Three-group numbers

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If $n \in \mathbb{N}$ then $f(n)$ is defined as the number of (isomorphism classes of) groups of order n. We call n a *k-group number*, if $f(n) = k$.

Suppose that $p_1, p_2, ..., p_t$ are distinct prime numbers. For $i \neq j$ we call p_i and p_j *related* if $p_i \mid (q_j - 1)$ or $p_j \mid (q_i - 1)$. We say that there are s *relations* between the p_j if *exactly* s pairs of the primes are related. Thus for example there are 2 relations between 3,7,13 and no relations (0 relations) between 5,7,13.

Proposition 1: *We have that n is a 1-group number if and only if n is a product of distinct prime numbers with no relations.*

Proof: Suppose that n is a 1-group number, and write $n = p_1^{a_1} p_2^{a_2}...$ If some $a_i \geq 2$ then there are at least 2 (nonisomorphic) abelian groups of order $p_i^{a_i}$, contradicting that n is a 1-group number. Thus n is a product of distinct prime numbers. If two of them are related, say p_i and p_j there exists also a nonabelian group of order $p_i p_j$, contradicting again that n is a 1-group number. Conversely if n is a product of distinct prime numbers with no relations, an abelian group of order n is cyclic. It is known that a group G of order n is metacyclic. If G is not abelian then G/G' and G' are nontrivial and there has to exist a relation between a prime divisor of $|G : G'|$ and a prime divisor of $|G'|$. Otherwise G' would be contained in the center of G and thus be a direct factor of G. This is clearly not possible. $\diamond$

Proposition 2: *Suppose that p, q are different prime numbers. Then*
(1) pq^2 is a 2-group number if and only if there is no relation between p, q and $p \nmid (q + 1)$.
(2) pq^2 is a 3-group number if and only if there is no relation between p, q and $p \mid (q + 1)$.

Proof: There are exacty two abelian groups of order pq^2. Suppose that we also have a nonabelian group G of this order. Let P be a p-Sylow subgroup and Q a q-Sylow subgroup of G. Then by Sylows theorem either $P \trianglelefteq G$ or $Q \trianglelefteq G$, but not both.
If $P \trianglelefteq G$ then Q acts nontrivially on P and this is only possible when $q \mid |Aut(P)| = p - 1$. But then there are two nonabelian groups of this order and pq^2 is a $k-$group number, $k \geq 4$. We assume now that $q \nmid (p - 1)$.

If $Q \trianglelefteq G$ then P acts nontrivially on Q so that $p \mid |Aut(Q)|$. We have $|Aut(Q)| = q(q-1)$ if Q is cyclic and $|Aut(Q)| = q(q-1)^2(q+1)$ otherwise. If $p = 2$ there are 3 non-abelian groups of order pq^2. Suppose $p \neq 2$. If $p \mid (q-1)$ we have at least two non-abelian groups of order pq^2. If $p \mid (q+1)$ we get exactly one nonabelian group. $\diamond$

Proposition 3: *We have that n is a 2-group number if and only if n is a product on one of the following forms:*
(1) $n = p_1p_2...p_tq^2$ for distinct prime numbers $p_1, .., p_t, q$ with no relations and such that in addition $p_i \nmid (q+1)$ for $i = 1, ..., t$.
(2) $n = p_1p_2...p_t$ for distinct prime numbers $p_1, .., p_t$, with 1 relation between the p_i's.

Proof: By considering the number of abelian groups of order n we see that n has the form $n = p_1p_2...p_tq^2$ or $n = p_1p_2...p_t$ for distinct prime numbers $p_1, .., p_t, q$. if n is a 2-group number.
Assume that n is a 2-group number. In the case of $n = p_1p_2...p_tq^2$ we see that there can be no non-abelian groups of order n. We then use the previous results to get that there are no relations and in addition $p_i \nmid (q+1)$ for $i = 1, ..., t$. Thus the conditions of (1) are fulfilled.
In the case of $n = p_1p_2...p_t$ each relation between the p_i's gives a new (isomorhism type) group of order n. Thus the condition of (2) is fulfilled.
Conversely assume first that n fulfils the conditions of (1). If $q = 2$ then q is related to any odd prime, forcing $n = 4$. If $q \neq 2$ then any group G of order n is solvable. By Propositions 1 and 2 we get that any π-Hall subgroup of G where $|\pi| = 2$ must be abelian. This forces G to be abelian.
If G has order n, fulfilling the condition of (2), again G is solvable. Suppose that p_1 and p_2 are related. By Proposition 1 we get that any π-Hall subgroup of G where $|\pi| = 2$ must be abelian unless $\pi = \{p_1, p_2\}$. Thus all p_i-subgroups for $i \geq 3$ are central in G and thus direct factors of G. The result follows, since there are exactly 2 groups of order p_1p_2. $\diamond$

Theorem 4: *We have that n is a 3-group number if and only if n is a product on one of the following forms:*
(1) $n = p_1p_2...p_tq^2$ for distinct prime numbers $p_1, .., p_t, q$ with no relations and such that in addition $p_i \mid (q+1)$ for exactly one $i \in \{1, ..., t\}$.
(2) $n = p_1p_2...p_t$ for distinct prime numbers $p_1, .., p_t$, with 2 relations between the p_i's which are on the form $p_i \mid (p_j - 1), p_j \mid (p_k - 1)$ for some i, j, k.

Proof: By considering the number of abelian groups of order n we see that n has the form $n = p_1p_2...p_tq^2$ or $n = p_1p_2...p_t$ for distinct prime numbers $p_1, .., p_t, q$, if n is a 3-group number.
We see that n is a 3-group number if and only if there exists exactly one non-abelian group of order n in the first case and exactly two in the second case.

Assume that n is a 3-group number. In the first case $n = p_1 p_2 ... p_t q^2$ we see that if there is a relation between $p_1, .., p_t, q$ we get at least two nonabelian groups of order n. By the previous result there has to exist at least one p_i dividing $(q+1)$, if n is not a 2-group number. If this happens more than once we get at least two nonabelian groups of order n by Proposition 2. Thus the conditions of (1) are fulfilled.

In the second case $n = p_1 p_2 ... p_t$ there must be 2 relations between the the p_i's, since 1 relation is not enough by the previous result and 3 relations give at least 3 non-abelian groups. We have the following possiblities for the relations:

(i) $p_i \mid p_j - 1, p_j \mid p_k - 1$

(ii) $p_i \mid p_j - 1, p_i \mid p_k - 1$

(iii) $p_i \mid p_j - 1, p_k \mid p_l - 1$,

with i, j, k, l different. But the two last possibilities give at least 3 non-abelian groups of order n. Thus the condition of (2) is fulfilled.

Conversely assume first that n fulfils the conditions of (1). If one of the primes is 2, it is related to all other primes, contradicting the conditions. Thus a group G of order n is solvable. Assuming that $p_1 \mid (q+1)$ we see that any any π-Hall subgroup of G where $|\pi| = 2$ must be abelian unless $\pi = \{p_1, q\}$. This forces all Sylow p_i-subgroups, $i \geq 2$ to be direct factors of G. By Proposition 2 we get that n is a 3-group number.

If n fulfils the conditions of (2), then again any G of order n is solvable. The p-Sylow subgroups of a group G of order n are direct factors of G for all $p \neq p_i, p_j, p_k$ and do not play a rôle. So assume G is a nonabelian group of order $n = p_1 p_2 p_3$ with $p_1 \mid (p_2 - 1), p_2 \mid (p_3 - 1)$. Let P_i be a p_i-Sylow group of G, $i = 1, 2, 3$. We have $p_1 < p_2 < p_3$ and thus $P_3 \trianglelefteq G$. Therefore also $C_G(P_3) \trianglelefteq G$. Since p_1 and p_3 are not related $P_1 \subseteq C_G(P_3)$. We now have one of the following:

(i) $C_G(P_3) = G$. Then P_3 is a direct factor of G.

(ii) $C_G(P_3) = P_1 P_3$. Then $P_1 char C_G(P_3) \trianglelefteq G$ and therefore $P_1 \trianglelefteq G$ must be a direct factor of G.

Each of these cases thus give exactly one non-abelian group of order n.