# Quotients of the Heisenberg group

Assignment due March 18, 2015 (counts for 25 % of the grade)

Let $G$ denote the Heisenberg group

$$G = \left\{ g_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbf{R} \right\}.$$

Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and let $\mathfrak{g}$ be the span of these matrices, with commutator bracket.

(i) Show that $\mathfrak{g}$ is the Lie algebra of $G$ and find all three brackets among $X$, $Y$ and $Z$.

(ii) Determine $\exp(sX + tY)$ and $\exp(uZ)$ for $s, t, u \in \mathbf{R}$ and verify the relation

$$\exp(sX)\exp(tY) = \exp(tY)\exp(sX)\exp(stZ)$$

(iii) Determine the center $\mathfrak{c}$ of $\mathfrak{g}$ and the center $C$ of $G$. Determine the Lie algebra of $G/C$.

(iv) Let $M = \mathbf{R}^2$ and define

$$\alpha(g_{x,y,z}, (u, v)) = (u + xv + z, v + y)$$

for $(u, v) \in M$. Show that this gives a transitive left action of $G$ on $M$, and determine a closed subgroup $H$ of $G$ such that $M \simeq G/H$.

(v) Let

$$\Gamma = \{g_{0,0,n} | n \in \mathbf{Z}\}.$$

Show that $L = G/\Gamma$ is a Lie group, and determine its Lie algebra. Show that $L$ has compact center.

(vi) Let $(\rho, V)$ be a finite dimensional complex continuous representation of $G$, and let $\rho_*$ be the derived representation of $\mathfrak{g}$. Let $\gamma \in \mathbf{C}$ be an eigenvalue for $\rho_*(Z)$, and let $V_\gamma \subset V$ denote the corresponding eigenspace. Show that $V_\gamma$ is a $G$-invariant subspace. Determine the action of the one-parameter group $\exp(uZ)$ on $V_\gamma$.

(vii) Let $\gamma$ and $V_\gamma$ be as above, and let $v \in V_\gamma$ be an eigenvector for $\rho(\exp(X))$, say with eigenvalue $\alpha \in \mathbf{C}$. Can $\alpha$ be zero? Show that for each $t \in \mathbf{R}$, $\rho(\exp(tY))v$ is an eigenvector for $\rho(\exp(X))$ (hint: (ii)). What is the eigenvalue?

(viii) Let $\gamma$ be as above. Show that $\gamma = 0$ (hint: $\dim V < \infty$).

(ix) Show that every finite dimensional continuous representation of the Lie group $L$ is trivial on its center. Deduce that there is no injective Lie homomorphism of $L$ into $\mathrm{GL}(N, \mathbf{R})$ for any $N \in \mathbf{N}$.