Iwasawa decomposition for $\mathrm{GL}(n, \mathrm{R})$.

(Non-mandatory) Assignment

Due February 18, 2015.

Let $G = \mathbf{GL}(n, \mathbf{R})$ and let A be the subset of G consisting of diagonal matrices with positive diagonal entries, N the subset of G consisting of upper triangular matrices with 1 in each diagonal entry, and B the subset of G consisting of upper triangular matrices with positive diagonal entries.

(i) Prove that A, N and B are closed Lie subgroups of G.

(ii) Prove that the map

$$A \times N \to B; \qquad (a, n) \mapsto an$$

is a diffeomorphism. Is this map a Lie isomorphism?

(iii) Let $K = \mathbf{O}(n)$ and show that the map

$$K \times B \to G; \qquad (k, b) \mapsto kb \tag{1}$$

is injective (hint: consider $K \cap B$).

(iv) Let $\{e_1, \cdots, e_n\}$ be the standard basis for $\mathbf{R}^n$, and let $E_i = \text{span}\{e_1, \ldots, e_i\}$. Then $\dim E_i = i$ for $i = 1, \ldots, n$ and

$$E_1 \subset E_2 \subset \cdots \subset E_n = \mathbf{R}^n\,.$$

Let $g \in G$ and assume that $gE_i = E_i$ and $ge_i \cdot e_i > 0$ for all $i = 1, \ldots, n$. Show $g \in B$.

(v) Show that if

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbf{R}^n$$

is an arbitrary chain of linear subspaces with $\dim F_i = i$ for all i, then there exists $k \in \mathbf{O}(n)$ such that $kE_i = F_i$ for all i (hint: find an appropriate orthonormal basis)

(vi) Use (iv) and (v) to show that the map (1) is surjective (hint: Given $g \in G$ let $F_i = gE_i$ for all i).

The identity $G = KAN$ is called the Iwasawa decomposition of G. It is named after the Japanese mathematician Kenkichi Iwasawa $(1917\ -\ 1998)$.