

Question: how to analyse families of minimal surfaces / Einstein 4-manifolds without bounds on Morse index + Area / Euler characteristic bounds + volume?

Unbounded families

Question: how to analyse families of minimal surfaces / Einstein 4-manifolds without bounds on Morse index + Area / Euler characteristic bounds + volume?

One cannot rely on compactness arguments anymore.

Unbounded families

Question: how to analyse families of minimal surfaces / Einstein 4-manifolds without bounds on Morse index + Area / Euler characteristic bounds + volume?

One cannot rely on compactness arguments anymore.

Idea: divide the object using a large/small decomposition and prove quantitative estimates on the simplest part.

New quantitative results

Theorem 1: Let (N, g) be a closed 3-manifold, there is a constant $C = C(N, g)$ such that for any minimal surface $\Sigma \subset N$ and $n_0 > 0$,

$$\text{genus}(\Sigma_{\leq n_0}) \leq Cn_0(\text{Morse index}(\Sigma) + 1).$$

New quantitative results

Theorem 1: Let (N, g) be a closed 3-manifold, there is a constant $C = C(N, g)$ such that for any minimal surface $\Sigma \subset N$ and $n_0 > 0$,

$$\text{genus}(\Sigma_{\leq n_0}) \leq C n_0 (\text{Morse index}(\Sigma) + 1).$$

Theorem 2: There are constants C, ϵ, v_0 , such that for any closed Einstein 4-manifold (M, g) , $M_{\geq v_0}$ admits a metric h with

$$|\text{Rm}_h| \leq 1, \quad \text{injr}_{ad}_h \geq 1 \text{ and } \text{Vol}(M_{\geq v_0}, h) \leq C \chi(M).$$

