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The Mumford conjecture, Madsen-Weiss and homological stability for mapping class groups of surfaces

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Introduction

The Mumford conjecture [**26**] says that the rational cohomology ring of the moduli space of Riemann surfaces is a polynomial algebra on the so-called Mumford-Morita-Miller classes, in a range of degrees increasing with the genus of the surface. This conjecture is now known to be true, using the following two theorems as main ingredients: *Harer's stability theorem* [**16**] which tells us that the rational cohomology of the moduli space is independent of the genus in a range of dimensions, and *Madsen-Weiss' theorem* [**23**] which identifies the stable cohomology with that polynomial algebra.

Harer's and Madsen-Weiss' theorems are both statements about the integral homology of the mapping class groups of surfaces, using, in the Madsen-Weiss' case, Earl and Eells' theorem [**8**] relating the diffeomorphism groups to the mapping class groups of surfaces. In Lecture 1, we describe the relationship between the moduli space of Riemann surfaces, the mapping class groups and the diffeomorphism groups of surfaces. We then give a definition of the Mumford-Morita-Miller classes, state the main part of Harer's stability theorem, and give a first statement of Madsen-Weiss' theorem.

The last three lectures are devoted to a sketch proof of Harer stability theorem, using improvements by Ivanov [**19, 20**], Hatcher [**17**], Boldsen [**4**] and Randal-Williams [**32**]. This proof uses two spectral sequences associated to the action of the mapping class groups on certain simplicial complexes of arcs in the surfaces. In Lecture 2, we give the general strategy, define the relevant arc complexes and study the properties of the action of the mapping class groups on these complexes. In Lecture 3, we give the spectral sequence argument (following Randal-Williams). Harer's stability theorem is proved by studying the spectral sequences carefully, using the properties of the action given in Lecture 2, as well as a connectivity property of the arc complexes, whose proof is sketched in Lecture 4. The last three are mostly based on the survey [**37**].

This series of lectures is supplemented by Galatius' lectures in this volume [**11**], which present a sketch proof of the Madsen-Weiss theorem.

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LECTURE 1

The Mumford conjecture and the Madsen-Weiss theorem

In this lecture, we give a brief introduction to important players in the proof of the Mumford conjecture by Madsen and Weiss. We introduce the moduli space of Riemann surface, the Teichmüller space and describe their relationship to diffeomorphism groups and mapping class groups of surfaces. We state the Mumford conjecture, the Madsen-Weiss theorem and Harer's stability theorem.

1. The Mumford conjecture

Let S_g be a closed, smooth, oriented surface of genus g and let $\mathrm{Diff}(S_g)$ denote the topological group of orientation preserving diffeomorphisms of S_g. The moduli space $\mathcal{M}_g$ can be defined in many ways:

$$\begin{aligned}
\mathcal{M}_g &= \textit{Moduli space of Riemann surfaces}\\
&= \text{Space of conformal classes of Riemannian metrics on } S_g\\
&= \{\text{Riemannian metrics on } S_g\}/\mathrm{Diff}(S_g)\\
&= \text{Isometry classes of hyperbolic structures on } S_g\\
&= \text{Biholomorphic classes of complex structures on } S_g\\
&= \text{Isomorphy classes of smooth algebraic curves homeomorphic to } S_g
\end{aligned}$$

We would like to describe $\mathcal{M}_g$, and, for example, compute its (co)homology. The present lectures, together with Galatius' lectures in the same volume [**11**], are centered around the following result about $\mathcal{M}_g$:

Theorem 1.1 (Mumford conjecture, proved by Madsen-Weiss [**23, 26**]).

$$H^*(\mathcal{M}_g;\mathbb{Q}) \cong_{(*)} \mathbb{Q}[\kappa_1,\kappa_2,\dots] \quad \textit{with} \quad |\kappa_i| = 2i$$

where the isomorphism $(*)$ *is in a range of dimension growing with* g.

We will reformulate this theorem in terms of diffeomorphism groups and mapping class groups of surfaces. The classes κ_i are called the *Mumford-Morita-Miller classes* and are defined below, and the range for the isomorphism is the homological stability range of the mapping class group of surfaces, also given explicitly below.

2. Moduli space, mapping class groups and diffeomorphism groups

Define the *Teichmüller space*

$$\mathcal{T}_g = \{\text{Riemannian metrics on } S_g\}/\,\mathrm{Diff}_0(S_g)$$

with $\mathrm{Diff}_0(S_g)$ the topological group of diffeomorphisms of S_g isotopic to the identity, i.e. the component of the identity in $\mathrm{Diff}(S_g)$, acting on the space of metrics by pull-back (see [**9**, 10.1]).

Let

$$\Gamma_g = \Gamma(S_g) := \mathrm{Diff}(S_g)/\mathrm{Diff}_0(S_g) = \pi_0 \mathrm{Diff}(S_g)$$

denote the *mapping class group* of S_g. (We note that Γ_g is also often denoted $\mathrm{Mod}(S_g)$, and sometimes refered to as the *modular group*.) The group Γ_g acts on $\mathcal{T}_g$, via the full action of $\mathrm{Diff}(S_g)$ on the space of metrics (see [**9**, 12.1]), and

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$$

As $\mathcal{T}_g \simeq *$ (in fact $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$, see [**9**, 10.6]) and Γ_g acts properly discontinuously on $\mathcal{T}_g$ with finite stabilizers [**9**, 12.1,12.3], we have

$$H^*(\mathcal{M}_g;\mathbb{Q}) \;\cong\; H^*(B\Gamma_g;\mathbb{Q})$$

where $B\Gamma_g$ is a *classifying space* for Γ_g, i.e. $B\Gamma_g = E\Gamma_g/\Gamma_g$, for $E\Gamma_g \simeq *$ with free properly discontinuous Γ_g–action. [See exercices after Lecture 3.]

Recall moreover that $H^*(B\Gamma_g;\mathbb{Z}) = H^*(\Gamma_g;\mathbb{Z})$ is the group cohomology of Γ_g (see [**5**, I.4]).

To relate these homology groups to the diffeomorphism group, we need the following

Theorem 1.2 (Earl-Eells [**8**]). *For $g \geq 2$, $\mathrm{Diff}(S_g)$ has contractible components.*

In other words, the theorem says that the homomorphism

$$\mathrm{Diff}(S_g) \to \pi_0 \mathrm{Diff}(S_g) = \Gamma_g$$

is a homotopy equivalence. As one can build compatible models of $E\,\mathrm{Diff}(S_g)$ and $E\Gamma_g$ (via the *standard resolution* and a topological version of it [**5**, I.5]), it follows that, when $g \geq 2$,

$$B\,\mathrm{Diff}(S_g) = E\,\mathrm{Diff}(S_g)/\mathrm{Diff}(S_g) \xrightarrow{\simeq} E\Gamma_g/\Gamma_g = B\Gamma_g$$

and thus

$$H^*(\mathcal{M}_g;\mathbb{Q}) \;\cong\; H^*(\Gamma_g;\mathbb{Q}) \;\cong\; H^*(B\Gamma_g;\mathbb{Q}) \;\cong\; H^*(B\,\mathrm{Diff}(S_g);\mathbb{Q})$$

giving us many formulations of the Mumford conjecture.

The space $B\,\mathrm{Diff}(S_g)$ is a *classifying space for S_g-bundles*: there is a 1-1 correspondence

$$\{S_g \to E \xrightarrow{\pi} X\}/_{\cong} \quad \longleftrightarrow \quad \mathrm{Maps}(X, B\,\mathrm{Diff}(S_g))/_{\simeq}$$

between isomorphism classes of bundles and homotopy classes of maps. Moreover, elements of $H^*(B\,\mathrm{Diff}(S_g))$ are *characteristic classes* for S_g–bundles: they give an assignment of a cohomology class $c(E,\pi) \in H^*(X)$ to any bundle $S_g \to E \xrightarrow{\pi} X$, which is natural in the sense that $c(g^*(E,\pi)) = g^*c(E,\pi)$ for any map $g : Y \to X$. Given a class $c \in H^*(B\,\mathrm{Diff}(S_g))$, the associated characteristic class is defined by $c(E,\pi) = f^*(c)$ for $f : X \to B\,\mathrm{Diff}(S_g)$ classifying (E,π) via the above correspondence. (See [**11**, Cor 1.5,1.6] for more details, using an embedding model for $B\,\mathrm{Diff}(S_g)$.)

3. The Mumford-Morita-Miller classes

The Mumford-Morita-Miller-classes κ_i, $i = 1, 2, \ldots$, are characteristic classes for surface bundles defined as follows: to the bundle of oriented surfaces

$$S_g \to E \xrightarrow{\pi} X,$$

one associates the *vertical tangent bundle*

$$\mathbb{R}^2 \to T_v E \xrightarrow{\pi} E$$

with fiber at $e \in E$ the tangent plane to the surface $\pi^{-1} \circ \pi(e) = F_{\pi(e)}$, the fiber over $\pi(e)$. (See Figure 1.)

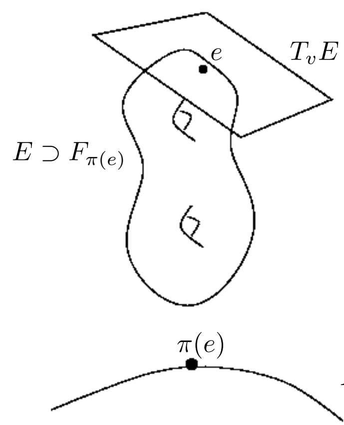

FIGURE 1. The vertical tangent bundle

This plane bundle has a first Chern class $c_1(T_v E) \in H^2(E)$. Then

$$\kappa_i := (-1)^{i+1} \pi_*(c_1(T_v E)^{i+1}) \in H^{2i}(X)$$

were $\pi_* : H^{2i+2}(E) \to H^{2i}(X)$ is the *Gysin homomorphism* or *integration along the fibers* (see e.g. [**28**, 4.2.1,4.2.3]). (Note that Mumford originally defined κ_i using the cotangent bundle.)

The cohomology class $\kappa_i \in H^{2i}(B\,\mathrm{Diff}(S_g))$ (or $H^{2i}(\mathcal{M}_g; \mathbb{Q})$) corresponding to this characteristic class is obtained by doing the same construction on the universal bundle

$$S_g \longrightarrow E\,\mathrm{Diff}(S_g) \times_{\mathrm{Diff}(S_g)} S_g \longrightarrow B\,\mathrm{Diff}(S_g)$$

or on the universal curve

$$S_g \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g$$

where $\mathcal{M}_{g,1}$ is the moduli space of Riemann surfaces with one marked point.

4. Homological stability

In the previous section, we have defined a class $\kappa_i \in H^{2i}(B\,\mathrm{Diff}(S_g))$ for each $i = 1, 2, \ldots$, and for each genus g, that is we have define a map of graded rings

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \longrightarrow H^*(B\,\mathrm{Diff}(S_g)).$$

The Mumford conjecture says that this map is an isomorphism in a range of dimensions growing with g. Note that $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ is independent of g, so part of the Mumford conjecture is a stability statement, which says that the cohomology of $\mathcal{M}_g$ in any given degree is independent of g if g is sufficiently large. This is known as Harer's stability theorem, which we state in this section.

A familly of groups

$$G_1 \hookrightarrow G_2 \hookrightarrow \dots \hookrightarrow G_n \hookrightarrow \dots$$

satisfies *homological stability* if the induced maps

$$H_i(G_n) \longrightarrow H_i(G_{n+1})$$

are isomorphisms in a range $i \ll n$, where $H_*(G_n)$ denotes the group homology of G_n.

Examples: Families of groups satisfying homological stability are $G_n =$ the symmetric group Σ_n [**29**], the braid group β_n [**1**], the linear group $GL_n(\mathbb{Z})$ [**13**].

Define $G_\infty = \bigcup_{n\geq 1} G_n$ to be the "*stable group*". If $\{G_n\}_{n\geq 1}$ satisfies homological stability, then

$$H_i(G_n) \cong H_i(G_\infty) \quad \text{in the range } i \ll n$$

and $H_*(G_\infty)$ is the "*stable homology*".

Let $S_{g,1}$ be a surface of genus g with one boundary component, and let $\Gamma_{g,1} = \pi_0 \operatorname{Diff}(S_{g,1} \text{ rel } \partial)$ be the group of components of the diffeomorphisms restricting to the identity on the boundary. Consider the family of groups

$$\Gamma_{1,1} \hookrightarrow \Gamma_{2,1} \hookrightarrow \dots \hookrightarrow \Gamma_{g,1} \hookrightarrow \dots$$

where the map $\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$ is induced by including $S_{g,1}$ inside $S_{g+1,1}$ as in Figure 2 and extending the diffeorphisms by the identity on $S_{g+1,1} \backslash S_{g,1}$.

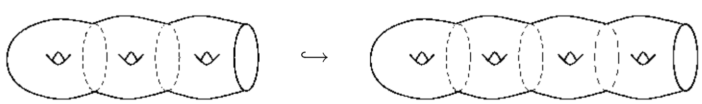

FIGURE 2. Inclusion $S_{3,1} \hookrightarrow S_{4,1}$

As we are interested in mapping class groups of closed surfaces, we also consider the map $\Gamma_{g,1} \to \Gamma_g$ induced by gluing a disc on the boundary component of $S_{g,1}$.

Theorem 1.3 (Harer's stability theorem, improved by Ivanov, Boldsen, and Randal-Williams [**16, 19, 4, 32**])**.**

$$H_i(\Gamma_{g,1};\mathbb{Z}) \xrightarrow{\cong} H_i(\Gamma_{g+1,1};\mathbb{Z}) \quad \textit{for } i \leq \frac{2}{3}(g-1)$$

and

$$H_i(\Gamma_{g,1};\mathbb{Z}) \xrightarrow{\cong} H_i(\Gamma_g;\mathbb{Z}) \quad \textit{for } i \leq \frac{2}{3}g$$

The range $i \leq \frac{2}{3}(g-1)$ is the range of degrees i in which the isomorphism in the Mumford conjecture holds. We will give a sketch proof of the stability theorem in the next three lectures.

5. The Madsen-Weiss theorem

The Madsen-Weiss theorem gives a computation of the stable (co)homology of mapping class groups, i.e. the group (co)homology of

$$\Gamma_\infty = \bigcup_{g\geq 1} \Gamma_{g,1}$$

or singular (co)homology of its classifying space $B\Gamma_\infty$. We give now a first formulation of this theorem, without defining all the players yet:

Theorem 1.4 (Madsen-Weiss [**23**]). *There is a homology isomorphism*

$$B\Gamma_\infty \longrightarrow \Omega_0^\infty MTSO(2)$$

where the target is the 0th component of the infinite loop space of the spectrum $MTSO(2)$.

The spectrum $MTSO(2)$, defined in Galatius' lectures [**11**], is build out of Grassmanians of 2-planes in $\mathbb{R}^n$, in the limit as $n \to \infty$, and the map uses a vertical tangent bundle type construction. (See Definition 1.7, Theorem 1.8 and Corollary 1.10 in [**11**].)

Using homological stability, this can be restated as saying that

$$H_i(\Gamma_g;\mathbb{Z}) \cong H_i(\Omega^\infty MTSO(2);\mathbb{Z}) \quad \text{for } i \leq \frac{2}{3}(g-1)$$

(and the same for cohomology using the universal coefficient theorem).

The target space is computable and $H^*(\Omega_0^\infty MTSO(2);\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\dots]$ with κ_i in degree $2i$ corresponding to the Mumford-Morita-Miller-class of the same name (see [**11**, 2.1]). Combining these two facts gives the Mumford conjecture, namely that

$$H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\Gamma_g,\mathbb{Q}) \cong_{(*)} \mathbb{Q}[\kappa_1,\kappa_2,\dots]$$

where the isomorphism $(*)$ is up to degree $\frac{2}{3}(g-1)$.

This type of theorem for the symmetric groups and braid groups were already proved in the early 70's:

Theorem 1.5 (Symmetric groups). $H_*(\Sigma_\infty) \cong H_*(\Omega_0^\infty S^\infty)$

Theorem 1.6 (Braid groups). $H_*(\beta_\infty) \cong H_*(\Omega_0^2 S^2)$

The first theorem is known as the *Barratt-Priddy theorem* [**31**]. Both theorems can be seen as special cases of the *approximation theorem*, which says that the map $C_nX \to \Omega^n\Sigma^n X$ is a group completion for C_n the little n-cubes monad. Here one needs to take $X = S^0$ and $n = \infty$ in the first case, and the same X but $n = 2$ in the second case. (See May [**24**, Thm 2.7] and [**7**, p.486 (15)], or Segal [**35**, Thm 1], for the approximation theorem—see also the work of Boardman-Vogt [**3**] and Barratt-Eccles [**2**]. See [**11**, Lec 4] or [**25**] for the group completion theorem.) The proof of the Madsen-Weiss Theorem presented in Galatius' lecture series [**11**] follows Galatius–Randal-Williams [**12**], which can be seen as a generalization of Segal's proof of the approximation theorem.

The Madsen-Weiss Theorem was generalized to other types of mapping class groups of surfaces (Non-orientable [**36**], framed, Spin and Pin mapping class groups [**33**]). Other examples are

Theorem 1.7 (Automorphisms of free groups [**10**]). $H_*(\mathrm{Aut}_\infty) \cong H_*(\Omega_0^\infty S^\infty)$

Here $\mathrm{Aut}_\infty = \bigcup_{n\geq 1} \mathrm{Aut}(F_n)$ with F_n the free group on n letters.

Theorem 1.8 (Handlebody groups [**18**]). $H_*(\mathcal{H}_\infty) \cong H_*(\Omega_0^\infty \Sigma^\infty (BSO(3)_+))$

Here $\mathcal{H}_\infty = \bigcup_{g\geq 1} \mathcal{H}_{g,1}$ with $\mathcal{H}_{g,1} = \pi_0 \,\mathrm{Diff}(H_g \text{ rel } D^2)$ the mapping class group of a handlebody H_g of genus g fixing a disc in the boundary of H_g.

All the above examples are computations of the homology of the stable group of a family of groups satisfying homological stability, so they all can be restated as computations of the homology of the unstable groups in a range of degrees. Stability is however not a necessary ingredient of a "Madsen-Weiss theorem"—it is rather an interpretational tool.

6. Exercices

1) Show that S^2 and T^2 do not satisfy the Earl-Eells theorem, i.e. that $\mathrm{Diff}(S^2)$ and $\mathrm{Diff}(T^2)$ do not have contractible components.
2) Give a definition of the group $\Gamma_\infty = \bigcup_{g\geq 1} \Gamma_{g,1}$ in terms of an infinite genus surface S_∞.
3) Let $\Gamma_g^1 = \pi_0 \,\mathrm{Diff}(S_g^1)$ denote the mapping class group of a once punctured genus g surface S_g^1. Use homological stability and a factorization of the map $\Gamma_{g,1} \to \Gamma_{g,0}$ to show injectivity of the map $H_i(\Gamma_{g,1}) \to H_i(\Gamma_g^1)$ in a range.

LECTURE 2

Homological stability: geometric ingredients

In this lecture, we briefly describe a general strategy for proving homological stability for families of groups and then give the main geometric ingredients needed for the case of the mapping class group of surfaces, with an emphasis on the case of surfaces with boundaries. We follow Randal-Williams [**32**] and the survey [**37**], which contains further details.

1. General strategy of proof

A *simplicial complex* $X = (X_0, \mathcal{F})$ is a set of vertices X_0 together with a collection $\mathcal{F}$ of subsets of X_0 closed under taking subsets and containing all the singletons. The subsets of cardinality $p+1$ are called the *p–simplices* of X.

To a simplicial complex X, one can associate its *realization* $|X|$ which has a topological p-simplex Δ^p for each p-simplex of X.

A space or simplicial complex X is called *n–connected* if $\pi_i(X) = 0$ for all $i \leq n$ (where $\pi_i(X) := \pi_i(|X|)$ if X is a simplicial complex). Note that, by Hurewicz theorem, a simply connected space X is n-connected, $n \geq 2$, if and only if $H_*(X) = 0$ for $2 \leq * \leq n$.

Given a familly of groups

$$G_1 \hookrightarrow G_2 \hookrightarrow \ldots \hookrightarrow G_n \hookrightarrow \ldots$$

we want to find a simplicial complex (or simplicial set) X_n for each n such that

- G_n acts on X_n,
- the stabilizer $\mathrm{Stab}(\sigma_p) \cong G_{n-p-1}$ for any p-simplex σ_p,
- the action is as transitive as possible,
- X_n is highly connected.

There is then a spectral sequence for the action of G_n on X_n which decomposes the homology of G_n in terms of the homology of the stabilizers. As these are assumed to be previous groups in the sequence, this spectral sequence allows an inductive argument. (See Lecture 3 for more details.)

2. The case of the mapping class group of surfaces

To prove homological stability for the groups Γ_g, we will need to consider surfaces with any number of boundary components. Let $S = S_{g,r}$ be a surface of genus g with r boundary components. We will consider 3 maps

$$\begin{aligned} \alpha_g\colon \Gamma(S_{g,r}) &\to \Gamma(S_{g+1,r-1}) \\ \beta_g\colon \Gamma(S_{g,r}) &\to \Gamma(S_{g,r+1}) \\ \delta_g\colon \Gamma(S_{g,r}) &\to \Gamma(S_{g,r-1}) \end{aligned}$$

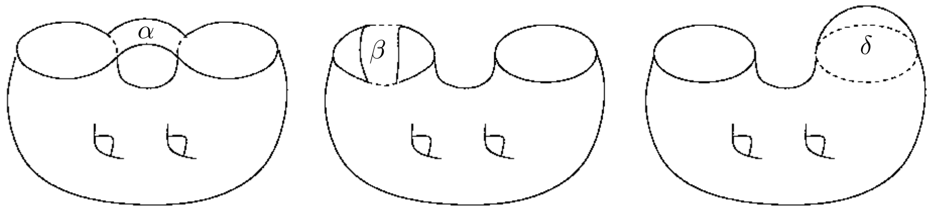

FIGURE 1. The maps α, β and δ

induced by gluing a strip which identifies arcs lying in different (for α) or the same (for β) boundary component(s) of S, and by gluing a disc (for δ). (See Figure 1.)

The proof of homological stability for mapping class groups presented here involves three simplicial spaces: an arc complex for each of α and β, and a disc space for δ. In the remainer of the lecture, we will define the two arc complexes and study their properties. These are the complexes $\mathcal{O}^2$ and $\mathcal{O}^1$ defined below.

3. The ordered arc complex

We will work here with collections of disjointly embedded arcs in a surface. We say that a collection of arcs $\langle a_0, \dots, a_p\rangle$ is *non-separating* if its complement $S\backslash(a_0 \cup \dots \cup a_p)$ is connected.

Given a surface S with points b_0, b_1 in its boundary, define $\mathcal{O}(S, b_0, b_1)$ to be the simplicial complex with

vertices =: isotopy classes of non-separating arcs with boundary $\{b_0, b_1\}$

p–**simplices =:** non-separating collections of $p+1$ distinct isotopy classes of arcs $\langle a_0, \dots, a_p\rangle$ disjointly embeddable (away from b_0, b_1) in such a way that the anticlockwise ordering of $a_0, \dots, a_p$ at b_0 agrees with the clockwise ordering at b_1.

Up to isomorphism, there are two such complexes:

$\mathcal{O}^1(S)$ **=:** $\mathcal{O}(S, b_0, b_1)$ with b_0, b_1 on the same boundary component,
$\mathcal{O}^2(S)$ **=:** $\mathcal{O}(S, b_0, b_1)$ with b_0, b_1 on different boundary components.

The mapping class group $\Gamma(S_{g,r}) = \Gamma_{g,r}$ acts on $\mathcal{O}^1(S_{g,r})$ and $\mathcal{O}^2(S_{g,r})$. We give now the properties of this action that are key for us.

Property 1. *$\Gamma(S)$ acts transitively on p–simplices of $\mathcal{O}^i(S)$ for each p.*

PROOF SKETCH. (See the proof of [**37**, Prop 2.2 (1)] for more details.) Consider the case $\mathcal{O}^1(S)$ and let $\sigma_p = \langle a_0, \dots, a_p\rangle$ be a p–simplex. Then $S\backslash\sigma_p = S\backslash\{a_0, \dots, a_p\}$ is a connected surface with $r+p+1$ boundary components (see Figure 2) and Euler characteristic $\chi(S)+p+1$. Thus it must have genus $g-p-1$ and any two $S\backslash\sigma_p$ and $S\backslash\sigma'_p$ are diffeomorphic. Moreover, a diffeomorphism $S\backslash\sigma_p \to S\backslash\sigma'_p$ can be chosen so that it identifies the arcs of σ_p with those of σ'_p and thus glues back to a diffeomorphism of S mapping σ_p to σ'_p. □

Property 2. *There is an isomorphism $Stab_{\mathcal{O}}(\sigma_p) \cong \Gamma(S\backslash\sigma_p)$, i.e.*

$$Stab_{\mathcal{O}^1(S_{g,r})}(\sigma_p) \cong \Gamma_{g-p-1,r+p+1} \quad \textit{and} \quad Stab_{\mathcal{O}^2(S_{g,r})}(\sigma_p) \cong \Gamma_{g-p,r+p-1}.$$

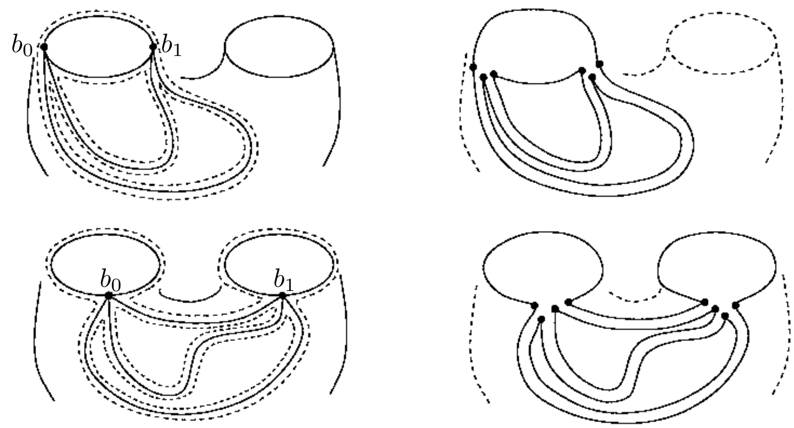

FIGURE 2. Cutting along simplices of $\mathcal{O}^1$ and $\mathcal{O}^2$

The proof of this second property will rely on two useful results of manifold toplogy which we first state.

For M, N two manifolds, let $\mathrm{Emb}(N, M)$ denote the space of embeddings of M into N (with the C^∞ topology).

Theorem 2.1 (*Fibering Theorem*, Palais and Cerf [**30, 6**])**.** *Let M, N be manifolds and V a compact submanifold of M. Then the restriction map*

$$\mathrm{Emb}(M, N) \longrightarrow \mathrm{Emb}(V, N)$$

is a locally trivial fibration.

For $M = N$, one gets a fibration

$$\mathrm{Diff}(M \,\mathrm{rel}\, V) \to \mathrm{Diff}(M) \to \mathrm{Emb}(V, M)$$

See [**6**, Chap. II 3.4.2], or the exercices for a sketch of proof.

This theorem has the following very useful corollary (see [**30**, Sec. 5]):

Theorem 2.2 (*Isotopy extension theorem*)**.** *For a compact submanifold $V \subset M$, and any path $\gamma : I \to \mathrm{Emb}(V, M)$ with $\gamma(0)$ the inclusion, there exits a path $\hat{\gamma} : I \to \mathrm{Diff}(M)$ with $\hat{\gamma}(0) = id$ and $\hat{\gamma}(t)|_V = \gamma(t)$.*

Note in particular that the theorem produces a diffeomorphism of M (namely $\hat{\gamma}(1)$) which takes V to its isotoped image $\gamma(1)(V)$ in M.

SKETCH OF PROOF OF PROPERTY 2. (See the proof of [**37**, Prop 2.2 (2)] for more details.) Suppose $\sigma_p = \langle a_0, \dots, a_p \rangle$. There is a map

$$\Gamma(S \backslash \sigma_p) \to \mathrm{Stab}_{\mathcal{O}}(\sigma_p)$$

induced by gluing the surface back together along the arcs.

Surjectivity: Consider $\phi \in \mathrm{Stab}_{\mathcal{O}}(\sigma_p)$. So $\phi(a_i)$ is isotopic to $a_{\theta(i)}$ for all i for some permutation $\theta \in \Sigma_{n+1}$. Applying the isotopy extension theorem to the isotopy $\phi(a_0) \simeq a_{\theta(0)}$ gives a diffeomorphism ψ_0 of S with $\psi_0 \simeq id$ and $\psi_0(\phi(a_0)) = a_{\theta(0)}$. Hence we can replace ϕ by the isotopic diffeomorphism $\phi_0 = \psi_0 \circ \phi$ which satisfies that $\phi_0(a_0) = a_{\theta(0)}$. Proceed in the same way with the other arcs, one by one, away from the arcs already dealt with. Hence we can replace ϕ with a diffeomorphism that fixes the arcs, possibly up to a permutation. But the permutation must be trivial as ϕ fixes the boundary. Then ϕ can be reinterpreted as a diffeomorphism of $S\backslash\sigma_p$.

Injectivity: Suppose $p = 0$ for simplicity. We would like to show that the map $\Gamma(S\backslash I) \to \Gamma(S)$ is injective for any (non-separating) arc I. A relative version of Theorem 2.1 above gives a fibration

$$\mathrm{Diff}(S \text{ rel } \partial S \cup I) \to \mathrm{Diff}(S \text{ rel } \partial S) \to \mathrm{Emb}^{\partial}(I, S)$$

where $\mathrm{Emb}^{\partial}(I, S)$ denotes the space of embeddings of an arc I in S with ∂I mapping to chosen points $A, B \in \partial S$. By [**15**, Thm 5], each component of $\mathrm{Emb}^{\partial}(I, S)$ is contractible. The result then follows from looking at the long exact sequence of homotopy groups of the fibration for the component of the non-separating arcs. □

Note that gluing strips identifying arcs of the boundary induce maps (still denoted α and β)

$$\alpha : \mathcal{O}^2(S_{g,r}) \to \mathcal{O}^1(S_{g+1,r-1}) \text{ and } \beta : \mathcal{O}^1(S_{g,r}) \to \mathcal{O}^2(S_{g,r+1})$$

(see Figure 3).

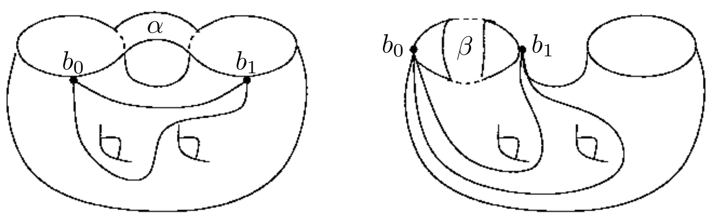

FIGURE 3. The maps α, β on the complexes of arcs

Property 3. *The map α on the complexes induces β on stabilizers and vice versa.*

(See [**37**, Prop 2.3] for a more detailed formulation.)

PROOF SKETCH. (See the proof of [**37**, Prop 2.3] for more details.) Patching together Figure 2 and Figure 3 shows how the strips defining α and β glue on the boundary components of the surface cut along the arcs of a simplex σ: in the case of α, the strip is glued on the cut surface to a unique boundary component of $S\backslash\sigma$, that is it induces β, while for β it is glued to two different boundaries of $S\backslash\sigma$, that is it induces α. □

Property 4. *Let $S = S_{g,r}$ and let $S_\alpha \cong S_{g+1,r-1}$ and $S_\beta \cong S_{g,r+1}$ denote S union a strip glued via α and β respectively as in Figure 3. Then for any vertex σ_0 of $\mathcal{O}^i(S)$, there are curves c_α, c_β (given in Figure 4) in S_α and S_β such that conjugation by Dehn twists t_{c_α} and t_{c_β} along these curves fits into commutative diagrams*

$$\begin{array}{ccc} St_{\mathcal{O}^2}(\sigma_0) & \longrightarrow & St_{\mathcal{O}^1}(\alpha(\sigma_0)) \\ \big\downarrow & \swarrow t_{c_\alpha} & \big\downarrow \\ \Gamma(S) & \xrightarrow{\alpha} & \Gamma(S_\alpha) \end{array} \qquad \begin{array}{ccc} St_{\mathcal{O}^1}(\sigma_0) & \longrightarrow & St_{\mathcal{O}^2}(\beta(\sigma_0)) \\ \big\downarrow & \swarrow t_{c_\beta} & \big\downarrow \\ \Gamma(S) & \xrightarrow{\beta} & \Gamma(S_\beta) \end{array}$$

i.e. there are conjugations $St_{\mathcal{O}^1}(\alpha(\sigma_0)) \sim_{t_{c_\alpha}} \alpha(\Gamma(S))$ *in* $\Gamma(S_\alpha)$ *relative to* $\alpha(St_{\mathcal{O}^2}(\sigma_0))$, *and* $St_{\mathcal{O}^2}(\beta(\sigma_0)) \sim_{t_{c_\beta}} \beta(\Gamma(S))$ *in* $\Gamma(S_\beta)$ *relative to* $\beta(St_{\mathcal{O}^1}(\sigma_0))$.

Note that it follows from the existence of these conjugation that the maps $\alpha\colon \Gamma(S) \to \Gamma(S_\alpha)$ and $\beta\colon \Gamma(S) \to \Gamma(S_\beta)$ are injective, as we already know by Property 2 that the stabilizers of vertices are abstractly isomorphic to $\Gamma(S)$.

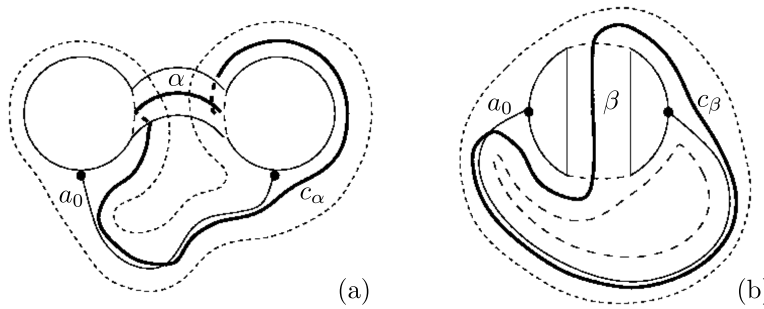

FIGURE 4. The curves c_α and c_β of Property 4 for $\sigma_0 = \langle a_0 \rangle$

PROOF SKETCH. (See the proof of [**37**, Prop 2.4] for more details.) Let $\sigma_0 = \langle a_0 \rangle$ be as in Figure 4. Check that the left Dehn twist along c_α (resp. c_β) takes a_0 to an arc whose stabilizer is $\Gamma(S)$. (Hint: to identify the stabilizer with $\Gamma(S)$, thiken a neighborhood of ∂S union the arc.) □

Property 5. $\mathcal{O}^i(S_{g,r})$ *is* $(g-2)$*–connected.*

The proof of this last property is the topic of Lecture 4.

4. Curve complexes and disc spaces

When the surface has no boundary components, one cannot work with arcs. Harer originally worked with embedded curves instead of arcs. These are not quite as well-behaved as arcs (see the exercices below). Randal-Williams introduced instead a space of embedded discs which gives a slightly better stability range for closing the last boundary component, but does require working with a semi-simplicial space build out of spaces of embedded discs, instead of a simplicial complex as we have done so far, because isotopy classes of embedded discs are not so interesting... (See [**32**, Sect. 10] or [**37**, Sect. 5].)

5. Exercices

1) Define the n-simplex Δ^n and its boundary $\partial\Delta^n$ as simplicial complexes.
2) Show that $\mathcal{O}(S, b_0, b_1)$ is indeed a simplicial complex.
3) Let S_g be a closed surface of genus g and let $\mathcal{C}_0(S_g)$ be the simplicial complex whose p-simplices are non-separating collections $\{[C_0], \dots, [C_p]\}$ of isotopy classes of disjoint embedded *curves* in S_g, i.e. embeddings $S^1 \hookrightarrow S_g$, for which there exist representatives whose images $\{C_1, \dots, C_p\}$ satisfy
   - $C_i \cap C_j = \emptyset$ whenever $i \neq j$,
   - the complement $S \setminus (C_0 \cup \cdots \cup C_p)$ is connected.

   (a) Construct a natural action of $\Gamma_g = \Gamma(S_g)$ on $\mathcal{C}_0(S_g)$.

(b) Show that this action is transitive on p-simplices for each $p \geq 0$. (Hint: as in the proof of Property 1 above, first use the classification of surfaces to prove that the complement of two such collections of $p+1$ circles are diffeomorphic.)

(c) Construct a map from $\Gamma_{g-1,2}$ to the stabilizer of a vertex of $\mathcal{C}_0(S_g)$ and prove that it is surjective using the isotopy extension theorem as in the proof of Property 2. Is it injective?

4) Complete the proof of Property 4.

5) Let M and N be smooth manifolds. Denote by $C^0(M,N)$ the set of continuous maps from M to N, and denote by $\mathrm{Emb}^0(M,N) \subset C^0(M,N)$ the subset consisting of topological embeddings. Inductively, for $k>0$ denote by $C^k(M,N) \subset C^{k-1}(M,N)$ the subset consisting of differentiable maps $f: M \to N$ for which the induced map on tangent spaces Tf is in $C^{k-1}(TM,TN)$. We topologize the set $C^k(M,N)$ inductively: we use the compact-open topology on $C^0(M,N)$, and we note that $D: C^k(M,N) \to C^{k-1}(TM,TN)$ is an inclusion, so we give $C^k(M,N)$ the subspace topology. We let $C^\infty(M,N)$ denote the inverse limit of the $C^k(M,N)$.

Denote by $\mathrm{Emb}^k(M,N) \subset C^k(M,N)$ the subspace consisting of topological embeddings e for which Te is in $\mathrm{Emb}^{k-1}(TM,TN)$, and write the inverse limit as $\mathrm{Emb}(M,N)$. Denote by $\mathrm{Diff}(M) \subset \mathrm{Emb}(M,M)$ the subspace consisting of those invertible maps ϕ for which $\phi^{-1} \in \mathrm{Diff}(M)$.

(a) Prove that a sequence of maps $f_n \in C^1(\mathbb{R},\mathbb{R})$ converges if and only if the sequences $f_n \in C^0(\mathbb{R},\mathbb{R})$ and $f_n' \in C^0(\mathbb{R},\mathbb{R})$ converge.

(b) Prove that this inclusion $\mathrm{Emb}^k([0,1],\mathbb{R}) \subset C^k([0,1],\mathbb{R})$ is open when $k=1$ but not when $k=0$.

(c) (difficult) In this exercise we will prove that if N is a compact submanifold of M, then the restriction map

$$j: \mathrm{Emb}(M,\mathbb{R}^n) \longrightarrow \mathrm{Emb}(N,\mathbb{R}^n)$$

is locally trivial fibration (i.e. a fibre bundle with structure group the full homeomorphism group of the fibre). This was first proved by Palais and Cerf, but we follow Lima [**22**].

(i) Given a $f \in \mathrm{Emb}(N,\mathbb{R}^n)$, show there is a neighbourhood U of f and a map $\xi: U \to \mathrm{Diff}(\mathbb{R}^n)$ such that $\xi(g) \circ f = g$. [Hint: note that $\mathrm{Diff}(\mathbb{R}^n)$ is an open subset of $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$.]

(ii) Hence construct a trivialisation of j over U.

Remark 2.3. Note that this implies it is in particular a Serre fibration, and hence the existence of a long exact sequence on homotopy groups

$$\cdots \to \pi_{n+1}F \to \pi_n \mathrm{Emb}(M,\mathbb{R}^n) \to \pi_n \mathrm{Emb}(N,\mathbb{R}^n) \to \pi_n F \to \cdots,$$

where F is the fiber of the restriction map over some base point $f \in \mathrm{Emb}(N,\mathbb{R}^n)$.

LECTURE 3

Homological stability: the spectral sequence argument

In this lecture, we present a spectral sequence argument (the one used by Randal-Williams in [**32**]) which allows to prove homological stability for the mapping class groups of surfaces in the range stated in Lecture 1. We will consider the two spectral sequences associated to a double chain complex build from a pair of groups acting on a pair of spaces. We will then use all the geometric properties presented in Lecture 2 to analyze these spectral sequences.

1. Double complexes associated to actions on simplicial complexes

To X a simplicial complex, one can associate a chain complex $(\tilde{C}_*, \partial)$ (its augmented cellular complex) which computes the reduce homology of its realization $|X|$. It has

- $\tilde{C}_p(X) = \mathbb{Z}X_p$, the free module on the set of p-simplices
- $\tilde{C}_{-1}(X) = \mathbb{Z}$

with boundary maps coming from the face maps and the augmentation.

We are interested here in simplicial complexes X admiring a simplicial G–action for some group G. For such, one can construct a double complex

$$E_*G \otimes_G \tilde{C}_*(X)$$

where $\cdots \to E_qG \to \cdots \to E_0G \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. This is the basic double complex commonly used to prove homological stability results. We will use here a relative version of it, which we construct now.

Suppose Y in addition is a simplicial complex with a simplicial H–action and $f : X \to Y$ a simplicial map equivariant with respect to a map $G \to H$. Then we get a map of double complexes

$$F : E_*G \otimes_G \tilde{C}_*(X) \longrightarrow E_*H \otimes_H \tilde{C}_*(Y)$$

(The two examples of interest to us are the maps $\alpha : \mathcal{O}^2 \to \mathcal{O}^1$ and $\beta : \mathcal{O}^1 \to \mathcal{O}^2$ of Lecture 2 and their companion maps on the mapping class groups.) We will use the double complex

$$C_{p,q} = (E_{q-1}G \otimes_G \tilde{C}_p(X)) \oplus (E_qH \otimes_H \tilde{C}_p(Y))$$

with horizontal differential $(a \otimes b, a' \otimes b') \mapsto (a \otimes \partial b, a' \otimes \partial b')$
and vertical differential $(a \otimes b, a' \otimes b') \mapsto (da \otimes b, da' \otimes b' + F(a \otimes b))$
(i.e. for each p we take the mapping cone of F in the q–direction).

The horizontal and vertical filtrations of such a double complex give two spectral sequences, both converging to the homology of the total complex. We now analyze these two spectral sequences.

2. The spectral sequence associated to the horizontal filtration

As $E_{q-1}G$ and E_qH are free G– (resp. H–)modules and the horizontal differential is that of $\tilde{C}(X)$ and $\tilde{C}(Y)$, taking first the homology in the p–direction computes copies of the reduced homology of X plus that of Y.

In particular, if X is $(c-1)$–connected and Y is c–connected, the E^1–term of the horizontal spectral sequence, which is the homology of $C_{p,q}$ with respect to the horizontal differential, is 0 in the range $p+q \leq c$ (noting that $\tilde{C}_p(X)$ only contributes to $C_{p,q}$ when $q > 0$).

It follows that the other spectral sequence, obtained using the vertical filtration instead, also converges to 0 in the range $p+q \leq c$.

3. The spectral sequence associated to the vertical filtration

For each p, the module $\tilde{C}_p(X) = \mathbb{Z}X_p$ is a G-module, which decomposes as a sum of modules corresponding to the orbits of the G-action on X_p. (We define here $X_{-1} = \{*\}$ with the trivial action.) Given an orbit $o \in O(X_p)$, the set of orbits of X_p, we let $\mathrm{Stab}(o) \leq G$ denote the stabilizer subgroup of some chosen simplex σ_p in the orbit o. Assuming that the stabilizer of a simplex fixes the simplex pointwise, we can rewrite the G-module $\tilde{C}_p(X)$ as

$$\tilde{C}_p(X) \cong \bigoplus_{o \in O(X_p)} G \otimes_{\mathrm{Stab}(o)} \mathbb{Z}$$

The chain complex $E_*(G) \otimes_G \tilde{C}_p(X)$, where p is now fixed, computes the homology of G with coefficients in that module. (This is the definition of the homology of a group with twisted coefficients.)

We will use a relative version (left as an exercise) of the following well-known lemma (see e.g. [**5**, III 6.2]):

Lemma 3.1 (*Shapiro's lemma*)**.** *Let $H < G$ be groups and M be an H–module, with $G \otimes_H M$ the induced G-module. Then*

$$H_*(G, G \otimes_H M) \cong H_*(H, M)$$

Hence for any G–simplicial complex X as above, we have, for each p, that

$$H_*(E_*G \otimes \tilde{C}_p(X)) \cong \bigoplus_{o \in O(X_p)} H_*(\mathrm{Stab}(o)).$$

The E^1–term of the vertical spectral sequence is the homology of the double complex $C_{p,q} = (E_{q-1}G \otimes_G \tilde{C}_p(X)) \oplus (E_qH \otimes_H \tilde{C}_p(Y))$ with respect to the vertical differential. This is the relative homology group

$$E^1_{p,q} = H_q\big(E_*H \otimes_H \tilde{C}_p(Y), E_*G \otimes_G \tilde{C}_p(X)\big)$$

as the columns of $C_{p,q}$ are the mapping cones of the map F (with p fixed).

Now if the actions of G and H are transitive on X and Y (which is the case we are interested in), a relative version of Shapiro's lemma identifies the E^1–term of the vertical spectral sequence with

$$E^1_{p,q} = H_q(\mathrm{Stab}_Y(\sigma_p), \mathrm{Stab}_X(\sigma_p))$$

where $\mathrm{Stab}_X(\sigma_p)$ and $\mathrm{Stab}_Y(\sigma_p)$ are the stabilizers in X and Y of some chosen p–simplex σ_p of X and its image in Y. Note that this formulation in the case $p = -1$ gives

$$E^1_{-1,q} = H_q(H, G)$$

4. The proof of stability for surfaces with boundaries

Recall from the previous lecture the maps

$$\alpha_g\colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r}) \quad \text{and} \quad \beta_g\colon \Gamma(S_{g,r}) \to \Gamma(S_{g,r+1}).$$

Denote by $H(\alpha_g)$ the relative homology group $H(\Gamma_{g+1,r}, \Gamma_{g,r+1}; \mathbb{Z})$ corresponding to the map α_g, and $H(\beta_g)$ the relative homology group $H(\Gamma_{g,r+1}, \Gamma_{g,r}; \mathbb{Z})$ corresponding to β_g. (The number of boundaries r here will not play a role here.) Harer's improved stability theorem (Theorem 1.3) can be restated as follows [exercise]:

Theorem 3.2. (1) $H_i(\alpha_g) = 0$ *for* $i \leq \frac{2g+1}{3}$ *and* (2) $H_i(\beta_g) = 0$ *for* $i \leq \frac{2g}{3}$.

The proof of this theorem uses the spectral sequences just described in the case of the maps α and β of Lecture 2. The argument will need Properties 1–5 of Lecture 2.

PROOF. We prove the theorem by induction on g. To start the induction, note that statements (1) for genus 0 and (2) for genus 0,1 are trivially true as they are just concerned with H_0. Let (1_g) and (2_g) denote the truth of (1) and (2) in the theorem for genus g. The induction will go in two steps:

Step 1: For $g \geq 1$, $(2_{\leq g})$ implies (1_g).
Step 2: For $g \geq 2$, $(1_{<g})$ and (2_{g-1}) imply (2_g).

Step 1: We consider the spectral sequence described above for the actions of $G = \Gamma_{g,r+1}$ on $X = \mathcal{O}^2(S_{g,r+1})$ and of $H = \Gamma_{g+1,r}$ on $Y = \mathcal{O}^1(S_{g+1,r})$ with the homomorphism $\phi\colon G \to H$ and the map $f\colon X \to Y$ both induced by the map $\alpha\colon S_{g,r+1} \to S_{g+1,r}$ of Figure 3. As the action is transitive in both cases (Property 1), we get, as explained above, that the vertical spectral sequence has the form $E^1_{p,q} = H_q(\mathrm{Stab}_Y(\alpha(\sigma_p)), \mathrm{Stab}_X(\sigma_p))$, with σ_p a chosen p-simplex of X and $\alpha(\sigma_p)$ its image in Y. That is, when $p = -1$, we have

$$E^1_{-1,q} = H_q(\Gamma_{g+1,r}, \Gamma_{g,r+1}) = H_q(\alpha_g)$$

which are the groups we are interested in. By Properties 2 and 3, the other groups are identified with

$$E^1_{p,q} = H_q(\beta_{g-p}) \quad \text{for} \quad p \geq 0.$$

Hence we will be able to apply induction to these terms of the spectral sequence. We want to deduce that $E^1_{-1,q} = 0$ for $q \leq \frac{2g+1}{3}$. This will follow from the following three claims:

Claim 1: $E^\infty_{-1,q} = 0$ for $q \leq \frac{2g+1}{3}$.

Claim 2: The E^1–term is as in Figure 1, i.e. there are no possible sources of differentials to kill classes in $E^1_{-1,q}$ with $q \leq \frac{2g+1}{3}$, except possibly for $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ when $q = \frac{2g+1}{3}$ (i.e. when the fraction is an integer).

Claim 3: The map $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ is the 0-map.

Claims 1 and 2 imply immediately that $E^1_{-1,q} = 0$ for $q < \frac{2g+1}{3}$ as "it must die by E^∞" (Claim 1) and "nobody can kill it" (Claim 2). Claim 3 gives that this also

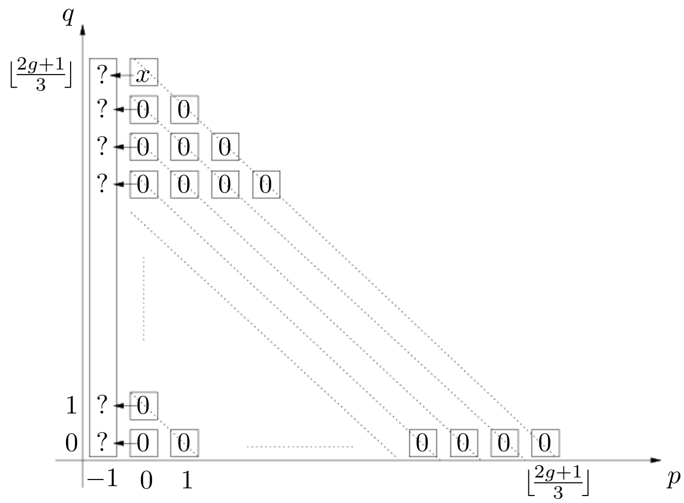

FIGURE 1. Spectral sequence for Step 1. The possible sources of differentials for the "?" are along the dotted diagonals.

holds when $q = \frac{2g+1}{3}$ as the only differential with a possibly non-trivial source is the zero map, and hence won't kill anything in the target.

Proof of Claim 1: By Property 5, X is $(g-2)$–connected and Y is $(g-1)$–connected. By the analysis of the horizontal spectral sequence above, and the fact that both spectral sequences converge to the same target, we get that $E^\infty_{p,q} = 0$ for $p+q \le g-1$. In particular, $E^\infty_{-1,q} = 0$ for $q \le g$, and $\frac{2g+1}{3} \le g$ when $g \ge 1$.

Proof of Claim 2: The sources of differentials to $E^1_{-1,q}$ are the terms $E^{p+1}_{p,q-p}$ for $p \ge 0$. As $E^1_{p,q} = H_q(\beta_{g-p})$ when $p \ge 0$, by induction we know that $E^1_{p,q} = 0$ when $q \le \frac{2(g-p)}{3} = \frac{2g-2p}{3}$ and $p \ge 0$. Hence $E^1_{p,q-p} = 0$ for $q \le \frac{2g+p}{3}$ and $p \ge 0$, i.e. they are all 0 for any $p \ge 0$ if $q \le \frac{2g}{3}$ or for $p \ge 1$ if $q = \frac{2g+1}{3}$.

Proof of Claim 3: The map $d^1 \colon E^1_{0,q} \to E^1_{-1,q}$ is the map

$$H_q(\mathrm{Stab}_{\mathcal{O}^1}(\alpha(\sigma_0)), \mathrm{Stab}_{\mathcal{O}^2}(\sigma_0)) \to H_q(\Gamma(S_\alpha), \Gamma(S))$$

for σ_0 a vertex of $\mathcal{O}^2(S)$ with $S = S_{g,r+1}$ and $S_\alpha = S_{g+1,r}$ the source and targets of α. This map comes precisely from mapping the top row to the bottom row of the first square in Property 4. It is tempting to deduce immediately from Property 4 that the map is zero but it does require a little extra argument: From the diagram, it follows that the map factors as

$$H_q(\mathrm{Stab}_{\mathcal{O}^1}(\alpha(\sigma_0)), \mathrm{Stab}_{\mathcal{O}^2}(\sigma_0)) \xrightarrow{\partial} H_{q-1}(\mathrm{Stab}_{\mathcal{O}^2}(\sigma_0)) \to H_q(\Gamma(S_\alpha), \Gamma(S))$$

where the second map "crosses" with the Dehn twist t_{c_α}. (See Lemma 2.5 in [**37**].) The triviality of the map then follows from the fact that c_α can be conjugated to a curve in S fixing the support of $\mathrm{Stab}_{\mathcal{O}^2}(\sigma_0)$. (This uses that c_α is non-separating in the neighborhood pictured in Figure 4.)

For Step 2, the argument is essentially the same, though proving Claim 3 is a little more subtle because c_β is separating in the corresponding neighborhood in

Figure 4. It is necessary to use induction an extra time here to finish the argument. (See the proof of [**37**, Thm 3.1 (2)] for the details.) □

5. Closing the boundaries

To prove that the map $\delta : \Gamma_{g,1} \to \Gamma_{g,0}$ also induces a homology isomorphism in a range, one uses a similar spectral sequence for the action of the mapping class groups on the disc semi-simplicial space of [**32**] (or the curve complex as in [**16**]) and compare the spectral sequence for each case (a comparison argument going back to [**20**]). See [**37**, Sect 5] for details.

6. Exercises

1) State and prove the relative version of Shapiro's lemma needed in the analysis of the vertical spectral sequence above.
2) Show that Theorem 3.2 implies the first part of Theorem 1.3 of Lecture 1.
3) This exercise sets up a way of approaching the proof of homological stability for the symmetric groups.
   (a) Let Δ^n denote the n–simplex. It can be thought of as a simplicial complex with $n+1$ vertices $\{0,\dots,n\}$, the set of p-simplices being the set of subsets of $\{0,\dots,n\}$ of cardinality $p+1$. Consider the action of the symmetric group Σ_{n+1} on Δ^n induced by permuting the vertices. Show that the action is transitive on the set of p-simplices for each p. What is the stabilizer of a vertex? of a p-simplex in general? (Note that the stabilizer of a simplex is the subgroup of symmetries that map the simplex to itself, as a *set* of vertices.)
   (b) Replace Δ^n in the above exercise by the semi-simplicial set (=simplicial set without degeneracies) X_{n+1} whose p-simplices are injective maps $\sigma : \{0,1,\dots,p\} \to \{0,\dots,n\}$. Is the action of Σ_{n+1} still transitive on the set of p-simplices for any p? What is the stabilizer of a p-simplex in this case?
   (c) For a semi-simplicial set $Y = Y_*$, let $||Y|| = \coprod_{p\geq 0} \Delta^p \times Y_p/\sim$ denote its realization, where the equivalence relation $\sim$ is induced by the face relations. For X_1, X_2, X_3 as in (b), show that $||X_1|| = *$, $||X_2|| \cong S^1$ and that $||X_3||$ is simply-connected.
   (d) Consider the cellular chain complex $C_*(X_{n+1})$ with its induced action of Σ_{n+1}. Pick a free resolution $E_*\Sigma_{n+1}$ of $\mathbb{Z}$ considered as a $\mathbb{Z}[\Sigma_{n+1}]$-module with a trivial action of Σ_{n+1}. Now consider the double complex $C_{p,q} = C_p(X_{n+1}) \otimes_{\Sigma_{n+1}} E_q\Sigma_{n+1}$. Write down the E^1-terms of the associated spectral sequences in both filtrations.

   Remark 3.3. Given that X_n is $(n-2)$-connected (see [**34**, Prop 3.2] or [**21**]), one can use the above spectral sequences to prove Nakaoka's stability theorem: the map $H_q(\Sigma_n) \to H_q(\Sigma_{n+1})$ induced by the inclusion of groups is an isomorphism for $q \leq n/2$.

4) The action of the mapping class group Γ_g on the Teichmüller space $\mathcal{T}_g$ satisfies that the stabilizers of points are finite (most often trivial) groups (see e.g. [**9**, 12.1]). Let $C_*(\mathcal{T}_g,\mathbb{Q})$ denote the singular chain complex of $\mathcal{T}_g$ with rational coefficients. The action of the mapping class group gives rise, as above, to a double complex $C_*(\mathcal{T}_g,\mathbb{Q}) \otimes_{\Gamma_g} E_*\Gamma_g$, with $E_*\Gamma_g$ now a free resolution of $\mathbb{Q}$ as a trivial $\mathbb{Q}[\Gamma_g]$-module. Using the spectral sequences associated to the double

complex, show that the coarse moduli space $\mathcal{T}_g/\Gamma_g$ is rationally a classifying space for Γ_g, i.e. that $H_*(\mathcal{T}_g/\Gamma_g, \mathbb{Q}) \cong H_*(B\Gamma_g, \mathbb{Q})$.

LECTURE 4

Homological stability: the connectivity argument

In this lecture, we give a sketch proof of the last ingredient of the proof of homological stability for mapping class groups of surfaces with boundaries, namely the fact that the complex $\mathcal{O}(S; b_0, b_1)$ of Lecture 2 is $(g-2)$-connected for any surface S of genus g. To close the surfaces, an analogous statement is needed for the disc space or curve complex, which is not presented here. We refer instead [**32**], [**37**], or [**20**] for that case.

1. Strategy for computing the connectivity of the ordered arc complex

We will prove that $\mathcal{O}(S; b_0, b_1)$ is highly connected by working our way through the following sequence of smaller and smaller simplicial complexes:

$$\mathcal{A}(S,\Delta) \overset{i_1}{\hookleftarrow} \mathcal{B}(S,\Delta_0,\Delta_1) \overset{i_2}{\hookleftarrow} \mathcal{B}_0(S,\Delta_0,\Delta_1) \overset{i_3}{\hookleftarrow} \mathcal{O}(S,b_0,b_1)$$

where $\Delta, \Delta_0, \Delta_1$ are sets of points in ∂S and

$\mathcal{A}(S,\Delta)$ is the simplicial complex whose vertices are isotopy classes of all non-trivial arcs in S with boundary in Δ. A p–simplex of $\mathcal{A}(S,\Delta)$ is a collection of $p+1$ distinct isotopy classes of arcs $\langle a_0,\dots,a_p\rangle$ representable by arcs with disjoint interiors.

$\mathcal{B}(S,\Delta_0,\Delta_1) \subset \mathcal{A}(S,\Delta_0\cup\Delta_1)$ is the subcomplex of arcs having each one boundary point in Δ_0 and one in Δ_1.

$\mathcal{B}_0(S,\Delta_0,\Delta_1) \subset \mathcal{B}(S,\Delta_0,\Delta_1)$ is the subcomplex of non-separating collections.

$\mathcal{O}(S,b_0,b_1) \subset \mathcal{B}_0(S,\{b_0\},\{b_1\}$ is the ordered subcomplex (defined in Lecture 2, Section 3).

The largest complex $\mathcal{A}(S,\Delta)$ is contractible in most cases, and one slowly looses connectivity as one goes down to smaller and smaller subcomplexes.

The connectivity arguments used are of three types:

(1) *direct calculation* showing contractibility,
(2) exhibition of a complex as a *suspension* (or wedge of such) of a "previous" complex,
(3) *inductive deduction* from the connectivity of a larger complex.

The argument for the connectivity of $\mathcal{A}(S,\Delta)$ is a mix of type (1) and (2), the deduction along i_1 in the sequence is the most intricate argument and is a mix of the three types of arguments, while deduction along i_2 and i_3 are purely (and simpler) type (3) arguments.

We do here, to exemplify, a type (1) and a type (3) argument and refer to [**37**] for the complete proofs.

2. Contractibility of the full arc complex

Theorem 4.1. [**16**, Thm 1.5] *Suppose S is not a disc or a cylinder with Δ included in a single boundary of S. Then $\mathcal{A}(S,\Delta)$ is contractible.*

Sketch of proof of the main case, following Hatcher [**17**]. (See also the proof of [**37**, Thm 4.1] for a complete proof.) We will prove the theorem under the extra assumption that S has at most one point of Δ in each boundary component. Reducing to that case requires an extra (type (2)) argument.

Non-emptiness: Clear if $|\Delta| > 1$ by choosing any arc in the surface between two distinct points of Δ—such an arc is always non-trivial as the points lie on different boundary components. For $|\Delta| = 1$, we have assumed that S is not a disc or a cylinder, i.e. S has non-zero genus or at least three boundary components. In both cases, there are non-trivial arcs.

In a simplicial complex X, the *star* of a simplex σ is the union of all the simplices τ of X such that $\sigma \cup \tau$ is again a simplex of X. This is a contractible subcomplex.

Contraction: As $\mathcal{A}(S,\Delta)$ is non-empty, we can choose an arc $a \in \mathcal{A}(S,\Delta)$. We will contract the complex to the star of a.

Let σ be a simplex of $\mathcal{A}(S,\Delta)$. It is in the star of a precisely if the interior of the arcs of σ do not intersect a. If they do intersect a, the idea is to modify σ by cutting the arcs one by one at the intersection points with a, replacing each time an arc by one or two arcs with fewer intersection points with a. This is illustrated by the following figure:

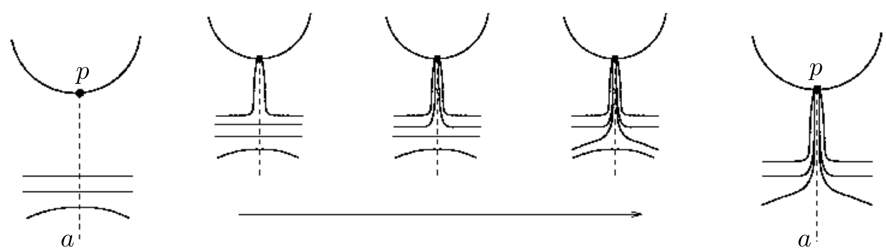

Figure 1. Retraction of $\mathcal{A}(S,\Delta)$ in the case of 3 crossings with a

More precisely, if σ has k intersection points with a, we produce a sequence of consecutive simplices $r_1(\sigma),\dots,r_k(\sigma)$ such that σ is a face of $r_1(\sigma)$ while a face of $r_k(\sigma)$ lies in the star of a. A continuous retraction of X can be defined from there using the barycentric coordinates (thought of as weights on the arcs of σ) to go through this sequence of simplices at a speed and with a weight on the arcs depending on those coordinates.

Note that, to be well-defined, the above argument requires that

- the intersection $\sigma \cap a$ is suitably independent of the chosen representative of σ
- the new arcs created during the deformation contain each time at least one non-trivial arc.

The first issue is addressed by choosing representative with minimal intersection with a, and the second is addressed by the additional assumption we worked with, namely that there is at most one point of Δ in each boundary component. $\square$

3. Deducing connectivity of smaller complexes

The connectivity of the subcomplex $\mathcal{B}(S, \Delta_0, \Delta_1)$ of arcs between two subsets Δ_0 and Δ_1 of Δ is deduced from that of $\mathcal{A}(S, \Delta)$ by a long argument... To be able to state the connectivity bound, we need a couple of definitions: Disjoint sets $\Delta_0, \Delta_1 \subset \partial S$ define a decomposition of ∂S into vertices (the points of $\Delta_0 \cup \Delta_1$), edges between the vertices, and circles without vertices. We say that an edge is *pure* if both its endpoints are in the same set, Δ_0 or Δ_1. We say that an edge is *impure* otherwise. Note that the number of impure edges is always even.

Theorem 4.2. [**16**, Thm 1.6] *The complex $\mathcal{B}(S, \Delta_0, \Delta_1)$ is $(4g+r+r'+l+m-6)$–connected, where g is the genus of S, r its number of boundary components, r' the number of components of ∂S containing points of $\Delta_0 \cup \Delta_1$, l is half the number of impure edges and m is the number of pure edges.*

See [**37**, Thm 4.3] for a detailed proof.

We will now use the connectivity of $\mathcal{B}(S, \Delta_0, \Delta_1)$ to deduce that of the subcomplex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ of non-separating simplices.

The *join* $X * Y$ of two simplicial complexes X and Y is the simplicial complex with vertices $X_0 \sqcup Y_0$ and a $(p+q+1)$–simplex $\sigma_X * \sigma_Y = \langle x_0, \dots, x_p, y_0, \dots, y_q \rangle$ for each p–simplex $\sigma_X = \langle x_0, \dots, x_p \rangle$ of X and q–simplex $\sigma_Y = \langle y_0, \dots, y_q \rangle$ of Y. Note that $|X * Y| = |X| * |Y|$, i.e. the realization of the join complex is the (topological) join of the realization of the two complexes. This follows from the fact that it is true for each pair of simplices.

Recall that a space (or simplicial complex) X is called *n–connected* if $\pi_i(X) = 0$ for all $i \leq n$ For $n = -1$, we use the convention that (-1)*–connected* means non-empty. (For $n \leq -2$, n–connected is a void property.)

The following proposition tells us how to compute the connectivity of a join in terms of the connectivity of the pieces.

Proposition 4.3. [**27**, Lem 2.3] *Consider the join $X = X_1 * \dots * X_k$ of k non-empty spaces. If each X_i is n_i–connected, then X is $\big(\big(\sum_{i=1}^{k}(n_i+2)\big)-2\big)$–connected.*

Theorem 4.4. [**16**, Thm 1.4] *If Δ_0, Δ_1 are two disjoint non-empty sets of points in ∂S, then the complex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ is $(2g+r'-3)$–connected, for g and r' as above.*

Proof. We prove the theorem by induction on the lexicographically ordered triple (g, r, q), where $r \geq r'$ is the number of components of ∂S and $q = |\Delta_0 \cup \Delta_1| \geq 2$. To start the induction, note that the theorem is true when $g = 0$ and $r' \leq 2$ for any $r \geq r'$ and any q, and more generally that the complex is non-empty whenever $r' \geq 2$ or $g \geq 1$.

So fix (S, Δ_0, Δ_1) satisfying $(g, r, q) \geq (0, 3, 2)$. Then $2g + r' - 3 \leq 4g + r + r' + l + m - 6$. Indeed, $r \geq 1$ and $l + m \geq 1$. Moreover we assumed that either $r \geq 3$ or $g \geq 1$.

Let $k \leq 2g + r' - 3$ and consider a map $f \colon S^k \to \mathcal{B}_0(S, \Delta_0, \Delta_1)$, which we may assume to be simplicial for some PL triangulation of S^k (see [**37**, Sect 6]). This map can be extended to a simplicial map $\hat{f}\colon D^{k+1} \to \mathcal{B}(S, \Delta_0, \Delta_1)$ by Theorem 4.2 and the above calculation, for a PL triangulation of D^{k+1} extending that of S^k. We call a simplex σ of D^{k+1} *regular bad* if $\hat{f}(\sigma) = \langle a_0, \dots, a_p\rangle$ and each a_j separates $S\backslash(a_0 \cup \dots \widehat{a_j} \cdots \cup a_p)$. Let σ be a regular bad simplex of maximal dimension p. Write $S\backslash\hat{f}(\sigma) = X_1 \sqcup \cdots \sqcup X_c$ with each X_i connected. By maximality of σ, $\hat{f}$ restricts to a map

$$\mathrm{Link}(\sigma) \longrightarrow J_\sigma = \mathcal{B}_0(X_1, \Delta_0^1, \Delta_1^1) * \cdots * \mathcal{B}_0(X_c, \Delta_0^c, \Delta_1^c)$$

where each Δ_ϵ^i is inherited from Δ_ϵ and is non-empty as the arcs of $\hat{f}(\sigma)$ are impure. Each X_i has $(g_i, r_i, q_i) < (g, r, q)$, so by induction $\mathcal{B}_0(X_i, \Delta_0^i, \Delta_1^i)$ is $(2g_i + r_i' - 3)$–connected. The Euler characteristic gives $\sum_i(2 - 2g_i - r_i) = 2 - 2g - r + p' + 1$, where $p'+1 \le p+1$ is the number of arcs in $\hat{f}(\sigma)$. We also have $\sum_i(r_i - r_i') = r - r'$, so $\sum(2g_i + r_i') = 2g + r' - p' + 2c - 3$. Now J_σ is $(\sum_i(2g_i + r_i' - 1) - 2)$–connected (using Proposition 4.3), that is $(2g + r' - p' + c - 5)$–connected. As $c \ge 2$ and $p' \le p$, we can extend the restriction of $\hat{f}$ to $\mathrm{Link}(\sigma) \simeq S^{k-p}$ to a map $F\colon K \to J_\sigma$ with K a $(k - p + 1)$–disc with boundary the link of σ. We modify $\hat{f}$ on the interior of the star of σ using $\hat{f} * F$ on $\partial\sigma * K \simeq \mathrm{Star}(\sigma)$. If a simplex $\alpha * \beta$ in $\partial\sigma * K$ is regular bad, β must be trivial since β does not separate $S\backslash\hat{f}(\alpha)$, so that $\alpha * \beta = \alpha$ is a face of σ. We have thus reduced the number of regular bad simplices of maximal dimension and the result follows by induction. □

From there, one can prove by a similar type of argument that the ordered subcomplex is also highly connected:

Theorem 4.5 (Property 5). *$\mathcal{O}(S, b_0, b_1)$ is $(g-2)$–connected.*

See [**37**, Thm 4.9] for a detailed proof.

4. Exercises

1) The complex $\mathcal{C}_0(S)$ of the exercises of Lecture 2 is a subcomplex of the complex $\mathcal{C}(S)$ with vertices all isotopy classes of non-trivial circles in S and p-simplices all disjointly embeddable collections of $p+1$ distinct isotopy classes. Assuming that $\mathcal{C}(S)$ is $(2g+r-4)$–connected if S has genus g and r boundary components, show that $\mathcal{C}_0(S)$ is $(g-2)$–connected.

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