SEMINAR NOTES ON PSEUDO-DIFFERENTIAL OPERATORS AND BOUNDARY PROBLEMS, I.A.S. PRINCETON 1965-66

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Introduction

This series of lectures[1] consists of two parts. The first is a study of pseudo-differential operators, and the second consists of applications to boundary problems for elliptic (pseudo-)differential operators. However, when sketching the aims of I and II change the order since the second part gives some of the motivation for the first.

The standard theory of boundary problems for elliptic differential equations (or systems), as it can be found for example in the last chapter of my book, runs as follows: One first considers a model for the local behavior in the case of an elliptic homogeneous differential equation $Pu = f$ in a half space with boundary conditions $B_j u = f_j$ involving some homogeneous constant coefficient differential equations. Fourier transformation along the boundary reduces the study to that of a boundary problem for ordinary differential equations involving as parameters an element of the bounding hyperplane. When these equations always have unique solutions, the problem is called elliptic. Using L^2 norms one then finds that the derivatives of u of order $m =$ order of P can be estimated in terms of the norm of f and suitable norms on f_j. Changing lower order terms or adding small perturbations in the leading ones in P and B_j leads to perturbations which are compact or of small norm, so one can immediately pass to local results for the case where P and B_j have variable coefficients and lower order terms. From there a partition of unity easily leads to global existence and regularity theorems for boundary problems in manifolds with boundary satisfying the ellipticity condition if viewed "microscopically" at any boundary point (and interior point).

The class of boundary value problems which is covered by this technique is called elliptic, coercive or of Lopatinski-Shapiro type. It is clearly stable under arbitrary perturbations of lower order terms and small perturbations in the leading terms, therefore for perturbations of the boundary which are small in the C^1 topology. This property is of course appealing but it is also shown that many important boundary problems must fail to be elliptic. For example, if we are interested in boundary problems like the $\overline{\partial}$ Neumann problem in the theory of functions of several complex variables we know that existence theorems can only be expected to hold when the boundary satisfies certain convexity conditions. These are not stable under small perturbations of the boundary in the C^1 topology, so in questions of this kind one will always encounter non-elliptic boundary problems.

[1] TEX-typed by G. Grubb in 2013 (Ch. II) and 2018 (Introduction).

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Another weak point is that if there is a jump in boundary conditions we cannot control the perturbations and the methods outlined break down, although there are problems of this kind which can easily be studied with variational methods.

A third drawback is that much work has to be done which is closely analogous to what one does in studying elliptic differential equations in an open manifold. This makes it natural to try to reduce the study of boundary problems to that of equations only inside the boundary and in that way also be able to exploit the techniques developed for the study of non-elliptic equations to the study of non-elliptic boundary problems. This is indeed possible by essentially classical arguments, which I wish to indicate in a special case.

Suppose we want to solve the boundary problem

$$\Delta u = 0, \quad b_0 u_0 + b_1 u_1 = f \text{ on } \omega,$$

where Ω is an open set in $\mathbb{R}^n$ with smooth boundary ω, Δ is the Laplacean, and b_0, b_1 are differential operators in ω acting respectively on the boundary value u_0 and the normal derivative u_1 of u. If $E = c|x|^{2-n}$ is a fundamental solution of the Laplacean, we obtain from Green's formula

$$u(x) = \int \partial E(x-y)/\partial n_y \, u_0(y)\, dS(y) - \int E(x-y)\, u_1(y)\, dS(y), \quad x \in \Omega.$$

Thus it suffices to determine u_0 and u_1. Letting x approach ω we obtain a relation between u_0 and u_1 of the form

$$u_0 = k_0 u_0 + k_1 u_1$$

where k_0 and k_1 are (singular) integral operators. Conversely, it is easily seen that this implies that the normal derivative of the expression defined by Green's formula is also equal to u_1. Our boundary problem is therefore reduced to the solution of the system of equations

$$\begin{aligned}(1-k_0)u_0 - k_1 u_1 &= 0\\ b_0 u_0 + b_1 u_1 &= f\end{aligned}$$

where the unknowns are functions in the manifold without boundary ω. What is involved is therefore the existence of solutions of systems of singular integral equations. If we had different boundary conditions on different parts of the boundary it is easily seen that we are led to a boundary problem for a system of singular integral equations. Such have recently been discussed by Vishik and Eskin. This in turn should be possible to reduce to the study of some other singular integral equations in the manifold separating regions with different boundary conditions.

A closer inspection of the argument just outlined will indicate how to choose a class of operators which is large enough to make possible the argument outlined and still lies sufficiently close to the class of partial differential operators so that one can hope to extend what is known about these to the more general class of operators. Thus let us now consider a more general elliptic operator than the Laplacean, an operator $P(D)$ where $D = -i\partial/\partial x$ and P is a homogeneous polynomial with $P(\xi) \neq 0$ when $\mathbb{R}^n \ni \xi \neq 0$. A fundamental solution is then formally given by

$$Eu(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{u}(\xi)/P(\xi)\, d\xi,$$

where we have to make some modification near zero to guarantee convergence — it does not matter very much how we do that since Fourier integrals over compact sets are C^∞ functions and we are mainly concerned with singularities anyway. In the classical tradition of singular integral operators one would now rewrite E as an integral operator acting on u — which will be of convolution type. When P is perturbed by an operator with small variable coefficients the solution of an integral equation will then yield a fundamental solution of the new equation in the form of a Neumann series. However, there is very little one can say about that kernel apart from the type of regularity properties it has. The reason is of course that for example convolution of functions should be expressed in terms of the Fourier transform where it appears simply as multiplication. The recent trend of the theory of integral equations has therefore been to forget the kernels almost entirely and thus also for equations with variable coefficients try to write fundamental solutions in the form of Fourier integrals

$$Eu(x) = (2\pi)^{-n} \int e(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi.$$

It turns out that indeed one will then be able to determine the function e almost exactly by algebraic calculations alone. Since no singularities are visible any longer it is natural to talk about pseudo-differential operators — a term suggested by Friedrichs — instead of singular integral operators.

Thus we are led to consider operators of the following form — I change e to p at this moment —

$$p(x,D)u(x) = (2\pi)^{-n} \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi, \quad u \in C_0^\infty(\Omega),\ x \in \Omega,$$

where Ω is an open subset of $\mathbb{R}^n$. The first question is what one should assume concerning the function p. The following should certainly be accepted:

a) $p(x,\xi) =$ arbitrary polynomial in ξ. Then $p(x,D)$ is just a differential operator with the characteristic polynomial p. This motivates the notation.

b) Any $p(x,\xi)$ which is a positively homogeneous function of ξ and is smooth when $\xi \neq 0$ ($\xi \in \mathbb{R}^n$), being suitably modified near $\xi = 0$.

We should also allow linear combinations of the preceding functions p and suitable limits. Starting from functions homogeneous of a real degree this is precisely what gives rise to the pseudo-differential operators of Mihlin-Calderón-Zygmund-Seeley-Kohn-Nirenberg and others; Seeley has also written a paper where he allows complex orders of homogeneity which is essential in some questions. However, we shall take a more general class. First note that the functions which we have allowed so far will satisfy estimates of the form

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\alpha|}, x \in K \subset\subset \Omega, \xi \in \mathbb{R}^n,$$

for arbitrary multi-indices α and β. (Notations!) Two students of Schwartz, Unterberger and Bokobza have carried out a study of the operators defined by arbitrary functions of this kind. However, I want to allow still greater generality in order not to restrict the usefulness of the machinery to elliptic equations. The next simplest class of operators is the class of hypoelliptic operators. If for example $P(D)$ is a constant coefficient differential operator, then it is known that $P(D)$ is hypoelliptic (that is, all solutions of $P(D)u = 0$ are C^∞) if and only if the derivatives $D^\alpha P(\xi)$ are decisively smaller than $P(\xi)$ at infinity. For the function $E(\xi) = 1/P(\xi)$ which occurs in the fundamental solution of P this means that for a suitable m

$$|D^\alpha E(\xi)| \le C_\alpha (1 + |\xi|)^{m-\varrho|\alpha|},$$

where $0 < \varrho \le 1$ is a number which is closely related to the regularity properties of the solutions of $P(D)u = 0$. (The solutions are of Gevrey class $1/\varrho$ but no better, if ϱ is the smallest number that can be used.)

Suppose we make a simple modification of E, by taking an invertible matrix $A(x)$ and forming $E(A(x)\xi)$. Then we notice that derivatives with respect to x may grow faster and faster,

$$|D_x^\beta D_\xi^\alpha E(A(x)\xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{m-\varrho|\alpha|+(1-\varrho)|\beta|}.$$

We allow for such behavior in the following definition: Enter page 1, Definition 1.1.1. [This seems to be Definition 2.1 of [1].][2]

Chapter I
Pseudo-differential operators

[Not available. The text seems to have been essentially incorporated in the contribution "Pseudo-differential operators and hypoelliptic equations" to the Symposium in Pure Mathematics X "Singular Integrals" 1966, listed at the end of these notes as [1].]

[2] Remarks made by G. Grubb during the typing are given in square brackets.

Chapter II
Boundary problems for "classical" pseudo-differential operators

2.1. Preliminaries. In this chapter we shall restrict the use of the term pseudo-differential operator to the subset $L^m_{1,0}$ consisting of operators with symbol $\sum_0^\infty p_j(x,\xi)$ where p_j is positively homogeneous of degree m_j, $\operatorname{Re} m_1 \geq \operatorname{Re} m_2 \geq \cdots \to -\infty$. (Note that any such sum, conveniently modified at the origin, satisfies the hypotheses of Theorem 1.1.5 [seems to be covered by Theorem 2.7 of [1]] with $\varrho = 1$, $\delta = 0$. Every such sum can therefore occur as symbol.) This class of operators is first defined in open subsets of $\mathbb{R}^n$ but since it is invariant for a change of variables in view of Theorem 1.1.11 [seems to be Theorem 2.16 of [1]], the extension to manifolds is immediate.

Let M be a fixed paracompact C^∞ manifold, and let Ω be an open subset of M with a C^∞ boundary $\partial\Omega$. Our purpose is to study boundary problems for the pseudo-differential operator P in Ω. This means that we shall look for distributions u with support in $\overline{\Omega}$ such that $Pu = f$ is given in Ω and u satisfies some conditions on $\partial\Omega$ in addition. In particular we shall make a detailed study of the regularity of u at the boundary when f and the boundary data are smooth. Examples involving α-potentials due to M. Riesz and extended in part by Wallin show that one should not expect u to be smooth up to the boundary but that one has to expect u to behave as the distance to the boundary raised to some power. This leads us to define a family of spaces of distributions $\mathcal{E}_\mu$ as follows.

If $\operatorname{Re}\mu > -1$ and if d is a real valued function in $C^\infty(M)$ such that

$$\Omega = \{x; d(x) > 0\}$$

and d vanishes only to the first order on $\partial\Omega$, then $\mathcal{E}_\mu(\overline{\Omega})$ consists of all functions u such that $u = 0$ in $\complement\overline{\Omega}$ and $u = d^\mu v$ in $\overline{\Omega}$ for some $v \in C^\infty(\overline{\Omega})$. This definition is independent of the choice of d for if d_1, d_2 are two functions with the required properties the quotient d_1/d_2 is positive and infinitely differentiable. In order to extend the definition to arbitrary μ we note that if D is a first order differential operator with C^∞ coefficients and if $\operatorname{Re}\mu > 0$ then $D\mathcal{E}_\mu \subset \mathcal{E}_{\mu-1}$, for $D(d^\mu v) = d^{\mu-1}V$ for some $V \in C^\infty$. The linear hull of the spaces $D\mathcal{E}_\mu$ when D varies is in fact equal to $\mathcal{E}_{\mu-1}$. It is sufficient to prove that it contains any element in $\mathcal{E}_{\mu-1}$ with support in a coordinate patch where Ω is defined by $x_n > 0$. Then we can take $D = \partial/\partial x_n$ noting that if $v \in C^\infty$ then

$$\int_0^{x_n} t^{\mu-1} v(x_1, \ldots, x_{n-1}, t)\, dt = x_n^\mu V(x),$$

where

$$V(x) = \int_0^1 t^{\mu-1} v(x_1, \ldots, x_{n-1}, x_n t)\, dt$$

is a C^∞ function. If $u = x_n^{\mu-1} v$ and $U = x_n^\mu V \chi$, both functions being defined as 0 when $x_n < 0$, and $\chi \in C_0^\infty$ is 1 in a neighborhood of $\operatorname{supp} u$, then $u = \partial U/\partial x_n$ is a C^∞ function [means "on $\mathbb{R}^n_+$"?] with support in $x_n \geq 0$, so $u \in \partial\mathcal{E}_\mu/\partial x_n + \mathcal{E}_\mu$. It is thus legitimate to define $\mathcal{E}_\mu$ successively for decreasing $\operatorname{Re}\mu$ so that $\mathcal{E}_{\mu-1}$ is always the linear hull of the spaces $D\mathcal{E}_\mu$ when D varies over the first order differential operators with C^∞ coefficients.

The spaces $\mathcal{E}_\mu$ so obtained have the local property that $u \in \mathcal{E}_\mu(\overline{\Omega})$ and $\varphi \in C^\infty(M)$ implies that $\varphi u \in \mathcal{E}_\mu(\overline{\Omega})$. In fact, if D again denotes a first order differential operator we have

$$\varphi D\mathcal{E}_{\mu+1} \subset D\varphi\mathcal{E}_{\mu+1} + \mathcal{E}_{\mu+1} \subset D\mathcal{E}_{\mu+1} + \mathcal{E}_\mu \subset \mathcal{E}_\mu$$

where we have assumed that the assertion is already proved with μ replaced by $\mu+1$. The spaces $\mathcal{E}_\mu$ are thus determined by local properties. Inside the set the condition $u \in \mathcal{E}_\mu$ only means that u is a C^∞ function.

To determine the meaning of the condition $u \in \mathcal{E}_\mu$ at a boundary point we consider the case when u has compact support in a coordinate patch where Ω is defined by the condition $x_n > 0$.

Lemma 2.1.1. *An element $u \in \mathcal{E}'(\mathbb{R}^n)$ belongs to $\mathcal{E}_\mu(\overline{\mathbb{R}}^n_+)$, where $\mathbb{R}^n_+$ is the half space of $\mathbb{R}^n$ where $x_n > 0$, if and only if u vanishes when $x_n < 0$ and one can find $u_0, u_1, \dots \in C_0^\infty(\mathbb{R}^{n-1})$ such that for every N*

$$\hat{u}(\xi) - \sum_0^{N-1} (\xi_n - i)^{-\mu-j-1}\hat{u}_j(\xi') = O(|\xi|^{-\operatorname{Re}\mu-N-1}),\ \xi \to \infty. \tag{2.1.1}$$

Conversely, given such $u_0, u_1, \dots$ one can find $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}^n_+)$ satisfying this condition.

Here the argument of $\xi_n - i$ is chosen so that it tends to 0 when $\xi_n \to +\infty$.

Proof. Any element $u \in \mathcal{E}_\mu$ can be written $u = v + \partial w/\partial x_n$ where v and w belong to $\mathcal{E}_{\mu+1}$. If the necessity of (2.1.1) has been proved when μ is replaced by $\mu+1$ it follows therefore for μ. Hence we may assume that $\operatorname{Re}\mu > 0$, thus $u = vx_n^\mu$ when $x_n > 0$, where $v \in C_0^\infty(\mathbb{R}^n)$. By forming a Taylor expansion of ve^{x_n} we can write for every N

$$v = e^{-x_n}\sum_0^N v_j(x')x_n^j + R_N(x)$$

where $v_j \in C_0^\infty(\mathbb{R}^{n-1})$ and $R_N(x) = O(x_n^N)$ when $x_n \to 0$, $R_N(x) = O(e^{-x_n/2})$ when $x_n \to \infty$. Set $R_N^0(x) = R_N(x)$ when $x_n > 0$ and $R_N^0(x) = 0$ when $x_n \le 0$. Then $R_N^0(x)$ [0 added] has integrable derivatives of order N, so the Fourier transform is $O(|\xi|^{-N})$. Now

$$\hat{u} = \sum_0^\infty \hat{v}_j(\xi')\int_0^\infty e^{-x_n(1+i\xi_n)}x_n^{j+\mu}\,dx_n + \int R_N^0(x)x_n^\mu e^{-i\langle x,\xi\rangle}\,dx.$$

If we set

$$u_j = v_j\Gamma(j+\mu+1)e^{-\pi i(j+\mu+1)/2}, \tag{2.1.2}$$

it follows that (2.1.1) holds with the error term $O(|\xi|^{-N})$. Taking a few additional terms in the left hand side of (2.1.1) and noting that they can all be estimated in terms of the quantity on the right, we thus conclude that (2.1.1) is valid. On the other hand, if u satisfies (2.1.1) we obtain with v_j defined by (2.1.2) that $u - e^{-x_n}\sum_0^{N-1} v_j x_n^{j+\mu}$ will be arbitrarily smooth if N is large. This proves the sufficiency of (2.1.1). To prove the last statement we again assume that $\operatorname{Re}\mu > 0$, take $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 when $|x_n| < 1$ [replaces $x_n > 0$] and define

$$u(x) = 0,\ x_n \le 0; \quad u(x) = \sum_0^\infty e^{-x_n} v_j(x') x_n^{\mu+j} \chi(x_n a_j),\ x_n > 0,$$

where a_j is chosen so large that the derivatives of the jth term of order $\le j$ are all $\le 2^{-j}$. This is possible since $(x_n a_j)^\nu \chi^{(k)}(x_n a_j)$ is bounded uniformly in x_n and a_j if $\operatorname{Re}\nu \ge 0$. This completes the proof.

The particular case where μ is an integer is of special importance. When $\mu \ge 0$ the space $\mathcal{E}_\mu$ then consists of all functions in $C^\infty(\overline{\Omega})$ which vanish to the order μ at the boundary (that is the derivatives of order $< \mu$ vanish there), extrapolated by 0 outside. When $\mu < 0$ we have the sum of a function in $C^\infty(\Omega)$, [probably means $C^\infty(\overline{\Omega})$] extrapolated as 0 in the complement of $\overline{\Omega}$, and multiple layers with C^∞ densities and of order $< -\mu$ on $\partial\Omega$. This is the only case when $\mathcal{E}_\mu$ contains elements supported by $\partial\Omega$; in other words, the restriction of an element in $\mathcal{E}_\mu$ to Ω determines it uniquely except when μ is a negative integer.

One final notation: we shall denote by $\overline{C}^\infty(\Omega)$ the set of restrictions to Ω of functions in $C^\infty(M)$.

It was convenient in the proof of Lemma 2.1.1 to work with powers of $(\xi_n - i)$ instead of powers of ξ_n, but it will be less convenient in the applications. With the notation $(\xi_n^-)^a$ for the boundary values from the lower half plane of z^a, defined to be real and positive on the positive real axis, we can rewrite (2.1.1) in the form

$$\hat{u}(\xi) - \sum_0^{N-1} (\xi_n^-)^{-\mu-j-1} \hat{u}_j'(\xi') = O(|\xi|^{-\operatorname{Re}\mu - N - 1}),\ \xi \to \infty,\ |\xi_n| > 1, \tag{2.1.1$'$}$$

where u_j' is a linear combination of $u_0, \dots, u_j$ with coefficient 1 for u_j. Thus the u_j' occurring in (2.1.1$'$) are in one to one correspondence with the u_j in (2.1.1) and can be chosen arbitrarily.

2.2. Regularity at the boundary. The first question we shall discuss in this paragraph is when a pseudo-differential operator P in M maps $\mathcal{E}_\mu$ into $\overline{C}^\infty(\Omega)$ (more precisely, the restrictions to Ω belong to $\overline{C}^\infty(\Omega)$). By the pseudo-local property we know that $Pu \in C^\infty(\Omega)$ for all $u \in \mathcal{E}_\mu$. We shall therefore only expect a restriction on P at points on $\partial\Omega$. Of course it is no restriction to assume P compactly supported when studying a regularity problem.

Theorem 2.2.1. *Let P be a compactly supported pseudo-differential operator in M. In order that $Pu \in \overline{C}^\infty(\Omega)$ for all $u \in \mathcal{E}_\mu(\overline{\Omega})$ it is necessary and sufficient that in any local coordinate system we have*

$$p_j{}_{(\beta)}^{(\alpha)}(x, -N) = e^{\pi i(m_j - |\alpha| - 2\mu)} p_j{}_{(\beta)}^{(\alpha)}(x, N),\ x \in \partial\Omega, \tag{2.2.1}$$

where $\sum p_j(x,\xi)$ is the symbol of P in the local coordinate systems, p_j is homogeneous of degree m_j, and N denotes the interior normal of $\partial\Omega$ at x.

Since every polynomial satisfies this hypothesis with $\mu = 0$ it follows from Theorem 1.1.11 that (2.2.1) is invariant under any change of variables. In the proof we may therefore use local coordinates such that Ω is defined by the inequality $x_n > 0$. The statement is local so it is enough to consider Pu for $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}^n_+)$ with compact support in the coordinate patch $U \subset \mathbb{R}^n$. After modifying P by an operator with symbol 0 we may assume that P is a compactly supported operator in U.

Proof of Theorem 2.2.1. Suppose that the theorem were already proved with μ replaced by $\mu+1$. The necessity of (2.2.1) is then obvious for it holds with μ replaced by $\mu+1$ and $e^{-2\pi i} = 1$. To prove its sufficiency we have to show that $PDu \in \overline{C}^\infty(\Omega)$ if $u \in \mathcal{E}_{\mu+1}$ and D is a first order differential operator. Since $PDu = DPu + [P,D]u$ and $[P,D]$ satisfies (2.2.1) if P does, the assertion follows. Hence we may assume in what follows that $\operatorname{Re}\mu > \operatorname{Re} m_0$. Then the product of $p(x,\xi)$ by the Fourier transform of any compactly supported $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}^n_+)$ is integrable, so by an obvious regularization we obtain

$$p(x,D)u = (2\pi)^{-n} \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi. \tag{2.2.2}$$

We shall introduce a Taylor expansion of p in (2.2.2),

$$p(x,\xi) = \sum_{|\alpha|<\nu} \partial^{|\alpha|} p(x',0,0,\xi_n)/\partial\xi^{\alpha'}\partial x_n^{\alpha_n}\, x_n^{\alpha_n}\xi^{\alpha'}/\alpha! + \sum_{|\alpha|=\nu} r^\alpha(x,\xi) x_n^{\alpha_n}\xi^{\alpha'}, \tag{2.2.3}$$

where

$$r^\alpha(x,\xi) = |\alpha|/\alpha! \int_0^1 (1-t)^{|\alpha|-1} p^{(\alpha')}_{(\alpha_n)}(x', tx_n, t\xi', \xi_n)\, dt,$$

where somewhat incorrectly we have used the notation α' for $(\alpha',0)$ and α_n for $(0,\alpha_n)$. When $|\alpha'| > \operatorname{Re} m_0$ we can estimate r^α by $(1+|\xi_n|)^{\operatorname{Re} m_0 - |\alpha'|}$ and when $|\alpha'| \le \operatorname{Re} m_0$ we can estimate by $(1+|\xi|)^{\operatorname{Re} m_0 - |\alpha'|}$ instead. Now we have

$$\int r^\alpha(x,\xi) x_n^{\alpha_n}\xi^{\alpha'}\hat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi = \int (i\partial/\partial\xi_n)^{\alpha_n}(r^\alpha(x,\xi)\xi^{\alpha'}\hat{u}(\xi))e^{i\langle x,\xi\rangle}\, d\xi.$$

[Moved a parenthesis.] Using (2.1.1) we conclude that the integral and its derivatives of order $\le k$ are absolutely convergent, thus the integral defines a C^l function, provided that

$$l + \operatorname{Re} m_0 - |\alpha'| - \alpha_n - \operatorname{Re}\mu < 0.$$

If we choose $\nu > k + \operatorname{Re}(m_0 - \mu)$ the error term in (2.2.3) will therefore only contribute a C^l term to $p(x,D)u$. The remaining problem is only to study the regularity of the partial sums of the series obtained by replacing $p(x,\xi)$ by its Taylor expansion in (2.2.2). Since $\hat{u}$ is rapidly decreasing when $\xi \to \infty$ with $|\xi_n| < 1$, this part of the integral in (2.2.2) is infinitely differentiable. In view of (2.2.1′) — where we drop the prime on u'_j — it only remains to examine when the partial sums of the series

$$\sum_{\alpha,j,k} (2\pi)^{-n} \int_{|\xi_n|>1} p_j{}^{(\alpha')}_{(\alpha_n)}(x',0,0,\xi_n) x_n^{\alpha_n}\xi^{\alpha'}\hat{u}_k(\xi')(\xi_n^-)^{-\mu-k-1}e^{i\langle x,\xi\rangle}\, d\xi/\alpha!$$

become arbitrarily smooth when the order of the sum goes to infinity. We can remove the factor $x_n^{\alpha_n}$ by an integration by parts with respect to ξ_n. The boundary terms which then occur will give rise to only C^∞ terms. Thus we are reduced to examining the differentiability of the partial sums of the series

$$\sum_{\alpha,j,k} D^{\alpha'}u_k(x')(2\pi)^{-1}\int_{|\xi_n|>1}(i\partial/\partial\xi_n)^{\alpha_n}\big(p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,\xi_n)(\xi_n^-)^{-\mu-k-1}\big)e^{ix_n\xi_n}\,d\xi_n/\alpha!.$$

[$1/\alpha!$ added.] Since the functions $D^{\alpha'}u_k$ can be chosen arbitrarily in the neighborhood of any point, or rather, linear combinations of them are arbitrary, we conclude that for P to have the required property it is necessary and sufficient that for any α' and $k = 0, 1, \dots$ the partial sums of higher order of the series

$$\sum_{\alpha_n,j}(2\pi)^{-1}\int_{|\xi_n|>1}(i\partial/\partial\xi_n)^{\alpha_n}\big(p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,\xi_n)(\xi_n^-)^{-\mu-k-1}\big)e^{i\langle x,\xi\rangle}\,d\xi/\alpha! \tag{2.2.3}$$

[label (2.2.3) occurs twice] are in $\overline{C}^\nu(\mathbb{R}_+)$ [seems to stand for $r^+C^\nu(\mathbb{R})$] for any given ν. We now need an elementary lemma.

Lemma 2.2.2. *Let q be a positively homogeneous function on $\mathbb{R}$ of degree σ, $\operatorname{Re}\sigma < -1$. For $t > 0$ we set $\varphi_\sigma(t) = t^{-\sigma-1}$ if σ is not an integer and $\varphi_\sigma(t) = t^{-\sigma-1}\log t$ if σ is an integer. Then*

$$\int_{|\tau|>1} e^{it\tau}q(\tau)\,d\tau,\ t>0,$$

is on $\mathbb{R}_+$ equal to the sum of a function in $\overline{C}^\infty(\mathbb{R}_+)$ and $C\varphi_\sigma(t)$, where $C = 0$ if and only if $q(-1) = e^{i\pi\sigma}q(1)$, that is, if $q(\tau) = q(1)(\tau^+)^\sigma$.

We postpone the proof of the lemma. Noting that a finite sum $\sum c_j\varphi_{\sigma_j}(t)$ with different σ_j is in $\overline{C}^\nu(\mathbb{R}_+)$ if and only if $c_j = 0$ when $-\sigma_j - 1 \le \nu$, we conclude that (2.2.3) has the desired differentiability properties if and only if for each complex number σ, each α' and $k = 0, 1, \dots,$ the sum

$$\sum_{m_j-|\alpha|-\mu-1=\sigma}(i\partial/\partial\xi_n)^{\alpha_n}\big(p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,\xi_n)(\xi_n^-)^{-\mu-k-1}\big)/\alpha_n! \tag{2.2.4}$$

[parentheses added] is proportional to $(\xi_n^+)^{\sigma-k}$. (The sum of course contains only finitely many terms.) Explicitly, this means that for fixed α', k and σ

$$\begin{aligned}\sum_{m_j-|\alpha|-\mu-1=\sigma} &(m_j-|\alpha'|-\mu-k-1)\dots(m_j-|\alpha|-\mu-k)p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,1)/\alpha_n!\,e^{\pi i(\sigma-k)}\\ &= \sum(-1)^{\alpha_n}(m_j-|\alpha'|-\mu-k-1)\dots(m_j-|\alpha|-\mu-k)p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,-1)/\alpha_n!\,e^{\pi i(-\mu-k-1)}.\end{aligned}$$

[Moved $(-1)^{\alpha_n}$ inside the summation. $e^{\pi i(-\mu-k-1)}$ should probably be $e^{\pi i(\mu+k+1)}$.] After the exponential factors have been moved to the same side we find that k occurs only in the polynomial factors, which are of degree α_n, all different. It follows that the coefficients have to agree, that is,

$$p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,1)e^{\pi i(m_j-|\alpha'|-2\mu)} = p_{j\,(\alpha_n)}^{(\alpha')}(x',0,0,-1) \tag{2.2.5}$$

is a necessary and sufficient condition for P to map $\mathcal{E}_\mu$ into $\overline{C}^\infty$. But (2.2.5) is a consequence of (2.2.1) and conversely, by differentiating (2.2.5) with respect to x' and using the homogeneity with respect to ξ_n we obtain (2.2.1). This completes the proof of Theorem 2.2.1.

Proof of Lemma 2.2.2. Let γ_+ (γ_-) consist of the real axis with the interval $(-1,1)$ replaced by a semi-circle in the upper (lower) half plane. Then the two functions

$$\int_{|\tau|>1} (\tau^\pm)^\sigma e^{it\tau}\,d\tau - \int_{\gamma^\pm} (\tau^\pm)^\sigma e^{it\tau}\,d\tau$$

are integrals of $e^{it\tau}$ over semi-circles, hence obviously entire analytic functions of t. By Cauchy's integral formula one concludes that the integral over γ_+ (γ_-) vanishes for $t>0$ ($t<0$) and that it is homogeneous of degree $-\sigma-1$ when $t<0$ ($t>0$). When σ is not an integer the two functions $(\tau^+)^\sigma$ and $(\tau^-)^\sigma$ are linearly independent, hence form a basis for positively homogeneous functions of degree σ. This proves the lemma for non-integral σ, and to complete the proof it only remains to study

$$\int_{|\tau|>1} (\tau^\pm)^{\sigma-1}|\tau|\,e^{it\tau}\,d\tau$$

when σ is an integer ≤ -2. When $\sigma=-2$ the last integral is equal to

$$2\int_1^\infty \tau^{-2}\sin t\tau\,d\tau = 2t\int_{1/t}^\infty \tau^{-2}\sin\tau\,d\tau.$$

A Taylor expansion of $\sin\tau$ shows that the integral is equal to $\log 1/t$ plus a function in $\overline{C}^\infty(\mathbb{R}_+)$. This proves the statement when $\sigma=-2$, and by successive integration it follows for all integers $\sigma<-2$.

2.3. The spaces $H_{(\sigma,\tau)}$. When studying boundary problems for the operator P we shall have to introduce topologies in the spaces $\mathcal{E}_\mu$. Estimates in the corresponding norms will be obtained locally at first, using coordinate systems where the boundary of Ω is flat. We then need to consider spaces of distributions in the half spaces $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ which are obtained by imposing conditions of tangential regularity in addition to a requirement of regularity in all variables. More precisely, as in my book, section 2.5, we denote by $H_{(\sigma,\tau)}(\mathbb{R}^n)$ the space of all tempered distributions u in $\mathbb{R}^n$ such that

$$\|u\|_{(\sigma,\tau)} = \Big((2\pi)^{-n}\int (1+|\xi|^2)^\sigma(1+|\xi'|^2)^\tau|\hat u(\xi)|^2\,d\xi\Big)^{\frac12} < \infty.$$

Here $\xi'=(\xi_1,\dots,\xi_{n-1})$. By $\overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ we denote the set of all $u\in\mathcal{D}'(\mathbb{R}^n_+)$ such that there exists a distribution $U\in H_{(\sigma,\tau)}(\mathbb{R}^n)$ with $U=u$ in $\mathbb{R}^n_+$; the norm of u is defined by

$$\|u\|^\bullet_{(\sigma,\tau)} = \inf\|U\|_{(\sigma,\tau)},$$

the infimum being taken over all such U. Further we set

$$\overset{\circ}{H}_{(\sigma,\tau)}(\overline{\mathbb{R}}^n_+) = \{u; u \in H_{(\sigma,\tau)}(\mathbb{R}^n), \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\};$$

this is a closed subspace of $H_{(\sigma,\tau)}(\mathbb{R}^n)$ (but not necessarily a subspace of $\mathcal{D}'(\mathbb{R}^n_+)$). (The notations differ slightly from those in my book; hopefully they are more clear.)

It is obvious that $\sigma = \tau = 0$ gives L^2 spaces. In my book it is proved that $\overline{C}_0^\infty(\mathbb{R}^n_+)$ (= set of restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$) is dense in $\overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ and that $C_0^\infty(\mathbb{R}^n_+)$ is dense in $\overset{\circ}{H}_{(\sigma,\tau)}(\overline{\mathbb{R}}^n_+)$. The spaces $\overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ and $\overset{\circ}{H}_{(-\sigma,-\tau)}(\overline{\mathbb{R}}^n_+)$ are dual with respect to an extension of the bilinear form

$$\langle u, v\rangle = \int uv\, dx, \quad u \in \overline{C}_0^\infty(\mathbb{R}^n_+), v \in C_0^\infty(\mathbb{R}^n_+).$$

Later on when we solve boundary problems for pseudo-differential operators by the Wiener-Hopf technique we shall sometimes have to consider the possibility of extending elements in $\overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ by setting them equal to 0 in the lower half space. More precisely, we shall have to know when the restriction mapping

$$\overset{\circ}{H}_{(\sigma,\tau)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+) \tag{2.3.1}$$

is injective or surjective. It is of course always continuous.

Lemma 2.3.1. *If an element $\mu \in H_{(\sigma,\tau)}(\mathbb{R}^n)$ has its support in the plane $\{x; x_n = 0\}$, it follows that $x_n^N \mu = 0$ if N is an integer with $\sigma + N + \frac{1}{2} \geq 0$.*

Proof. The Fourier transform $\hat{\mu}$ of μ must be a polynomial in ξ_n with

$$\int |\hat{\mu}(\xi', \xi_n)|^2 (1 + |\xi|^2)^\sigma (1 + |\xi'|^2)^\tau \, d\xi < \infty.$$

so the degree must be lower than $-\sigma - \frac{1}{2}$, hence lower than N. It follows that $\partial^N \hat{\mu}/\partial \xi_n^N = 0$, which proves the lemma.

On the other hand, a measure on x_n with density in $C_0^\infty(\mathbb{R}^{n-1})$ is an element in $H_{(\sigma,\tau)}(\mathbb{R}^{n-1})$ for all σ and τ with $\sigma < -\frac{1}{2}$. The map (2.3.1) is therefore injective if and only if $\sigma \geq -\frac{1}{2}$.

Lemma 2.3.2. *The mapping* (2.3.1) *is an isomorphism onto if and only if $-\frac{1}{2} < \sigma < \frac{1}{2}$.*

Proof. The adjoint of (2.3.1) is the analogous map with (σ, τ) replaced by $(-\sigma, -\tau)$. To prove the necessity it is therefore sufficient to show that (2.3.1) is not a homeomorphism if $\sigma = \frac{1}{2}$. Thus take $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ and a nonnegative function $\psi \in C_0^\infty(\mathbb{R}_+)$. Set $u_\varepsilon(x) = \varphi(x')\psi(x_n/\varepsilon)\varepsilon^{-1}$. Then it is easily seen that $\|u_\varepsilon\|_{(-\frac{1}{2},\tau)}$ grows like $\log 1/\varepsilon$ when $\varepsilon \to 0$. Moreover, the norm of the restriction to $\mathbb{R}^n_+$ is at most equal to $\|U_\varepsilon\|_{-\frac{1}{2},\tau)}$ where $U_\varepsilon(x) = \varphi(x')(\psi(x_n/\varepsilon) - \psi(-x_n/\varepsilon))\varepsilon^{-1}$ and using the fact that the Fourier transform vanishes when $\xi_n = 0$ it is easily proved that this is bounded when $\varepsilon \to 0$.

To prove the sufficiency it remains to show that (2.3.1) is a homomorphism when $0 \le \sigma < \frac{1}{2}$, that is, we have to prove the inequality

$$\|u_+\|_{(\sigma,\tau)} + \|u_-\|_{(\sigma,\tau)} \le C\|u_+ + u_-\|_{(\sigma,\tau)}, \tag{2.3.2}$$

when $u_+, u_- \in \overset{\circ}{H}_{(\sigma,\tau)}(\overline{\mathbb{R}}^n_\pm)$. For $\sigma = 0$ the statement follows immediately from the fact that

$$\|u\|^2_{(0,\tau)} = \int_{-\infty}^{\infty} \|u(\cdot, x_n)\|'_{(\tau)}{}^2 \, dx_n$$

where $\|u(\cdot, x_n)\|'_{(\tau)}$ denotes the norm in $H_{(\tau)}(\mathbb{R}^{n-1})$ of u as a function of x' for fixed x_n. If $0 < \sigma < 1$ we have instead by a vector valued version of Lemma 2.6.1 in my book that $\|u\|^2_{(0,\sigma+\tau)}$ is equivalent to

$$\|u\|^2_{(0,\sigma+\tau)} + \iint \|u(\cdot, x_n) - u(\cdot, y_n)\|'_{(\tau)}{}^2 |x_n - y_n|^{-1-2\sigma} \, dx_n dy_n.$$

To prove (2.3.2) it therefore suffices to show that if $0 < \sigma < \frac{1}{2}$ we have

$$\int \|u(x_n)\|^2 |x_n|^{-2\sigma} \, dx_n \le C \int\int \|u(\cdot, x_n) - u(\cdot, y_n)\|'_{(\tau)}{}^2 |x_n - y_n|^{-1-2\sigma} \, dx_n dy_n. \tag{2.3.3}$$

(We have dropped the subscripts on the norm for brevity.) This estimate is due to Aronszajn and Hardy; the following proof is used by Adams, Aronzajn and Smith. We may assume that u vsnishes for large $|x_n|$. Let t be a real number with $|t| > 1$. Then we have

$$\begin{aligned}\Big(\int \|u(x_n) - u(t^N x_n)\|^2 x_n^{-2\sigma} \, dx_n\Big)^{\frac{1}{2}} &\le \sum_{k=0}^{N-1} \Big(\int \|u(t^k x_n) - u(t^{k+1} x_n)\|^2 x_n^{-2\sigma} \, dx_n\Big)^{\frac{1}{2}} \\ &\le \Big(\int \|u(x_n) - u(t x_n)\|^2 x_n^{-2\sigma} \, dx_n\Big)^{\frac{1}{2}} \sum_{k=0}^{N-1} |t|^{k(\sigma - \frac{1}{2})}.\end{aligned}$$

Letting $N \to \infty$ we obtain

$$\int \|u(x_n)\|^2 x_n^{-2\sigma} \, dx_n \le \int \|u(x_n) - u(t x_n)\|^2 x_n^{-2\sigma} (1 - |t|^{-\sigma - \frac{1}{2}})^2 \, dx_n.$$

Now a change of variables gives

$$\int \|u(x_n) - u(y_n)\|^2 |x_n - y_n|^{-1-2\sigma} \, dx_n dy_n = 2 \iint_{|t|>1} \|u(x_n) - u(t x_n)\|^2 x_n^{-2\sigma} |1 - t|^{-1-2\sigma} \, dx_n dt.$$

This gives (2.3.3) and so completes the proof of the lemma.

If $u \in \overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ for some (σ, τ) with $|\sigma| < \frac{1}{2}$, the unique element in $H_{(\sigma,\tau)}(\mathbb{R}^n)$ which equals u in $\mathbb{R}^n_+$ and vanishes in $\mathbb{R}^n_-$ will be denoted by $e_0 u$. This linear operator is thus defined on the union of all the spaces $\overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ in question, which we denote by $\overline{H}_{(-\frac{1}{2}+0,-\infty)}(\mathbb{R}^n_+)$.

Lemma 2.3.3. *Let N be an integer ≥ 0 and σ a real number with $-\frac{1}{2} < \sigma < N + \frac{1}{2}$. If $u \in \overline{H}_{(\sigma,\tau)}(\mathbb{R}^n_+)$ and has compact support, it then follows that $x_n^N e_0 u \in \overset{\circ}{H}_{(\sigma,\tau)}(\overline{\mathbb{R}}^n_+)$.*

Proof. We know already the result when $-\frac{1}{2} < \sigma < \frac{1}{2}$. When $\sigma = \frac{1}{2}$ it follows by application of the norm in $\overset{\circ}{H}_{(\sigma,\tau)}$ given in the proof of Lemma 2.3.2, which shows that this norm is equivalent to

$$\int_0^\infty \|u\|'_{(\tau)}{}^2\, dx_n + \int_0^\infty \|u\|'_{(\tau+\frac{1}{2})}\, dx_n + \int\int \|u(\cdot, x_n) - u(\cdot, y_n)\|'_{(\tau)}{}^2 |x_n - y_n|^{-2}\, dx_n dy_n.$$

[Inserted $\cdot$, in $u(y_n)$.] An induction is now immediately obtained by noting that

$$D_j x_n^N e_0 u = x_n(x_n^{N-1} e_0 D_j u) + \delta_{jn} N x_n^{N-1} e_0 u \in \overset{\circ}{H}_{(\sigma-1,\tau)};$$

here we have used the fact that when $\sigma > \frac{1}{2}$ we have $D_n e_0 u = e_0 D_n u + u(0) \otimes \delta_{x_n}$, where $u(0)$ is the restriction of u to $x_n = 0$, and this is annihilated by x_n^N.

2.4. The homogeneous Dirichlet problem. We shall now again consider a C^∞ manifold M, a relatively compact subset Ω with C^∞ boundary $\partial\Omega$, and a classical pseudodifferential operator P in M. The operator P we assume to be *elliptic* in M, that is, in a local coordinate system where the symbol is $\sum p_j(x, \xi)$, the terms being homogeneous of degree m_j, we have $\operatorname{Re} m_j < \operatorname{Re} m_0$ when $j \neq 0$ and

$$p_0(x, \xi) \neq 0, 0 \neq \xi \in \mathbb{R}^n. \tag{2.4.1}$$

Further we assume that the hypothesis of Theorem 2.2.1 [parentheses removed] is fulfilled at least for $j = \alpha = \beta = 0$, that is, we assume that there is a number μ such that

$$p_0(x, -N) = e^{\pi i(m_0 - 2\mu)} p_0(x, N),\ x \in \partial\Omega, \tag{2.4.2}$$

where N denotes the interior normal of $\partial\Omega$ at x. If $n > 2$ the set $\{\xi; \xi \in \mathbb{R}^n, \xi \neq 0\}$ is simply connected, so for fixed x we can define $\log p(x, \xi)$ uniquely by fixing the value at one point. When $n = 2$, we impose this as a condition on p, called the root condition in analogy with the corresponding condition in the case of differential equations. Then we have

$$\log p_0(x, \xi + \tau N) - \log p_0(x, \tau N) = \log\big(p_0(x, \xi + \tau N)/p_0(x, \tau N)\big) \to 0,\ \tau \to \infty.$$

Hence

$$\log p_0(x, \xi + \tau N) - m_0 \log|\xi| \to a_\pm(x),\ \tau \to \pm\infty,$$

where $\exp a_\pm = p_0(x, \pm N)$. It follows from (2.4.2) that $e^{a_-} = e^{\pi i(m_0 - 2\mu) + a_+}$, that is, $\mu \equiv m_0/2 + (a_+ - a_-)/2\pi i \pmod 1$. We set

$$\mu_0 = m_0/2 + (a_+ - a_-)/2\pi i$$

noting that for reasons of continuity this number which is always congruent to μ must be a constant. (We assume here that $\partial\Omega$ is connected. In fact, it would make little difference if μ_0 takes different values on different components of $\partial\Omega$, and most of what follows goes through with light modifications when m_0 and μ_0 are both variable. See the second note by Vishik and Eskin and the definition of the spaces $H_{(m)}$ for variable m given in Chapter I. [Taken up in Section 5 of [1].]) Note that we may replace μ by μ_0 in (2.4.2). [Changed (2.4.1) to (2.4.2).]

We can now state the basic existence theorem for the Dirichlet problem, due to Vishik and Eskin. The spaces $\overset{\circ}{H}_{(s)}(\overline{\Omega})$ and $\overline{H}_{(s)}(\Omega)$ which occur in the statement are of course defined as in section 2.3.

Theorem 2.4.1. *Let P be elliptic of order m_0 satisfying the root condition if $n = 2$, and assume the number μ_0 introduced above to be constant on $\partial\Omega$. Then the mapping*

$$\overset{\circ}{H}_{(s)}(\overline{\Omega}) \ni u \mapsto Pu \in \overline{H}_{(s-\operatorname{Re} m_0)}(\Omega) \tag{2.4.4}$$

is a Fredholm operator if s is a real number with $|s - \operatorname{Re}\mu_0| < \frac{1}{2}$.

Proof. The mapping (2.4.4) is obviously continuous. The theorem will be proved if we show that it is a homomorphism with finite dimensional kernel. Indeed, the adjoint mapping is

$$\overset{\circ}{H}_{(\operatorname{Re} m_0 - s)}(\overline{\Omega}) \ni u \mapsto {}^tPu \in \overline{H}_{(-s)}(\Omega).$$

The operator ${}^tP$ satisfies the same conditions as P but with μ_0 replaced by $m_0 - \mu_0$. Since $|(\operatorname{Re} m_0 - s) - (\operatorname{Re} m_0 - \mu_0)| = |\operatorname{Re}\mu_0 - s| < \frac{1}{2}$, the adjoint must therefore also be a homomorphism with a finite dimensional kernel and the theorem will be proved.

By a standard argument it suffices then to prove the a priori estimate

$$\|u\|_{(s)} \le C(\|Pu\|^{\bullet}_{(s-\operatorname{Re} m_0)} + \|u\|_{(s')}),\ u \in \overset{\circ}{H}_{(s)}(\overline{\Omega}), \tag{2.4.5}$$

for some $s' < s$. If $s' \ge s - 1$ this estimate can be localized by use of a partition of unity. In the local situation we may drop all lower order terms from p if $s' \ge s + \operatorname{Re}(m_1 - m_0)$, and using Theorem 1.2.2 [possibly covered by localization arguments in [1]] we can reduce to an operator with constant coefficients. Clearly there is no trouble in the interior of $\overline{\Omega}$ so we have to look at the boundary only.

The situation is now the following: We have

$$Pu(x) = (2\pi)^{-n} \int p(\xi)\hat{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi,$$

where $p \in C^\infty$ and $p(\xi) = p_0(\xi)$ when $|\xi| > 1$, for example. Here p_0 is homogeneous of degree m_0, satisfies the root condition if $n = 2$, so a number μ_0 is defined by (2.4.3). We have to prove that if $|s - \operatorname{Re}\mu_0| < \frac{1}{2}$, then

$$\|u\|_{(s)} \le C(\|Pu\|^{\bullet}_{(s-\operatorname{Re} m_0)} + \|u\|_{s-1}),\ u \in C_0^\infty(\Omega), \tag{2.4.6}$$

where $\Omega = \{x; x_n > 0, |x| < 1\}$. (Note that the kernel of P is smooth and rapidly decreasing at ∞ so if $\varphi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 in a neighborhood of $\overline{\Omega}$ we can replace Pu by φPu in this estimate, incorporating the error committed in the term $\|u\|_{(s-1)}$.) The proof of

(2.4.6) is based on the Wiener-Hopf technique, so we shall now study the factorization of p in a factor analytic and $\neq 0$ when $\operatorname{Im}\xi_n < 0$ and another which has the same property when $\operatorname{Im}\xi_n > 0$.

Let $\chi \in C^\infty$ be equal to 1 when $|\xi| > 1$ and 0 when $|\xi| < \frac{1}{2}$. We may assume without restriction that $p(\xi) = \exp(\chi(\xi)\log p_0(\xi))$, so our aim is to represent $\chi(\xi)\log p_0(\xi)$ as a sum of functions analytic when $\operatorname{Im}\xi_n \gtrless 0$. First note that when $\xi_n \to \pm\infty$ for fixed ξ' we have with the notations in (2.4.3)

$$\chi(\xi)\log p_0(\xi) = m_0 \log|\xi_n| + a_\pm + O(1/\xi_n).$$

We eliminate the main terms by introducing the difference

$$\psi(\xi) = \chi(\xi)\log p_0(\xi) - \mu_0 \log(\xi_n - i\lambda) - (m_0 - \mu_0)\log(\xi_n + i\lambda) - a_+$$

where $\lambda = (1 + |\xi'|^2)^{\frac{1}{2}}$. Since $a_- = a_+ + i\pi(m_0 - 2\mu_0)$ we obtain $\psi(\xi) = O(1/\xi_n)$ when $\xi_n \to \infty$, uniformly for ξ' in a bounded set. Noting that for $|\xi'| > 1$

$$\psi(\xi) = \log p_0(\xi'/\lambda, \xi_n/\lambda) - \mu_0 \log(\xi_n/\lambda - i) - (m_0 - \mu_0)\log(\xi_n/\lambda + i) - a_+,$$

where $|\xi'|/\lambda$ lies between 1 and $1/\sqrt{2}$, it follows more precisely that

$$|\psi(\xi)| \leq C\lambda(|\xi_n| + \lambda)^{-1},\ |\partial\psi/\partial\xi_n)| \leq C\lambda(|\xi_n| + \lambda)^{-2}. \tag{2.4.7}$$

Now set for $\xi' \in \mathbb{R}^{n-1}$ and complex ξ_n, $(\xi', \xi_n) \neq 0$, [limits for $\operatorname{Im}\xi_n \searrow 0$ resp. $\operatorname{Im}\xi_n \nearrow 0$?]

$$\psi_+(\xi) = (2\pi i)^{-1} \int \psi(\xi', \tau)(\tau - \xi_n)^{-1}\, d\tau,\ \operatorname{Im}\xi_n \geq 0,$$
$$\psi_-(\xi) = -(2\pi i)^{-1} \int \psi(\xi', \tau)(\tau - \xi_n)^{-1}\, d\tau,\ \operatorname{Im}\xi_n \leq 0.$$

From (2.4.7) it follows that these functions are uniformly bounded, in view of the following lemma:

Lemma 2.4.2. *Let $f \in L^2(\mathbb{R})$, $f' \in L^2(\mathbb{R})$, and denote by $\tilde{f}$ the conjugate function of f. Then f and $\tilde{f}$ are both uniformly continuous and bounded by $\|f\|\|f'\|$, [should be $(\|f\|\|f'\|)^{\frac{1}{2}}$?], where the norms are L^2 norms. The Cauchy integrals $(2\pi i)^{-1}\int f(\tau)(\tau - z)^{-1}\, d\tau$ [plural? refers to the two integrals above?], whose boundary values are $(f \pm \tilde{f})/2$ therefore have the same bounds.*

Proof. See e.g. Beurling, Helsingfors congress 1938.

From (2.4.7) it now follows immediately that the functions ψ_+ and ψ_- are uniformly bounded and continuous. Their sum is of course equal to $\psi(\xi)$ when ξ is real. Setting

$$p_+(\xi) = (\xi_n + i\lambda)^{m_0 - \mu_0} \exp(a_+ + \psi_+(\xi)),\ p_-(\xi) = (\xi_n - i\lambda)^{\mu_0} \exp(\psi_-(\xi)),$$

we therefore have that $p(\xi) = p_+(\xi)p_-(\xi)$ for real arguments. Furthermore p_+ (p_-) is by the Paley-Wiener theorem the Fourier transform of a distribution with support in the half space $x_n \leq 0$ ($x_n \geq 0$). (At this point our normalization of the Fourier transform turns out to be somewhat unfortunate but we do not wish to change signs. Vishik and Eskin have a different normalization of the Fourier transform.)

We are now ready to prove (2.4.6). Write P_+ and P_- for the convolution operators corresponding to multiplication by $p_+(\xi)$ and $p_-(\xi)$ of the Fourier transforms. Write $Pu = f$, where u is chosen as in (2.4.6), and choose F equal to f in $\mathbb{R}^n_+$ with $\|F\|_{(s-\operatorname{Re} m_0)} = \|f\|^\bullet_{(s-\operatorname{Re} m_0)}$. [Equality or similarity?] From the equation

$$P_+P_-u = Pu = f$$

valid in the whole space, we obtain

$$P_-u = (P_+)^{-1}f.$$

Now P_-u has its support in $\overline{\mathbb{R}}^n_+$ and belongs to C^∞ in the whole space. On $\mathbb{R}^n_+$ we have $(P_+)^{-1}f = (P_+)^{-1}F$ since the support of P_+ lies in the lower half space. [Means probably that P_+ preserves support in $x_n \le 0$.] Hence, with the notation introduced at the end of section 2.3 [e_0 is an extension by zero], we have

$$P_-u = e_0(P_+)^{-1}F.$$

Since $|s - \operatorname{Re}\mu_0| < \frac{1}{2}$ we can apply Lemma 2.3.2 and obtain

$$\|P_-u\|_{(s-\operatorname{Re}\mu_0)} \le C\|P_+^{-1}F\|^\bullet_{(s-\operatorname{Re}\mu_0)} \le C'\|F\|^\bullet_{(s-\operatorname{Re} m_0)},$$

which immediately implies (2.4.6), since P_- is [elliptic] of order μ_0.

Our next task is to examine the smoothness properties of a solution u of the Dirichlet problem when Pu lies in a smaller space that $\overline{H}_{(s-\operatorname{Re} m_0)}$. Interior regularity is of course covered by the results of Chapter I, so we can confine our attention to the boundary. Our first step is then to obtain results on "tangential regularity" which follow by classical arguments due to Nirenberg.

Let Ω be the half ball $\{x; x \in \mathbb{R}^n, |x| < 1, x_n > 0\}$. The unit ball we denote by $\widetilde{\Omega}$. By $\mathring{H}^{\mathrm{loc}}_{(s)}(\Omega')$ and $\overline{H}^{\mathrm{loc}}_{(s-\operatorname{Re} m_0)}(\Omega)$ we denote the distributions which multiplied with functions in $C_0^\infty(\widetilde{\Omega})$ give elements in the analogous spaces in $\mathbb{R}^n_+$. Here $\Omega' = \{x; x \in \mathbb{R}^n, |x| < 1, x_n \ge 0\}$.

Theorem 2.4.3. *Let P satisfy the hypotheses of Theorem* 2.4.1. *If $|s - \operatorname{Re}\mu_0| < \frac{1}{2}$ and t_0, t_1 are real numbers then*

$$u \in \mathring{H}^{\mathrm{loc}}_{(s,t_0)}(\Omega'), Pu \in \overline{H}^{\mathrm{loc}}_{(s-\operatorname{Re} m_0,t_1)}(\Omega) \tag{2.4.8}$$

implies that

$$u \in \mathring{H}^{\mathrm{loc}}_{(s,t_1)}(\Omega'), \tag{2.4.9}$$

Proof. It is no restriction to assume that $t_1 - t_0$ is a positive integer, for we may always decrease t_0. It suffices to prove the theorem when $t_1 - t_0 = 1$. Now we claim that for every compact subset K of Ω, and every real number t there is a constant C such that

$$\|u\|_{(s,t)} \le C(\|Pu\|^\bullet_{(s-\operatorname{Re} m_0,t)} + \|u\|_{(s-1,t)}) \tag{2.4.5$'$}$$

for all $u \in C_0^\infty(K)$, hence for all $u \in \overset{\circ}{H}_{(s,t)}$ with support in K. In fact, this follows from the proof of (2.4.5) or else by applying (2.4.5) to $|D'|^t u$, cut off conveniently. We may replace the last term in (2.4.5$'$) by the larger quantity $\|u\|_{(s,t-1)}$. Now assume that (2.4.8) is fulfilled with $t_0 = t$, $t_1 = t+1$. Then φu satisfies the same hypothesis if $\varphi \in C_0^\infty(\widetilde{\Omega})$. (Note that a repetition of the simple case of the proof of Theorem 1.2.1 shows that P is continuous from $H_{(s,t)}$ to $H_{(s-\operatorname{Re} m_0,t)}$ for all s,t.) Let therefore u have compact support in Ω'. Denote by u_h the convolution of u by the Dirac measure at $(h_1,\dots,h_{n-1},0) = h$, that is, u_h is a tangential translation of u. Let P_h be the analogous translation of P. Then

$$P(u_h - u)/|h| = (f_h - f)/|h| + (P - P_h)/|h|\, u$$

where $f = Pu$. Since

$$\|(f - f_h)/|h|\|^\bullet_{(s,t)} \le \|f\|^\bullet_{(s,t+1)},$$

and since $(P - P_h)/|h|$ is continuous from $H_{(s,t)}$ to $H_{(s-\operatorname{Re} m_0,t)}$ uniformly when $h \to 0$, we conclude using (2.4.5)$'$ that $\|(u_h - u)/|h|\|_{(s,t)}$ is bounded when $h \to 0$. Hence $\|D_j u\|_{(s,t)} < \infty$ when $j < n$, which proves that $u \in \overset{\circ}{H}_{(s,t+1)}$. The proof is complete.

To study the regularity properties of u in non-tangential directions is much more delicate and requires the full force of the conditions (2.2.1). This will be done in the next section. Theorems 2.4.1 and 2.4.3 contain all the information we have to extract from the ellipticity, so we shall consider more general operators again in the next paragraph.

2.5. Completion of the spaces $\mathcal{E}_\mu$, and "partial hypoellipticity at the boundary". Our purpose is to introduce a topology in $\mathcal{E}_\mu$ which is analogous to that in $\overline{H}_{(s)}(\Omega)$. ($\Omega$ still denotes a relatively compact subset with smooth boundary of a manifold M.) As semi-norms in Ω we therefore introduce all semi-norms

$$\mathcal{E}_\mu(\overline{\Omega}) \ni u \to \|Pu\|^\bullet_{(s-m_0)} \tag{2.5.1}$$

where P is of type[3] μ and order m_0. [In (2.5.1), $s - m_0$ should be replaced by $s - \operatorname{Re} m_0$.] The norm $\|\quad\|^\bullet_{(t)}$ is that of $\overline{H}_{(t)}(\Omega)$, defined by extension to all of M.

Lemma 2.5.1. *The topology on $\mathcal{E}_\mu$ is stronger than that of $\mathcal{E}'(\overline{\Omega})$. For every $\varphi \in C^\infty(M)$, the mapping $u \to \varphi u$ is continuous from $\mathcal{E}_\mu$ to $\mathcal{E}_\mu$.*

Proof. If φ, $\psi \in C_0^\infty(M)$, we can take $Pu = \psi u(\varphi)$, for this is an operator with symbol 0, hence of type μ. This proves that $u \to u(\varphi)$ is a seminorm of type (2.5.1), hence the topology is stronger than that of $\mathcal{E}'(\overline{\Omega})$. For every P of type μ and order m_0 the operator $u \to P\varphi u$ is also of type μ and order m_0. Hence the second assertion follows.

The completion $H_{\mu(s)}$ of $\mathcal{E}_\mu$ in the toplogy just defined is therefore a subspace of $\mathcal{E}'(\overline{\Omega})$, and it is determined by local properties. When studying its elements more closely we may therefore restrict ourselves to a coordinate patch. For interior coordinate patches we get of course the space $H_{(s)}$ so we only have to study what happens at the boundary.

[3] By this we mean that (2.2.1) is fulfilled.

Theorem 2.5.2. *Let $s - \operatorname{Re}\mu > -\frac{1}{2}$. Then an element $u \in \mathcal{E}'(\overline{\mathbb{R}}^n_+)$ is in $H_{\mu(s)}$ if and only if*

$$\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u} \in \overset{\circ}{H}_{(-\frac{1}{2})}, \quad \|\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u}\|^{\bullet}_{(s-\operatorname{Re}\mu)} < \infty. \tag{2.5.2}$$

Here $\lambda = (1 + |\xi'|^2)^{\frac{1}{2}}$. When the support of u belongs to a fixed compact set the last expression in (2.5.2) *defines the topology in $H_{\mu(s)}$. When $s - \operatorname{Re}\mu \le \frac{1}{2}$ also [$< \frac{1}{2}$?], then $C_0^\infty(\mathbb{R}^n_+)$ is a dense subset.*

[Better say $\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u} \in \overset{\circ}{H}_{(-\frac{1}{2}+0)}$, since that is shown in the proof and is more useful.] From the last statement we conclude immediately that replacing μ by $\mu + j$ where j is a positive integer gives the same space if $-\frac{1}{2} < s - \operatorname{Re}\mu \le \frac{1}{2}$. Hence

Corollary 2.5.3. *If $s - \operatorname{Re}\mu \le \frac{1}{2}$, then $H_{\mu(s)} = H_{\mu+j(s)}$ for every integer $j \ge 0$.*

In view of the corollary the theorem therefore describes $H_{\mu(s)}$ for all s, μ.

Proof of Theorem 2.5.2. To prove the necessity of (2.5.2) we note that $(1-\Delta)^\mu$ is of type μ, so we must have $(1-\Delta)^\mu u \in \overline{H}_{(s-2\operatorname{Re}\mu)}$. We can factor $(1+|\xi|^2)^\mu$ into two factors $(\xi_n - i\lambda)^\mu$ and $(\xi_n + i\lambda)^\mu$ which are analytic and $\neq 0$ in the lower and upper half spaces in $\mathbb{R}^{n-1} \times \mathbb{C}$ respectively, hence Fourier transforms of distributions with support in $\overline{\mathbb{R}}^n_+$ and $\overline{\mathbb{R}}^n_-$. Now the convolution of a function f with a distribution with support in $\overline{\mathbb{R}}^n_-$ is in $\mathbb{R}^n_+$ independent of the values of f on $\mathbb{R}^n_-$, so we can factor out the distribution with Fourier transform $(\xi_n + i\lambda)^\mu$ and obtain

$$\mathcal{F}^{-1}((\xi_n - i\lambda)^\mu \hat{u}) \in \overline{H}_{(s-\operatorname{Re}\mu)}$$

the embedding being of course topological. When $u \in \mathcal{E}_\mu$ it follows from (2.1.1) that we have an element in $\overset{\circ}{H}_{(\sigma)}$ for any $\sigma < \frac{1}{2}$. If $-\frac{1}{2} < \sigma < \min(\frac{1}{2}, s - \operatorname{Re}\mu)$ it follows therefore that the same is true for the completion $H_{\mu(s)}$. Hence the necessity of (2.5.2). The proof of the sufficiency on the other hand requires a few preparations.

Lemma 2.5.4. *If u satisfies* (2.5.2) *and N is an integer ≥ 0, it follows that $x_n^N u \in \overset{\circ}{H}_{(\sigma, s-\sigma)}$ provided that $\sigma \le s$ and $\sigma < \operatorname{Re}\mu + N + \frac{1}{2}$.*

Proof. Define v so that $\hat{v} = (\xi_n - i\lambda)^\mu \hat{u}$. Then $v \in \overline{H}_{(s-\operatorname{Re}\mu)}$ and $v \in \overset{\circ}{H}_{(-\frac{1}{2})}$ by hypothesis. From Lemma 2.3.3 it follows therefore that

$$x_n^j v \in \overset{\circ}{H}_{(\sigma, s-\sigma-\operatorname{Re}\mu)} \text{ if } \sigma \le s - \operatorname{Re}\mu \text{ and } \sigma < j + \tfrac{1}{2}.$$

Now $\hat{u} = (\xi_n - i\lambda)^{-\mu}\hat{v}$, so we obtain

$$x_n^N u = (-1)^N \sum_0^N \binom{N}{k} \mathcal{F}^{-1}\big((D_n^k(\xi_n - i\lambda)^{-\mu}) D_n^{N-k}\hat{v}\big).$$

The kth term is in $\overset{\circ}{H}_{(\sigma, s-\sigma)}$ if $x_n^{N-k} v \in \overset{\circ}{H}_{(\sigma-k-\operatorname{Re}\mu, s-\sigma)}$ and we have seen that this is true if $\sigma - k - \operatorname{Re}\mu \le s - \operatorname{Re}\mu$, $\sigma - k - \operatorname{Re}\mu < N - k + \frac{1}{2}$, which follows by the assumptions since $k \ge 0$.

If $N > s - \operatorname{Re}\mu - \frac{1}{2}$ we can take $\sigma = s$ and conclude that $x_n^N u \in \overset{\circ}{H}_{(s)}$, and that $x_n^{N-j} u \in \overset{\circ}{H}_{(s-j)}$ for $0 \le j \le N$. If P is a pseudo-differential operator of order m, we can conclude that $x_n^N Pu \in H_{s-\operatorname{Re} m}$. In fact, using Leibniz rule for the adjoints we can write

$$x_n^N Pu = \sum P_j x_n^{N-j} u$$

where P_j is of order $m-j$. The norm of Pu in $H_{s-\operatorname{Re} m}$ can of course be estimated by the semi-norm occurring in (2.5.2).

To study a general P of type μ we can take a Taylor expansion of order N of the symbol; since the error term contains a factor x_n^N it has already been discussed. Each term in the Taylor expansion contains a factor which is a power of x_n and otherwise an operator which is independent of x_n. Thus it only remains to consider the case when P is of type μ and the symbol is independent of x_n.

Since $(\xi_n + i\lambda)^\mu$ is the Fourier transform of a distribution with support on the negative axis we have, reversing the first part of the proof of Theorem 2.5.2:

$$\|(1-\Delta)^\mu u\|^\bullet_{(s-\operatorname{Re} 2\mu)} \le \|(\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u}\|^\bullet_{(s-\operatorname{Re}\mu)} = N(u),$$

where the last inequality is a definition we shall use until the completion of the proof of Theorem 2.5.2. [One should omit the parenthesis before $\mathcal{F}$ and replace "inequality" by "equality".] If P is a pseudo-differential operator of order m_0 which is a linear combination of convolutions with distributions having support in $\{x; x_n \le 0\}$, we conclude that

$$\|P(1-\Delta)^\mu u\|^\bullet_{(s-\operatorname{Re}(m_0+2\mu))} \le CN(u). \tag{2.5.3}$$

Furthermore when $\sigma < \frac{1}{2}$ and $\sigma \le s$ we have $u \in \overset{\circ}{H}_{(\sigma, s-\sigma)}$, and it follows from Lemma 2.5.4 that

$$\|Pu\|_{(s-\operatorname{Re} m_0)} \le CN(u) \tag{2.5.4}$$

if P is now any operator of order m_0 which has order $\le \operatorname{Re} m_0 - s + \sigma$ [replaced m_0 by $\operatorname{Re} m_0$] in the ξ_n direction, that is,

$$|p(x,\xi)| \le C(1+|\xi|)^{\operatorname{Re} m_0 - s + \sigma}(1+|\xi'|)^{s-\sigma}. \tag{2.5.5}$$

(This is a simple case of the proof of Theorem 1.2.1.) Conditions on some derivatives $p^{(\alpha)}_{(\beta)}$ seem needed also.]

We shall combine these two types of results to complete the proof of the theorem. However, first we have to see that there exist "sufficiently many" operators which can be used in (2.5.3).

Lemma 2.5.5. *For any μ one can find a homogeneous function $p \in C^\infty(\mathbb{R}^n - \{0\})$ which is homogeneous of degree μ, $\ne 0$ at any given $\xi_0 \ne 0$, and can be extended analytically to $\{\xi; \xi' \in \mathbb{R}^{n-1}, \operatorname{Im}\xi_n \ge 0, \xi \ne 0\}$ as a C^∞ homogeneous function.*

Proof. Choose $g \in C_0^\infty(\mathbb{R}^n_+)$ and set when $\xi' \in \mathbb{R}^{n-1}$, $\operatorname{Im}\xi_n > 0$

$$p(\xi) = \int \langle x, \xi\rangle^\mu g(x)\,dx. \tag{2.5.6}$$

(We define z^μ in $\operatorname{Im} z \geq 0$ so that $1^\mu = 1$.) Then p is obviously homogeneous and C^∞ when $\operatorname{Im}\xi_n > 0$. To prove that p assumes boundary values when $\operatorname{Im}\xi_n = 0$, in the C^∞ topology, we integrate by parts, using the fact that

$$\langle x, \xi\rangle^\mu(\mu+1) = \sum \bar{\xi}_j |\xi|^{-2}\partial/\partial x_j \langle x, \xi\rangle^{\mu+1}.$$

This gives if $\mu \neq -1$,

$$p(\xi) = -(\mu+1)^{-1}\int \langle x, \xi\rangle^{\mu+1}|\xi|^{-2}\sum \bar{\xi}_j \partial g/\partial x_j\,dx.$$

If μ is not a negative integer we conclude after repeating this integration by parts k times that p is C^k for $\xi \neq 0$ in the domain in question provided that $\operatorname{Re}\mu + k - \nu > -1$. [$C^k$ should perhaps be C^ν.] Hence p is in C^∞. If μ is a negative integer we can also integrate by parts in the same way until we reach an integral involving $\langle x, \xi\rangle^{-1}$. The next time we then have to allow $\log\langle x, \xi\rangle$ as a factor but apart from the occurrence of a logarithmic factor nothing is changed in the continued integration by parts,

Remark. It does not seem clear that one can choose p elliptic and C^∞. [This has been clarified more recently.] If one drops the latter condition one can of course take $(\xi_n + i|\xi'|)^\mu$. The following arguments could be simplified if we had an elliptic p with the properties in the lemma.

End of proof of Theorem 2.5.2. Let now P be any operator of type μ with symbol independent of x_n. Denote the order by m_0 and form

$$Q = P(1-\Delta)^{-\mu}$$

which is a pseudo-differential operator of type 0 and order $m_0 - 2\mu$. In view of (2.2.1) and the fact that we have reduced consideration to the case when the symbol in independent of x_n, this means that if $\sum q_k$ is the symbol of Q, then $q_k^{(\alpha)}(x, 0, \dots, 0, \xi_n)$ is an analytic function of ξ_n when $\operatorname{Im}\xi_n \geq 0$, $\xi_n \neq 0$, for all k and α. By repeated application of Lemma 2.5.5 it follows that there exists an operator R of order $m_0 - 2\mu$ whose symbol is a finite linear combination of homogeneous functions with the properties listed in the lemma, so that $Q - R$ is of order $< \operatorname{Re} m_0 - 2\mu + s = \sigma$ with respect to ξ_n, where σ is the number in (2.5.5). Now

$$P = R(1-\Delta)^\mu + (Q-R)(1-\Delta)^\mu.$$

To the first term on the right we can apply (2.5.3) and to the second we can apply (2.5.4), and conclude that $Pu \in \overline{H}_{(s-\operatorname{Re} m_0)}$. [$m_0$ replaced by $\operatorname{Re} m_0$.]

Summing up, we have now proved that if u satisfies (2.5.2) and P is any operator of type μ and order m_0, then $Pu \in \overline{H}_{(s-\operatorname{Re} m_0)}$. Moreover, the proof (or the closed graph theorem) gives an estimate

$$\|Pu\|^\bullet_{(s-\operatorname{Re} m_0)} \leq C\|\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u}\|^\bullet_{(s-\operatorname{Re}\mu)},$$

at least when P is compactly supported. [Dots added.] Thus it only remains to show that $\mathcal{E}_\mu \cap \mathcal{E}'(\overline{\mathbb{R}}^n_+)$ is dense in the set of all u satisfying (2.5.2). We first take a sequence $v_j \in \overline{C}_0^\infty(\mathbb{R}^n_+)$ approximating $\mathcal{F}^{-1}(\xi_n - i\lambda)^\mu \hat{u}$ in the norm $\| \, \|^\bullet_{(s-\operatorname{Re}\mu)}$, and also in the topology of $\mathcal{S}$ outside a neighborhood of $\operatorname{supp} u$, which is possible since the function to approximate agrees with a function in $\mathcal{S}$ there. When $\sigma - \operatorname{Re}\mu \leq \frac{1}{2}$ we can take $v_j \in C_0^\infty(\mathbb{R}^n_+)$. Define $v_j = 0$ in $\mathbb{R}^n_-$. Set $u_j = \mathcal{F}^{-1}((\xi_n - i\lambda)^{-\mu}\hat{v}_j)$. [Replaced v_j by $\hat{v}_j$.] This is an element of $\mathcal{E}_\mu$ in view of Lemma 2.1.1 (the Fourier transform is the product of that of v_j and $(\xi_n - i\lambda)^{-\mu}$, and the behavior of the Fourier transform of v_j is described by Lemma 2.1.1 with $\mu = 0$). Then $u_j \to u$ in the norm in (2.5.2) and also in the topology of $\mathcal{S}$ outside a neighborhood of $\operatorname{supp} u$. Hence we can cut off u_j there without disturbing the convergence in order to obtain an approximating sequence with compact supports.

The following is also an immediate consequence of Theorem 2.5.2.

Corollary 2.5.6. *The intersection of $H_{\mu(s)}$ for all s is equal to $\mathcal{E}_\mu$.*

Closely related to the proof of the sufficiency of (2.5.2) is the following result which combined with Theorem 2.4.3 will complete our study of the regularity properties of solutions of the Dirichlet problem.

Theorem 2.5.7. *Assume that, with the notations of Theorem* 2.4.3, *$u \in \overset{\circ}{H}{}^{\mathrm{loc}}_{(\sigma,s-\sigma)}(\Omega')$ for some $\sigma \geq \operatorname{Re}\mu - \frac{1}{2}$, and that $Pu \in \overline{H}{}^{\mathrm{loc}}_{(s-m_0)}(\Omega)$ for some operator P of type μ and order m_0 such that the plane $x_n = 0$ is non-characteristic, that is, $p_0(x, 0, \ldots, 0, \xi_n) \neq 0$ when $0 \neq \xi_n$, $x_n = 0$, $x \in \Omega'$. Then $u \in H_{\mu(s)}$.*

The proof of this theorem will follow the second part of the proof of Theorem 2.5.2, but first we have to find a substitute for Lemma 2.5.4.

Lemma 2.5.8. *Let P be compactly supported, of order m_0, and assume that the plane $x_n = 0$ is non-characteristic. If $u \in \overset{\circ}{H}{}^{\mathrm{loc}}_{(\sigma,\tau)}(\Omega')$ and $x_n Pu \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+1-\operatorname{Re} m_0,\tau-1)}(\Omega)$ [m_0 changed to* $\operatorname{Re} m_0$*] it follows then that $x_n u \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+1,\tau-1)}(\Omega')$.*

Proof. The proof can immediately be reduced to the case when u has compact support in Ω', which we assume from now on. If Q is an operator of order $-m_0$ as constructed in Lemma 2.5.5, for which the plane $x_n = 0$ is noncharacteristic, we obtain $Qx_nPu \in \overline{H}_{(\sigma+1,\tau-1)}$, and since $[Q, x_n]P$ is of order -1 it follows that $x_nQPu \in \overline{H}_{(\sigma+1,\tau-1)}$. This reduces the proof to the case where P is of order 0. Since we may add to P any operator of order 0 and order -1 with respect to ξ_n we may reduce P to a finite sum of convolution operators multiplied to the left by C^∞ functions. [Builds on unavailable statement.] This we do in order to have no difficulties in operating with convolution operators which do not quite have the regularity asked for in Chapter I. First an application of $(1 + |D'|^2)^{\tau-1}$ [should be its square root] reduces the proof to the case $\tau = 1$. Secondly, setting $(D_n - i\sqrt{D'^2+1}\,)^\sigma u = v$ we have $v \in \overset{\circ}{H}_{(0,1)}$ and

$$(D_n + i\sqrt{D'^2+1}\,)^{+\sigma} x_n P (D_n - i\sqrt{D'^2+1}\,)^{-\sigma} v \in \overline{H}_{(1)}(\mathbb{R}^n_+),$$

from which we conclude [how?] that $Px_nv \in \overline{H}_{(1,0)}(\mathbb{R}^n_+)$. [Changed $\overline{\mathbb{R}}^n_+$ to $\mathbb{R}^n_+$.]

Now for any $w \in L^2$ with support in the upper half space and any P of order 0 we have $Px_nw \in H_{(1,0)}(\mathbb{R}^n_-)$. [overline seems missing] Indeed, Px_nw is [in $\mathbb{R}^n_-$] given by an integral operator with kernel homogeneous of degree $-n+1$, which can be estimated by $|x-y|^{-n}y_n \le |x-y|^{1-n}$ when x is in the lower and y in the upper half space. The first order derivatives are bounded by $C|x-y|^{-n}$ [illigible, guessed the power $-n$]. Now one can find a kernel k which is homogeneous of degree $-n$ such that the mean value of k over the unit sphere is 0 and $k(x) = |x|^{-n}$ in the lower half space. [It seems to be used that the convolution kernel of P for $x_n < 0$, $y_n > 0$, depends only on $z_n = x_n - y_n < 0$, hence can be chosen conveniently for $z_n > 0$.] From the original Calderon-Zygmund estimates we then obtain the required estimate. Hence $Px_nv \in \overline{H}_{(1,0)}$ in the upper as well as the lower half space. It remains to show that the boundary values from each half space are identical, that is, that D_nPx_nv is in L^2; a priori we only know that

$$D_nPx_nv = w + h \otimes \delta(x_n)$$

where $w \in L^2$, $h \in H_{(\frac{1}{2})}$. Thus take a function φ with compact support and set $\varphi_\varepsilon(x) = \varphi(x', x_n/\varepsilon)$. Then

$$\int h\varphi(x',0)\,dx' = -\int w\varphi_\varepsilon\,dx - \int vx_n\,{}^tPD_n\varphi_\varepsilon\,dx.$$

Since $x_n\,{}^tPD_n\varphi_\varepsilon = {}^tPx_nD_n\varphi_\varepsilon + [x_n, {}^tP]D_n\varphi_\varepsilon$, the L^2 norm of this quantity is $O(\varepsilon^{\frac{1}{2}})$. Since $v \in L^2$ it follows when $\varepsilon \to 0$ that $\int h\varphi(x',0)\,dx' = 0$, hence $h = 0$. Having proved now that $Px_nv \in H_{(1,0)}$, we apply $\overline{P}$ and conclude that $\overline{P}Px_nv \in H_{(1,0)}$ and adding to this a multiple of $|D'|^2/|D|^2x_nv \in H_{(1,0)}$, we conclude that some elliptic operator of order 0 applied to x_nv gives an element in $H_{(1,0)}$. Hence $x_nv \in H_{(1,0)}$, and this implies the statement.

A more general version of the lemma is the following

Theorem 2.5.9. *Let $u \in \overset{\circ}{H}{}^{\mathrm{loc}}_{(\sigma,\tau)}(\Omega')$, $x_n^NPu = f \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N-\operatorname{Re}m,\tau-N)}(\Omega)$, where P is compactly supported [of order m] and the plane $x_n = 0$ is non-characteristic for P. Then it follows that $x_n^Nu \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N,\tau-N)}$; if q is of order μ' then $x_n^Nqu \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N-\operatorname{Re}\mu',\tau-N)}$.*

Proof. When $N = 1$ the theorem follows from Lemma 2.5.8 since $x_nqu = qx_nu + [q, x_n]u$ where $[q, x_n]$ is of order $\mu' - 1$. Assume the theorem already proved for integers smaller than N, $N > 1$. Then

$$x_n^{N-1}P(x_nu) = x_n^NPu + x_n^{N-1}[P, x_n]u \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N-\operatorname{Re}m,\tau-N)}$$

since $[P, x_n] = q$ is of order $m-1$. Furthermore, $x_nu \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+1,\tau-1)}$. Hence $x_n^Nu \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N,\tau-N)}$ and $x_n^Nqu = x_n^{N-1}qx_nu + x_n^{N-1}[x_n, q]u \in \overline{H}{}^{\mathrm{loc}}_{(\sigma+N-\operatorname{Re}\mu',\tau-N)}$ since $[x_n, q]$ is of order $\mu' - 1$.

Proof of Theorem 2.5.7. Choose N so large that $\sigma + N \ge s$. Then it follows from Theorem 2.5.9 that $x_n^NQu \in \overline{H}{}^{\mathrm{loc}}_{(s-\operatorname{Re}m')}$ if Q is of order m'. Hence in proving that $Qu \in \overline{H}{}^{\mathrm{loc}}_{(s-\operatorname{Re}m')}$ for Q of order m' and type μ it is enough to show that this is true for some Q' which agrees with Q of order N when $x_n = 0$. Now by a formal computation of symbols as in the proof of Theorem 2.5.2 we can find R with symbol analytic when $\operatorname{Im}\xi_n > 0$, so that RP differs from $(1-\Delta)^\mu$ only by terms which vanish to arbitrarily high order when $x_n = 0$ or when $\xi' = 0$. The proof is then completed as before.

We are now ready to prove a theorem on the regularity of solutions of the Dirichlet problem.

Theorem 2.5.10. *Let P satisfy the hypotheses of Theorem* 2.4.1. *[It must be assumed also that P is of type μ_0.] If $u \in \overset{\circ}{H}_{(\sigma)}(\overline{\Omega})$ for some $\sigma > \operatorname{Re}\mu_0 - \frac{1}{2}$ and $Pu \in \overline{H}_{(s-\operatorname{Re}m_0)}(\Omega)$, where $s > \operatorname{Re}\mu_0 - \frac{1}{2}$, it follows that $u \in H_{\mu_0(s)}(\overline{\Omega})$. In particular, if $Pu \in \overline{C}^\infty$ it follows that $u \in \mathcal{E}_{\mu_0}$; the mapping*

$$H_{\mu_0(s)}(\overline{\Omega}) \ni u \to Pu \in \overline{H}_{(s-\operatorname{Re}m_0)}(\Omega)$$

is a Fredholm operator for every $s > \operatorname{Re}\mu_0 - \frac{1}{2}$.

Proof. This follows immediately from Theorem 2.4.1, Theorem 2.4.3 and Theorem 2.5.7, if we also recall Corollary 2.5.6.

2.6. The inhomogeneous Dirichlet problem. Let P satisfy the hypotheses of Theorem 2.4.1 and choose a number $\mu < \mu_0$ which differs from μ_0 by an integer.

If we introduce the natural mapping

$$\gamma_\mu \colon \mathcal{E}_\mu \to \mathcal{E}_\mu/\mathcal{E}_{\mu_0},$$

the inhomogeneous Dirichlet problem can be stated as the study of the mapping

$$u \to \{Pu, \gamma_\mu u\} \in \overline{C}^\infty \oplus (\mathcal{E}_\mu/\mathcal{E}_{\mu_0}). \tag{2.6.1}$$

It follows immediately from Theorem 2.5.10 that this mapping has finite index. However we must discuss the dependence of the solution on the boundary data rather closely in order to be able to handle more general boundary problems in the next section.

The first step is to represent $\mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ as the space of sections of a trivial bundle and introduce norms in it. To do so we first choose a Riemannian metric in M and then a C^∞ function d in $\overline{\Omega}$ which is equal to the distance from $\partial\Omega$ sufficiently close to the boundary and is positive and C^∞ throughout Ω. Set $I^\mu(x) = d(x)^\mu/\Gamma(\mu+1)$ in $\overline{\Omega}$ and $I^\mu = 0$ in $\complement\Omega$ when $\operatorname{Re}\mu > 0$. This definition can be uniquely extended modulo $C_0^\infty(\Omega)$ to arbitrary values of μ so that $D_n I^\mu = I^{\mu-1}$, where D_n denotes differentiation along the geodesics perpendicular to $\partial\Omega$ sufficiently close to $\partial\Omega$ and is defined as a C^∞ function elsewhere. By our definition of $\mathcal{E}_\mu$ it follows easily that every class in $\mathcal{E}_\mu/\mathcal{E}_{\mu+1}$ contains an element of the form $I^\mu(x)f$ where $f \in \overline{C}^\infty(\Omega)$, and that such elements are congruent to 0 if and only if $f = 0$ on the boundary. By repeated application of this fact we conclude that any element $u \in \mathcal{E}_\mu$ can be written

$$u = u_0 I^\mu + u_1 I^{\mu+1} + \cdots + u_{\mu_0-\mu-1} I^{\mu_0-1} + v, \tag{2.6.2}$$

where $u_j \in \overline{C}^\infty(\Omega)$ are constant close to $\partial\Omega$ on normal geodesics, and $v \in \mathcal{E}_{\mu_0}$. The boundary values of u_j are uniquely determined by u, and it is natural to write

$$D_n^{j+\mu} u = u_j|_{\partial\Omega}.$$

The mapping
$$u \to \{D_n^{j+\mu} u\}_{j=0}^{\mu_0-\mu-1}$$
has null space $\mathcal{E}_{\mu_0}$ and identifies $\mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ with $C^\infty(\partial\Omega)^{\mu_0-\mu}$. The identification depends of course on the choice of the Riemannian structure but we shall keep it fixed in all that follows. (It would of course be more natural to regard $\mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ as the space of sections of a bundle on $\partial\Omega$ with fiber dimension $\mu_0 - \mu$.) We can now think of γ_μ as a mapping of $\mathcal{E}_\mu$ into $C^\infty(\partial\Omega)^{\mu_0-\mu}$.

We shall now introduce in $\mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ the quotient of the topology of $H_{\mu(s)}$. In view of Corollary 2.5.3 we must then require that $s - \mathrm{Re}(\mu_0 - 1) > \frac{1}{2}$, for otherwise $\mathcal{E}_{\mu_0-1}$ is dense in $\mathcal{E}_{\mu_0}$ in that topology and the quotient topology would not be Hausdorff. Thus we assume that $s - \mathrm{Re}\,\mu_0 > -\frac{1}{2}$; note that this is the same condition as in Theorem 2.5.10. When discussing the quotient topology it is by Lemma 2.5.1 sufficient to consider sections with support in a local coordinate patch.

Thus let $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'(K)$ where K is a compact set, and let $d(x) = x_n$. Writing u in the form (2.6.2) we have for large ξ_n (cf. (2.1.1$'$)) and any N
$$\hat{u}(\xi) = \sum_{j=0}^{\mu_0-\mu-1} \hat{u}_j(\xi')\xi_n^{-\mu-j-1} + O((1+|\xi'|)^{-N}\xi_n^{-\mu_0-1}).$$
[Here ξ_n^{-a} should be $(\xi_n^-)^{-a}$. Moreover, exponential factors are missing, since $\mathcal{F}I^\mu(x_n) = e^{i\pi(\mu+1)/2}(\xi_n^-)^{-\mu-1}$ by [H83], Ex. 7.1.17.] Here u_j denotes $D_n^{j+\mu}u$. Hence
$$\hat{u}(\xi)(\xi_n - i\lambda)^\mu = \sum_{j=0}^{\mu_0-\mu-1}\sum_{k=0}^{j} c_{jk}\hat{u}_k(\xi')\lambda^{j-k}\xi_n^{-j-1} + O((1+|\xi'|)^{-N}\xi_n^{-(\mu_0-\mu)-1}).$$
where c_{kj} are constants, $c_{jj} = 1$. [Changed $(\xi_n + i\lambda)^\mu$ to $(\xi_n - i\lambda)^\mu$, as done by LH in the next expression. Moreover, replaced λ^{k-j} by λ^{j-k}, also in the next statements, after checking calculations.]

This means that the boundary values of
$$D_n^j\mathcal{F}^{-1}(\hat{u}(\xi)(\xi_n - i\lambda)^\mu)$$
are equal to
$$\mathcal{F}^{-1}\sum_{k=0}^{j} c_{jk}\hat{u}_k(\xi')\lambda^{j-k}.$$
Since $s - \mathrm{Re}\,\mu > \mu_0 - \mu - 1 + \frac{1}{2}$ by assumption, it follows from Theorem 2.5.2 here and Corollary 2.5.4 in my book that
$$\sum_{j=0}^{\mu_0-\mu-1} \|\mathcal{F}^{-1}(\sum_{k=0}^{j} c_{jk}\hat{u}_k(\xi')\lambda^{j-k})\|_{(s-\mathrm{Re}\,\mu-j-\frac{1}{2})} \le C\|u\|_{\mu(s)}.$$
By induction for increasing j we find that this is equivalent to

$$\sum_{j=0}^{\mu_0-\mu-1} \|u_j\|_{(s-\mathrm{Re}\,\mu-j-\frac{1}{2})} \le C\|u\|_{\mu(s)}. \tag{2.6.3}$$

Conversely, given u_j, $j = 0, \dots, \mu_0 - \mu - 1$, we can determine a function $u \in H_{\mu(s)}$ so that $D_n^{\mu+j} u = u_j$ and the opposite inequality holds [$D_n^{\mu+j}$ replaced by $D_n^{\mu+j} u$]. In fact, we can construct $(D_n - i\sqrt{|D'|^2+1}\,)^{\mu} u$ with normal derivatives $\sum_{k=0}^{j} c_{jk} \sqrt{|D'|^2+1}^{\,j-k} u_k$ according to Theorem 2.5.7 in my book. [$k-j$ replaced by $j-k$.]

Thus we have proved

Theorem 2.6.1. *If $s > \operatorname{Re}\mu_0 - \frac{1}{2}$, then the topology induced in $\mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ by that in $H_{\mu(s)}$ is defined by*

$$|||\gamma_\mu u|||_s = \sum_{j=0}^{\mu_0-\mu-1} \|D_n^{j+\mu} u\|_{(s - \operatorname{Re}\mu - j - \frac{1}{2})}.$$

The solution of the nonhomogeneous Dirichlet problem now takes the form

Theorem 2.6.2. *Let P satisfy the hypotheses of Theorem* 2.4.1. *[It must be assumed also that P is of type μ_0.] For every $\mu \le \mu_0$ which is congruent to μ_0 (mod 1) and every $s > \operatorname{Re}\mu_0 - \frac{1}{2}$ the mapping*

$$H_{\mu(s)}(\overline{\Omega}) \ni u \to \{Pu, \gamma_\mu u\} \in \overline{H}_{(s-\operatorname{Re} m_0)}(\Omega) \times \prod_{0}^{\mu_0-\mu-1} H_{(s-\operatorname{Re}\mu - j - \frac{1}{2})}(\partial\Omega)$$

is a Fredholm operator.

In order to be able to pass from the Dirichlet problem to more general boundary problems we must consider the dependence of the solutions on the boundary data more closely.

Theorem 2.6.3. *Let $\mu_0 - \mu$ be a nonnegative integer, and let P satisfy the hypotheses of Theorem* 2.4.1. *There exists a linear mapping*

$$Q \colon \mathcal{E}_\mu/\mathcal{E}_{\mu_0} \to \mathcal{E}_\mu$$

such that $\gamma_\mu Q$ differs from the identity by a pseudo-differential operator with symbol 0 *and PQ can be extended to a continuous mapping from $\mathcal{E}'(\partial\Omega)^{\mu_0-\mu}$ to $\overline{C}^\infty(\Omega)$. If B is any pseudo-differential operator of type μ, then $BQ|_{\partial\Omega}$ is a pseudo-differential operator. In a local coordinate system where $\partial\Omega$ is defined by the inequality $x_n > 0$, the principal symbol is*

$$\Big\{ \sum_{k=0}^{\mu_0-\mu-1} \overline{\oint} b_0(x,\xi)/p_0^- \xi_n^k \, d\xi_n \pi_{\mu_0-\mu-1-k-j} \Big\}_{j=0}^{\mu_0-\mu-1}$$

or else of lower order. (Here the integral is the Lebesgue integral when the integrand is integrable and will otherwise be defined later.) We consider a_j [?] to have order $-j$ in the sense of Douglis-Nirenberg. The notation π_ν, ν integer, shall mean 0 *when $\nu < 0$, an elliptic symbol of order* 0 *when $\nu = 0$ and a symbol of order ν when $\nu > 0$, all depending only on P and not at all on B.) Q has the pseudo-local properties*

(1) *when the support of u lies in a compact part K of $\partial\Omega$ then the mapping $u \to Qu$ is continuous with the $\mathcal{E}'$-topology on u and the $\mathcal{E}_\mu$ topology on Qu near $\partial\Omega \setminus K$.*
(2) *if B is a pseudo-differential operator with symbol vanishing of infinite order on $\partial\Omega$ then BQ is continuous from ${\mathcal{E}'}^{\mu_0-\mu}$ to $\overline{C}^\infty$.*

To complete the statement it remains to clarify the notation $\overline{\oint} f(\tau)\,d\tau$ where f is a function defined on the real axis. If $f \in L^1(-\infty,\infty)$ it is the Lebesgue integral. If f is the boundary value of a function analytic in the upper half plane and $O(e^{|z|^\alpha})$ for some $\alpha < 1$, then we set $\overline{\oint} f(\tau)\,d\tau = 0$. These two definitions are compatible. For suppose that $f \in L^1$ and f has an analytic extension with this property. Take β with $\alpha < \beta < 1$. Then $f(z)e^{-(\varepsilon z/i)^\beta}$ tends to 0 fast at infinity, so Cauchy's integral formula applied to half circles yields that $\int f(x)e^{-(\varepsilon x/i)^\beta}\,dx = 0$. Since $f \in L^1$ we obtain $\int_{-\infty}^{\infty} f(x)\,dx = 0$ by letting $\varepsilon \to 0$ and using Lebesgue's theorem on dominated convergence. Having proved the compatibility we can extend the definition of $\overline{\oint}$ by linearity to the linear hull of the two spaces where it is defined. Whenever $\overline{\oint} f\,d\tau$ is defined we have for $\alpha < 1$ sufficiently close to 1

$$\overline{\oint} f\,d\tau = \lim_{\varepsilon\to 0} \int f(\tau)e^{-(\varepsilon\tau/i)^\alpha}\,d\tau.$$

Many other ways of summation can of course be used.

In view of the pseudo-local property (1) it suffices to construct Q locally and then form a global Q as a sum of the form $\sum \psi_i Q_i \varphi_i$ where $\{\varphi_i\}$ is a partition of unity on the boundary, Q_i a solution in a neighborhood of the support of φ_i which also contains the support of ψ_i although $\psi_i = 1$ in some neighborhood of the support of φ_i.

The local situation will be obtained by a more detailed study of the factorization of P first used in section 2.4. We keep the notations of Theorem 2.4.3. We start by modifying P slightly so that (2.2.1) will be fulfilled not only when $x_n = 0$. To do so we shall construct p'_k homogeneous of the same order as p_k so that $p'_k - p_k$ vanishes of infinite order when $x_n = 0$ and satisfies (2.2.1) in Ω'. Such a function can be obtained by setting

$$p'_k = \sum_0^\infty \partial^j p_k(x',0,\xi)/\partial x_n^j\, x_n^j \chi(a_j x_n)/j!$$

where $\chi \in C_0^\infty(\mathbb{R})$ is equal to 1 near 0 and a_j is so rapidly decreasing that

$$|D_{x'}^\alpha D_\xi^\beta D_n^j p_k(x',0,\xi)| \le C_{\alpha,\beta} a_j,\ |\xi| = 1.$$

[D_x replaced by $D_{x'}$] [$C_{\alpha,\beta}$?] Then p'_k is a C^∞ function of x and ξ when $\xi \neq 0$, which is homogeneous in ξ, and it is obvious that p'_k satisfies the condition (2.2.1) everywhere. Let P' be a pseudo-differential operator, compactly supported and with symbol $\sum p'_k$. If we can find Q having the desired properties relative to P' we will be through for $(P'-P)Q$ maps $\mathcal{E}'^{\mu_0-\mu}$ into $\overline{C}^\infty$ by the last pseudo-local property stated in the theorem.

In order to make notations less heavy we drop the prime, thus assume that P already satisfies condition (2.2.1) for all $x \in \Omega$.

Lemma 2.6.4. *Let $p_0(x,\xi)$ be homogeneous and elliptic in ξ, of degree m_0 and C^∞ when $\xi \neq 0$. Assume that a number μ_0 independent of x is defined by* (2.4.3)*, and that* (2.2.1) *is valid for all x (in the domains considered). Then*

$$p_0(x,\xi) = p_0^+(x,\xi)p_0^-(x,\xi)$$

where p_0^- and p_0^+ are homogeneous of degree μ_0 and $m_0 - \mu_0$ respectively, analytic when $\xi \neq 0$ and $\operatorname{Im}\xi_n \le 0$ *resp.* $\operatorname{Im}\xi_n \ge 0$*. Furthermore, the functions belong to C^∞ when*

$\xi' = (\xi_1, \dots, \xi_{n-1}) \neq 0$. *Instead of differentiability when* $\xi' = 0$ *we can only claim that for example* p_0^- *has an asymptotic expansion*

$$p_0^- \sim \sum_{j=0}^{\infty} (\xi_n - i)^{\mu_0 - j} r_j(x, \xi'),\ \xi_n \to \infty,\ |\xi'| = 1,$$

which remains valid after differentiations. Here $r_0 \equiv 1$ *is independent of* ξ'. *If* r_j *is extended as a homogeneous function of degree* j *we have of course for all* ξ',

$$p_0^- \sim \sum_{j=0}^{\infty} (\xi_n - i|\xi'|)^{\mu_0 - j} r_j(x, \xi'),\ \xi_n \to \infty.$$

[x inserted.] (Obviously we could express the asymptotic behavior with other factors than powers of $\xi_n - i|\xi'|$, *but these are convenient and were used by Vishik-Eskin.)*

Proof. Suppressing the variable x in the notations we write

$$\psi(\xi) = \log p_0(\xi) - \mu_0 \log(\xi_n - i|\xi'|) - (m_0 - \mu_0) \log(\xi_n + i|\xi'|) - a_+.$$

The logarithms in the later terms are defined so that their imaginary parts $\to 0$ when $\xi_n \to +\infty$. Then

$$\psi(\xi) = \log p_0(\xi'/|\xi_n|, \pm 1) - \mu_0 \log(\xi_n - i|\xi'|) - (m_0 - \mu_0) \log(\xi_n + i|\xi'|) - a_+ - m_0 \log|\xi_n|.$$

Now we have

$$p_0(\xi'/|\xi_n|, \pm 1) = e^{a_\pm}(1 - \exp(-a_\pm) \sum_{\alpha = \alpha' \neq 0} \xi'^{\alpha} |\xi_n|^{-|\alpha|} p_0^{(\alpha)}(0, \pm 1)),$$

[coefficients?] where the sum is an asymptotic expansion uniformly for ξ' in a compact set not containing 0. This follows from Taylor's formula. As in section 2.4 we conclude from (2.2.1) that $\psi(\xi', \xi_n)$ has an asymptotic expansion in powers of $1/\xi_n$ when $\xi_n \to \infty$; the asymptotic expansions hold in the C^∞ topology with respect to x and $\xi' \neq 0$. We can now write for any k

$$\psi(x, \xi', \xi_n) = \sum_{j=1}^{k} a_j(x, \xi')(\xi_n - i|\xi'|)^{-j} + \psi_k(x, \xi', \xi_n)$$

where $\psi_k = O(\xi_n^{-k-1})$ and $\psi_k' = O(\xi_n^{-k-2})$. The sum is already analytic in the half plane $\operatorname{Im} \xi_n < 0$. To see the effect of splitting the remainder term by means of a Cauchy integral as in section 2.4 we choose $\chi \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$ equal to 1 in $(-\frac{1}{4}, \frac{1}{4})$ and consider separately the two integrals

$$\int \psi_k(x, \xi', \tau)(\tau - \xi_n)^{-1} \chi(\tau/|\xi_n|)\, d\tau,\ \int \psi_k(x, \xi', \tau)(\tau - \xi_n)^{-1}(1 - \chi(\tau/|\xi_n|))\, d\tau.$$

In the former integral we have $|\tau|/|\xi_n| < \frac{1}{2}$ so we can use the Taylor expansion

$$(\tau - \xi_n)^{-1} = -1/|\xi_n| - \tau/|\xi_n|^2 - \tau^2/|\xi_n|^3 - \dots$$

In view of the fact that

$$\int \psi_k(x, \xi', \tau)\tau^j(1 - \chi(\tau/|\xi_n|))\, d\tau = O(|\xi_n|^{j+1-k})$$

we conclude that the first integral has an expansion in powers of $1/\xi_n$ with error $O(\xi_n^{-k})$. Using Lemma 2.4.2 it is easy to show that the second integrand is $O(\xi_n^{-k})$ too. The existence of an asymptotic expansion for the two parts of the Cauchy integral of ψ has thus been proved, both of them starting with $1/\xi_n$. If we set $p^-(\xi) = (\xi_n - i|\xi'|)^{\mu_0} \exp \psi_-$ [changed ψ_-^- to ψ_-, cf. p.11] and define p^+ similarly, the lemma is now proved.

We shall next continue the factorization of the symbol to a formal factorization of the operator P in the sense of the formulas given in section 1.1 [not available] — whose validity for symbols that are not smooth is of course by no means clear a priori.

Lemma 2.6.5. *There exist formal sums $\sum p_k^+$ and $\sum p_k^-$ whose terms are homogeneous, C^∞ when $\xi' \neq 0$, analytic respectively when $\operatorname{Im} \xi_n \geq 0$ and $\operatorname{Im} \xi_n \leq 0$, such that*

$$\sum p_l(x, \xi) = \sum_{j,k,\alpha} p_k^{+(\alpha)}(x, \xi) D_x^\alpha p_j^-(x, \xi)/\alpha!\,. \tag{2.6.4}$$

Moreover, $p_k^-(x, \xi)$ has an asymptotic expansion when $\xi_n \to \infty$, $\xi' \neq 0$,

$$p_k^-(x, \xi) \sim \sum_0^\infty (\xi_n - i|\xi'|)^{\mu_0 - j} r_{kj}(\xi'), \text{ where } r_{0,k} = 0, k \neq 0.$$

[probably means $r_{k,0}$] and r_{jk} is homogeneous of degree $j - k$. Also p_k^+ has asymptotic expansions in powers of $(\xi_n + i|\xi'|)$ but which powers occur we can not specify.

Proof. We have already discussed the construction of the leading terms. The other terms can then be determined successively by choosing solutions of problems like

$$p_0^+ p_k^- + p_k^+ p_0^- = \text{ function given by terms in } \sum p_l \text{ and earlier terms } p_j^\pm.$$

We divide out by $p_0^+ p_0^- = p$. Noting that for the previous p_j^- the quotient of any of their derivatives by p_j^- has an asymptotic expansion in integral powers of ξ_n, and using (2.2.1), we conclude that in the problem to solve

$$p_k^-/p_0^- + p_k^+/p_0^+ = F$$

the homogeneous function F has an asymptotic expansion in powers of $(\xi_n + i|\xi'|)$. (It is convenient to take $|\xi'| = 1$ during the discussion.) As in the proof of the previous lemma we conclude that the problem has a solution (unique!) such that p_k^-/p_0^- has an asymptotic expansion in integral powers of $1/\xi_n$ and p_k^+/p_0^+ has an asymptotic expansion in powers of $(\xi_n + i|\xi'|)$. This completes the proof, for it is clear that the real part of the orders will tend to $-\infty$ so that the formal sums make sense.

It is now standard to find an inverse $\sum q_j(x,\xi)$ of $\sum p_j^-(x,\xi)$, that is, find these functions homogeneous satisfying conditions analogous to those for p_k^- except that μ_0 is replaced by $-\mu_0$, so that

$$\sum p_j^{-(\alpha)}(x,\xi) D_x^\alpha q_k(x,\xi)/\alpha! = 1. \tag{2.6.5}$$

Now the associative law for composition of operators leads to the associative law for the composition formula for symbols — which can also be verified by direct computation. [Inserted handwritten half-page describing the associative law for symbols, not typed here.] Hence we obtain from (2.6.4) and (2.6.5) that [indexation?]

$$\sum_{j,k,\alpha} p_j^{-(\alpha)}(x,\xi) D_x^\alpha q_k(x,\xi)/\alpha! = \sum_l p_l^+. \tag{2.6.6}$$

If s_k is the order of q_k we have $\operatorname{Re} s_k \to -\infty$, and

$$|D_x^\alpha D_\xi^\beta q_k(x,\xi)| \le C|\xi|^{-\operatorname{Re}\mu_0 - \beta_n} |\xi'|^{\operatorname{Re}(\mu_0+s_k)-|\beta'|}.$$

Let $\varphi \in C^\infty(\mathbb{R}^n)$ be equal to 0 when $|\xi| < 1/2$ and equal to 1 when $|\xi| > 1$. As in the proof of Theorem 1.1.5 [seems covered by Theorem 2.7 of [1]] we can select a sequence $t_k \to +\infty$ so rapidly increasing that

$$q(x,\xi) = \sum \varphi(\xi'/t_k) q_k(x,\xi)$$

converges everywhere, also after differentiation, and that the remainder terms have the natural estimates,

$$|D_x^\alpha \xi^\beta \sum_{k \ge N} \varphi(\xi'/t_k) q_k(x,\xi)| \le C_{N,\alpha,\beta}(1+|\xi|)^{-\operatorname{Re}\mu_0-\beta_n}(1+|\xi'|)^{\operatorname{Re}(\mu_0+s_k)-|\beta'|}.$$

Let K be a compact subset of $\omega = \{x; x_n = 0, |x'| \le 1\}$, and let $\psi \in C_0^\infty(\widetilde{\Omega})$ be equal to 1 in K.With $v_j \in C_0^\infty(K)$, $j = 0, \dots, \mu_0 - \mu - 1$, we set

$$Q_1 v(x) = (2\pi)^{-n} \int q(x,\xi) \hat{v}_j(\xi') \xi_n^j e^{i\langle x,\xi\rangle}\, d\xi.$$

The Fourier transform shall here be understood in the sense of Schwartz. We shall later on cut off Q_1 by multiplication with ψ, but first we have to study the properties of Q_1.

a) Q_1 maps $C_0^\infty(K)$ into $\mathcal{E}_\mu$. This is perfectly obvious if we introduce the asymptotic expansion of q with respect to ξ_n; in view of Theorem 2.5.2 the terms in the asymptotic expansion give rise to elements in $\mathcal{E}_\mu$ and the remainder gives a highly differentiable function with support in $\overline{\mathbb{R}}_+^n$.

b) Q_1 is continuous from the topology of $H_{\sigma+j}(\mathbb{R}^{n-1})$ on v_j to $H_{\mu(\sigma-\operatorname{Re}\mu_0+1/2)}$. [$\mu$ was inserted.] This follows by the same argument as in a).

c) $(-ix_n)^N Q_1$ is obtained by replacing $q(x,\xi)\xi_n^j$ by its Nth derivative with respect to ξ_n. Thus we get continuity from the same topologies on v to $H_{\mu+N(\sigma+N-\operatorname{Re}\mu_0)}$. This implies the localization property (2) in the theorem. To prove the other one we note that $\int q(x,\xi)\xi_n^j e^{i\langle x,\xi\rangle}\,d\xi$ is in C^∞ with respect to x' when $x'\neq 0$. (Multiply by ${x'}^\alpha$ and integrate.) For suitable σ and all τ we then have a continuous mapping $\mathcal{E}'(K)^{\mu_0-\mu}\to H_{(\sigma,\tau)}(\complement K)$. The localization property (1) therefore follows from Theorem 2.5.7 — if we discuss what σ can be used — in view of [text missing]

d) PQ_1 is a continuous map from ${\mathcal{E}'}^{\mu_0-\mu}$ into $\overline{C}^\infty$. Indeed, the proof of Theorem 1.1.6 [seems related to Theorem 2.10 of [1]] does not require the symbol of p to be a smooth function of $|\xi|$. Hence we obtain that PQ_1 is given by

$$PQ_1 v = \sum_0^{\mu_0-\mu-1}(2\pi)^{-n}\int r(x,\xi)\hat{v}_j(\xi')\xi_n^j e^{i\langle x,\xi\rangle}\,d\xi$$

where $r(x,\xi)\sim\sum p_k^+(x,\xi)$. Since the terms are analytic in the upper half plane they do not contribute anything to $PQ_1 v$ when $x_n>0$, which gives the statement.

In the same way we can form BQ_1, which is defined by means of a kernel [symbol?] $\sum {b_j}^{(\alpha)}(x,\xi)D_x^\alpha q_k(x,\xi)/\alpha!$. This has an asymptotic expansion in integral powers of ξ_n, hence in powers of $(\xi_n+i|\xi'|)$, which proves that it maps $C_0^\infty(K)^{\mu_0-\mu}$ into $\overline{C}^\infty(\Omega)$. The boundary values are obtained by subtracting a term analytic in the upper half plane until the remainder can be integrated. Thus we obtain the symbol

$$\sum(2\pi)^{-1}\overline{\oint}{b_j}^{(\alpha)}(x,\xi)D_x^\alpha q_k(x,\xi)\xi_n^j/\alpha!\,,\, j=0,\dots,\mu_0-\mu-1.$$

It remains to consider $\gamma_\mu Q_1$. To do so we first note that in each term of the expansion

$$q(x,\xi)=\sum x_n^j\partial^j q(x',0,\xi)/\partial x_n\,/j!$$

we can replace x_n by $-i\partial/\partial\xi_n$, and obtain instead

$$q_1(x',\xi)\sim\sum(-i\partial^2/\partial x_n\partial\xi_n)^j q(x',0,\xi)/j!.$$

This sum has an asymptotic expansion in ξ_n,

$$\xi_n^{-\mu_0}s_0(x',\xi')+\xi_n^{-\mu_0-1}s_1(x',\xi')+\dots$$

where $s_0(x',\xi')=\varphi(\xi'/t_0)$. Hence

$$q_1(x',\xi)\sum\hat{v}_j(\xi')\xi_n^j\sim\sum V_j(\xi')\xi_n^{-\mu-j-1}$$

where

$$V_j(\xi')=s_0 v_{\mu_0-\mu-1-j}+s_1 v_{\mu_0-\mu-j}+\dots$$

Thus

$$D_n^j\gamma_\mu Q_1 v=s_0(x',D')v_{\mu_0-\mu-1-j}+s_1(x',D')v_{\mu_0-\mu-j}+\dots \tag{2.6.7}$$

Given $u_j \in C_0^\infty(K)$, $j = 0, \dots, \mu_0 - \mu - 1$, we wish to choose v so that $\gamma_\mu Q_1 v$ is approximately equal to u_j. Noting that $1 - s_0$ is a pseudo-differential operator with symbol 0, we get successively for increasing j

$$v_{\mu_0-\mu-1-j} = u_j - S_1(x', D')v_{\mu_0-\mu-j} - \dots \tag{2.6.8}$$

where S_j are compactly supported with the symbol s_j. Solving this system of equations for v_j we conclude that

$$D_n^j \gamma_\mu Q_1 v - u_j$$

is a pseudo-differential operator with symbol 0 acting on the u_k's. Putting

$$Qu = \psi Q_1 v$$

with v given by (2.6.8) we have constructed an operator with all the required properties.

[The rest is copied from handwritten text:]

2.7. General boundary problems.

Let

$$\begin{cases} Pu = f \in C^\infty & u \in H_{\mu(s)} \\ B_j u = \varphi_j \in C^\infty & j = 1, \dots, \mu_0 - \mu, \end{cases}$$

where B_j is of type μ. Ellipticity! Well defined? We take $s > \operatorname{Re}\mu_0 - \frac{1}{2}$, and, if the order μ_j of B_j has the am[?] [analogous?] form $s > \operatorname{Re}\mu_j + \frac{1}{2}$.

(It would be sufficient to have such a condition rel. to the "transmission order".)

Then we claim that $u \in C^\infty$. [Probably means $\mathcal{E}_\mu$.] In fact, for such s,

$$\|u\|_{\mu(s)} \le C(\|f\|^\bullet_{s-\operatorname{Re}\mu_0} + \sum \|\varphi_j\|^{\partial\Omega}_{s-\operatorname{Re}\mu_j-\frac{1}{2}} + \|u\|_{\mu(s-1)}).$$

Proof as follows: Form

$$v = u - Q\gamma_\mu u.$$

Then

$$\begin{cases} Pv = Pu - PQ\gamma_\mu u = f + \text{ rem. in } C^\infty \text{ (cont.)} \\ \gamma_\mu v \simeq 0, \end{cases}$$

: $v \in C^\infty$ or at least[?] in $H_{\mu(s)}$ under appropriate hypotheses on f. Now $B_j u = B_j v + B_j Q\gamma_\mu u$; hence $B_j Q \gamma_\mu u = \varphi_j -$ smooth.

If $B_j Q$ elliptic system we are through.

Existence thm. Try $u = \mathcal{L}_1 f + Q\mathcal{L}_2\psi$ where $\psi \in \mathcal{E}_\mu/\mathcal{E}_{\mu_0}$ and $\mathcal{L}_1$ sol. of int.[?] hom. Dirichlet problem, $\mathcal{L}_2$ sol. of ps. diff. eq. $B_j Q\varphi = \psi$. Gives[?] Fredholm eq.[?]

$$\begin{cases} g - K_1\{g, \psi\} = f \\ \psi - K_2\{g, \psi\} = \varphi \end{cases}$$

So existence with finite codimension.

More sophisticated reduction: See my paper in Annals.

2.8. Jump in boundary conditions.

Cbln[?] in boun[?] prb[?] like

$$\begin{cases} P_1 u = f_1 & \omega_1 \\ P_2 u = f_2 & \omega_2 \end{cases}$$

[drawing:] $\omega = \omega_1 \cup \omega_2$. P_1, P_2 elliptic ps.d.op. Set $P_1 u = v$; where

$$\begin{cases} v & = f_1 \text{ in } \omega_1 \\ P_2 P_1^{-1} v & = f_2 \text{ in } \omega_2 \end{cases}$$

Can substitute $w = v - f_1$

$$\begin{cases} w & = 0 \text{ in } \omega_1 \\ P_2 P_1^{-1} w & = f_2 - P_2 P_1^{-1} f_1 \text{ in } \omega_2 \end{cases}$$

Boundary problem for ps.d.op. $P_2 P_1^{-1}$!

References

[1]. L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*, Singular integrals (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 138-183.