ØKSENDAL

ITO STOCHASTIC CALCULUS FOR B.M. · STOCHASTIC CONTROL · OPTIMAL STOPPIN

↳ extension to semimartingales, Levy processes

↳ Malliavin calculus

↳ non semimartingales models (fractional BM)

↳ Anticipating calculus (insider trading)

MOTIVATING EXAMPLE

Population growth

* $N(t)$ : size of population at time $t$

$$\frac{dN(t)}{dt} = a\,N(t) \qquad a \text{ costant}$$

Solution : $N(t) = N_0 e^{at}$

If $a = a(t)$ continuous then we get

$$N(t) = N(0)\, e^{\int_0^t a(s)\,ds}$$

* More realistic model : $N(t)$ is subject to random perturbation

$a(t) = r(t) + \alpha$ "noise" $\qquad r(t)$ continuous

"noise" random fluctuation that we are unable to model including temperature changes

What is the solution then?

Suggestion: $N(t) = N(0)\, e^{\int_0^t r(s)\,ds + \int_0^t \alpha \text{ "noise" } ds}$ → WRONG !

The classical calculus doesn't applie with noise.

BASICS OF PROBABILITY

$\Omega$ a given set with $\omega$

A $\sigma$-algebra $\mathcal{F}$ of $\Omega$ is a collection of sets

If $X$ is $\mathcal{F}$-measurable we define $\mathbb{E}(X)$ as

$$\int_\Omega |X(\omega)| \, d\mathbb{P}(\omega)$$

If $\mathbb{E}[|X|]$ is finite we define $\mathbb{E}[X] = \int X(\omega) \, d\mathbb{P}(\omega)$

The space $L^2(\Omega, \mathbb{R})$ is the set of all RV st

$$\|X\|_{L^2(\Omega,\mathbb{R})} = \left(\mathbb{E}\left[X^2\right]\right)^{1/2} < +\infty$$

Two events $F, G$ are called independent if

$$\mathbb{P}(F \cap G) = \mathbb{P}(F)\,\mathbb{P}(G) \quad *$$

Independence of $\sigma$-algebras:

$\mathcal{H}, \mathcal{G}$ are independent if the same as $*$ holds $\forall \; F \in \mathcal{H}$ and $G \in \mathcal{G}$

Independence of RV. $X, Y$ are independent if $\sigma(X), \sigma(Y)$ are independent

$$\rightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$$

STOCHASTIC PROCESS A stochastic process is a collection $\{X_t\}_{t \in \mathcal{T}}$ of RV $X_t$.

For each $t$, $X_t(\omega)$ is a r.v.

For each $\omega$ the function $t \to X_t(\omega)$ is called a path of the process $X_t$

Usually $\mathcal{T} = [0,T]$ or $[0,\infty)$

EXAMPLE: Brownian Motion starting at $x \in \mathbb{R}^n$ is a stochastic process with the following properties

(i) $B_0(\omega) = x$ a.s.

(ii) $t \mapsto B_t(\omega)$ is continuous for almost all $\omega$

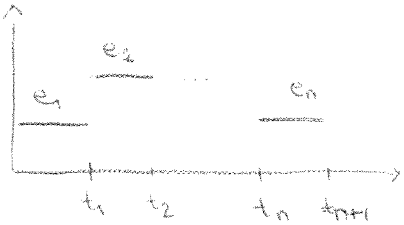
(iii) $B_t$ has independent, stationary increments

Increments $B_{t_2} - B_{t_1}$

Indep. Increments $(B_{t_1} - B_0, \; B_{t_2} - B_{t_1}, \dots)$ are independent for any choice $0 \le t_1 < t_2 < \dots$

Natural strategy for defining the Ito's integral:

$\rightarrow \quad f(s,\omega) = \sum_{i=1}^{m} e_i(\omega)\, \chi_{[t_i, t_{i+1})}(s)$ when we have simple integrands




NB. $e_i$ depends on $\omega$

$\rightarrow \quad \int f(s,\omega)\, dB_s(\omega) = \sum e_i(\omega)\left(B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right)$

EXAMPLE: With the above definition we have

$$* \quad I_1 = \int_0^T \Big( \sum \underbrace{B_{t_i}(\omega)}_{e_i(\omega)}\, \chi_{[t_i, t_{i+1})}(s) \Big)\, dB_s(\omega) =$$

$$= \sum_{i=1}^{m} B_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right)$$

$$* \quad I_2 = \int_0^T \sum_{i=1}^{m} B_{t_{i+1}}(\omega)\, \chi_{[t_i, t_{i+1})}(s)\, dB_s(\omega) =$$

$$= \sum_{i=1}^{m} B_{t_{i+1}}\left(B_{t_{i+1}} - B_{t_i}\right)$$

Note that the two integral are completely different. Infact

$$\mathbb{E}[I_1] = \sum \mathbb{E}\left[B_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right)\right] =$$

independent

$$= \sum \mathbb{E}[B_{t_i}]\, \underbrace{\mathbb{E}\left[B_{t_{i+1}} - B_{t_i}\right]}_{= 0} = 0$$

$$\mathbb{E}[I_2] = \sum \mathbb{E}\left[B_{t_{i+1}}\left(B_{t_{i+1}} - B_{t_i}\right)\right] =$$

this R.V. are not independent

$$= \sum \left\{ \mathbb{E}\left[(B_{t_{i+1}} - B_{t_i})^2\right] + \mathbb{E}\Big[\underbrace{B_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right)}_{= 0}\Big] \right\} =$$

$$= \sum_{i=1}^{m} \Delta t_i = T$$

Hence we can conclude that when defining Ito integral it does matter the choice that we make for the point in the subinterval.

<u>Step 3</u> : Define

$$\int_0^T f(t)\, dB_t = \lim_{n \to +\infty} \int_0^T \varphi_n(t)\, dB_t \quad (\text{limit in } L^2(\Omega, \mathbb{P}))$$

<u>NB</u> : This is a good definition because of the Ito isometry

EXAMPLE : What is $\int_0^T B_t\, dB_t$

By definition we have to approximate $B_t$ with sample functions and then consider the limit

$$\int_0^T B_t\, dB_t = \lim_{n \to +\infty} \int_0^T \varphi_n(t)\, dB_t$$

where $\varphi_n = \sum_i B_{t_i^{(n)}} \chi_{[t_i^{(n)}, t_{i+1}^{(n)})}$

$$= \lim_{n \to +\infty} \sum_i B_{t_i^{(n)}} \left( B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}} \right)$$

Some computations give

$$\int_0^T B_t\, dB_t = \frac{1}{2} B_T^2 - \frac{1}{2} T$$

Recall that $\int_0^T s\, ds = \frac{1}{2} T^2$. Then we conclude that the Ito integral doesn't behave like ordinary integrals

[illegible] (BM is not BV)