Quantum Geometry

Bergfinnur Durhuus

It is a generally accepted hypothesis that all physical phenomena can be described in terms of four fundamental forces: the electro-magnetic force, the weak and strong nuclear forces, and the gravitational force. Of these, the first three can be described by a unified so-called gauge theory, the *standard model*, whose mathematical framework is the geometric theory of principal and associated bundles, at the classical level. The quantum theory of the standard model has been well understood since the beginning of the 1970'ies at the perturbative level, meaning that fundamental physical quantities such as scattering cross sections can be calculated to any order as a power series in a certain parameter, the gauge coupling constant, yielding results that agree with experiments to an extraordinay accuracy. However, due to short-distance singularities the theory is not completely understood at the non-perturbative level. In particular, the problem of quark confinement, i.e. the phenomenon that the fundamental building blocks of the proton, the neutron, and other strongly interacting particles cannot be observed as free particles, is still unsolved. One way to resolve the short-distance singularities just mentioned is to replace continuum space-time by a lattice, say $\mathbb{Z}^d$, with some lattice spacing $a > 0$. This leads to so-called lattice gauge theory within which the confinement problem can be given a precise mathematical meaning in terms of a so-called area law for a certain observable quantity. In my thesis work I investigated various aspects of this problem, years before it obtained the prominent status as one of the milennium problems.

The gravitational force is weaker than the the other three forces by many orders of magnitude, e.g. the ratio between the magnitude of the electric Coulomb force and the gravitational force between an electron and a proton is 10^{37}. It turns out that the short distance singularities of gravitation are nevertheless more severe than for the standard model, and consequently a consistent quantum theory of gravity is not available, even on a perturbative level. Considering the concept of quantum fluctuations inherent in quantum theory, reflected e.g. by the Heisenberg uncertainty relations, one expects space-time to be subject to fluctuations in a quantum theory of gravitation, since space-time according to Einstein is the fundamental degree of freedom of gravitation. Following the previously mentioned philosophy of lattice gauge theory it is natural to replace continuum space-time with some kind of discretised manifold in order to avoid short-distance singularities. For this purpose there are various possible approaches. In 1961 T. Regge proposed to consider piecewise linear triangulated manifolds as approximations to continuum space-time in an attempt to formulate a semi-classical approximation to a quantum theory of gravity. Partly inspired by this work I developed in 1985 together with my collegues J. Ambjørn, NBI, and J. Fröhlich, ETH-Zürich, a different approach in terms of finite randomly triangulated manifolds with a fixed edge length $a > 0$. Although originally thought of as a regularised string theory, the model has a natural interpretation as two-dimensional gravity, and generalisations to higher dimensions are rather straight-forward (I will say a bit more about this towards the end). As it turns out, the two-dimensional model is but one out of a large class of similar models describing gravity interacting with various kinds of matter that are exactly solvable in the sense that certain statistical sums over surfaces with a fixed topology can be computed as functions of the area and the lengths of their boundary components. In fact, those functions can be thought of as generating functions for the number of geometric combinatorial objects (surfaces) of specific types and they can be computed either by applying purely combinatorial arguments or, in some ways more efficiently, by applying matrix model techniques. You can find more information about this topic and on how the continuum limit $a \to 0$ can be evaluated in the monograph *J. Ambjørn, B. Durhuus and T. Jonsson: Quantum Geometry.*

Although the two-dimensional case may seem somewhat academic from a physical point of view, this is in no way the case for at least three reasons: 1) it serves as a useful testbed for what to expect in more interesting higher dimensional cases, 2) it turns out to provide interesting examples of statistical mechanical models in a random environment, 3) it has significant independent interest from a purely combinatorial or probabilistic point of view. To illustrate the last point, a natural question to ask is what the generic local behaviour of a space-time surface looks like, a question that requires more refined probabilistic techniques than those previously mentioned. In 2002 O. Angel and O. Schramm constructed a probability measure on the space of triangulations of the plane by taking the local infinite area limit of the ensemble of triangulated finite surfaces described above. This limit is called the uniform infinite planar triangulation. By a different method I constructed together with P. Chassaing, Université H. Poincaré, Nancy, the uniform infinite planar quadrangulation in 2003. These measures turn out to be well suited to investigate local properties of the surfaces in question. Just to mention one result, we showed that generic surfaces have a fractal behaviour in the sense that the average area A of a disc of radius R around a fixed point obeys

$$A \sim R^4$$

as $R \to \infty$, where the exponent 4 is called the *Hausdorff dimension*. An almost sure version of this result has been established by other researchers.

Another indicator for the local characteristics of generic surfaces is obtained by considering a random walker on the individual surfaces and letting $q(n)$ be the probability for the walker to return to the origin after n steps. Then the *spectral dimension* d_s is defined by

$$q(n) \sim n^{-\frac{d_s}{2}}$$

as $n \to \infty$. For the standard hypercubic quadrangulation of the plane (or any regular tessalation) it is easy to show that $d_s = 2$. Together with my collaborators T. Jonsson, University of Iceland, and J. Wheater, University of Oxford, we have determined d_s as well as the Hausdorff dimension for a large variety of (random) geometric structures, in particular for various classes of random trees. However, for the uniform infinite planar triangulation and quadrangulation the valsue of d_s is still an open problem, although it has been shown that $d_s \leq 2$.

Finally, let us return briefly to gravity in space-time dimension $d > 2$. In 1991 we proposed together with J. Ambjørn and T. Jonsson a tensor model for three-dimensional gravity analogous to the matrix models mentioned above for $d = 2$. Although this models has been revived by various researchers in recent years and more sophisticated versions have been proposed, it seems that significant analytic progress is still awaiting new ideas. It is a crucial requirement within the combinatorial approach, in order to define the relevant physical quantities properly, that an exponential upper bound of the form

$$N(V) \leq C^V$$

holds, where $N(V)$ denotes the number (up to combinatorial equivalence) of triangulated piecewise linear d-manifolds of volume V, i.e. with V d-simplices, and C is some finite positive dimension dependent constant. It is easy to see that in order for such a bound to hold the topology of manifolds must be restricted. Exactly what kind of restriction is most suitable is not evident, but even under the strong assumption that the manifolds be homeomorphic to the d-dimensional sphere (or ball) the bound has so far not been established for $d \geq 3$, although numerical simulations seem to suggest that it holds, at least for $d = 3$ and $d = 4$. Some progress on certain modified but still interesting models of three-dimensional gravity has been obtained recently, but I shall not elaborate further on it here.

I hope this short essay conveys an indication of parts of what I have been doing in the past as well as what keeps me busy at present.