Actions on the unit disk

Assignment due March 3, 2015
(counts for 25 % of the grade)

Let

$$G = \left\{ g = g_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \Big| \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\},$$

and $P = \{g_{\alpha,\beta} \in G \mid \alpha + \beta \in \mathbb{R}\}$, $A = \{g_{\alpha,\beta} \in P \mid \alpha \geq 1\}$.

(i) Show that G (with matrix multiplication) is a Lie group, and that P and A are Lie subgroups.

(ii) Determine an element Y in the Lie-algebra of G for which A is the corresponding one-parameter group (hint: express the elements in A by means of $\cosh t$ and $\sinh t$, and then differentiate).

(iii) Let

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \, V = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}.$$

Determine $\exp(sX)$ and $\exp(uV)$ for $s, u \in \mathbb{R}$.

Let $\mathbf{D}$ be the complex unit disk, $\{z \in \mathbb{C} | \, |z| < 1\}$, equipped with the smooth structure induced by the usual structure on $\mathbb{C} \simeq \mathbb{R}^2$, and let $\mathbf{B}$ denote its boundary, the unit circle. For $g \in G$, we write $g_{i,j}$ with $1 \leq i, j \leq 2$ for the matrix coefficients of g,

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}.$$

(iv) Let $z \in \mathbb{C}$. Show that $g_{2,1}z + g_{2,2} \neq 0$ for every $g \in G$ if and only if $z \in \bar{\mathbf{D}} = \mathbf{D} \cup \mathbf{B}$.

(v) Define $\phi : G \times \bar{\mathbf{D}} \to \mathbb{C}$ to be the map given by

$$\phi(g, z) = \frac{g_{1,1}z + g_{1,2}}{g_{2,1}z + g_{2,2}} \qquad (g \in G, z \in \bar{\mathbf{D}}).$$

Show that ϕ maps $\mathbf{D}$ into $\mathbf{D}$ and $\mathbf{B}$ into $\mathbf{B}$, and that it gives a smooth left-action of G on each of these spaces. We denote these actions by $g.z = \phi(g, z)$.

(vi) Show that the 1-parameter group $N = \{\exp(uV) \mid u \in \mathbb{R}\}$ has exactly 2 orbits in $\mathbf{B}$.

(vii) Determine the stabilizer $G_z := \{g \in G | g.z = z\}$ of each of the elements $z = 0 \in \mathbf{D}$ and $z = 1 \in \mathbf{B}$.

(viii) Describe all the orbits of the 1-parameter subgroup $K = \{\exp(sX) | s \in \mathbb{R}\}$ in $\mathbf{D}$, and show that it has only one orbit on $\mathbf{B}$ (it acts transitively).

(ix) Show that G has only one orbit on $\mathbf{D}$ (hint: Determine the A-orbit through 0, and exploit that every K-orbit passes through it).

(x) Use (vii)-(ix) to show that P has only one orbit on $\mathbf{D}$.