$\mathbf{SL}(2, \mathbf{R})$ and the upper half-plane

Assignment due March 5, 2014 (counts for 25 % of the grade)

Let $G = \mathbf{SL}(2, \mathbf{R})$ and $K = \mathbf{SO}(2)$.

(i) Write $\mathfrak{s}$ for the 2-dimensional linear space of the symmetric matrices in $\mathfrak{sl}(2, \mathbf{R})$.

Let $X \in \mathfrak{s}$ and assume $X \neq 0$. Show that $\det X < 0$ and

$$\exp(X) = \cosh \lambda I + \frac{\sinh \lambda}{\lambda} X$$

where $\lambda = \sqrt{-\det(X)}$. (Hint: one can reduce to the case of a diagonal matrix).

(ii) Write S for the set consisting of the symmetric matrices in $\mathbf{SL}(2, \mathbf{R})$ with positive eigenvalues.

Prove that S is a 2-dimensional submanifold of $\mathbf{SL}(2, \mathbf{R})$ and that $\exp : \mathfrak{s} \to S$ is a diffeomeorphism.

(iii) Show that $x \mapsto x^2$ is a diffeomorphism of S onto itself.

(iv) Prove that

$$(x, k) \mapsto xk, \qquad S \times K \to G$$

is a diffeomorphism. (Hint: show $gg^t \in S$ for all $g \in G$.)

Let $\mathcal{H}$ be the upper half-plane, i.e., $\mathcal{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ equipped with the topology induced by the usual topology on $\mathbf{C}$. For $g \in G$, we write $g_{i,j}$ with $1 \leq i, j \leq 2$ for the matrix coefficients of g,

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}.$$

(v) Let $g \in G$. Show that $g_{2,1} z + g_{2,2} \neq 0$ for every $z \in \mathcal{H}$.

(vi) Define $\phi : G \times \mathcal{H} \to \mathbf{C}$ to be the map given by

$$\phi(g, z) = \frac{g_{1,1} z + g_{1,2}}{g_{2,1} z + g_{2,2}} \qquad (g \in G, z \in \mathcal{H}).$$

Show that ϕ maps into $\mathcal{H}$, and that it gives a smooth left-action of G on $\mathcal{H}$. We write it as $\phi(g, z) = g \cdot z$.

(vii) Prove that the action is transitive.

(viii) Prove that

$$X \mapsto \exp(X) \cdot i, \qquad \mathfrak{s} \to \mathcal{H}$$

is a diffeomorphism.