A theorem on the cores of partitions

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Abstract: If s and t are relatively prime positive integers we show that the s-core of a t-core partition is again a t-core partition. A similar result is proved for bar partitions under the additional assumption that s and t are both odd.

Suppose that $s, t \in \mathbb{N}$ are relatively prime positive integers. In the study of block inclusions between s- and t-blocks of partitions [5] we introduced an (s,t)-abacus to study relations between s- and t-cores of partitions. This is because the cores determine the blocks.

Before we state the main result of this paper let us mention that the basic facts about partitions, hooks and blocks of partitions may be found in [3], Chapter 2 or [6], Chapter 1. You may get to the s-core $\lambda_{(s)}$ of a partition λ by removing a series of s-hooks (i.e. hooks of length s) until all s-hooks are removed. The s-core is independent of the order in which the s-hooks are removed. A partition has by definition *s-weight w*, if you need to remove exactly w s-hooks to get to its s-core. It also equals the number of hooks in the partition of length divisible by s ([3], 2.7.40). Thus a partition is an s-core if and only if it has s-weight 0. Two partitions of n are said to be in the same s-block if they have the same s-core. This definition is inspired by a theorem about irreducible characters of the symmetric groups, which is still referred to as the Nakayama conjecture. It states that if p is a prime number, then two irreducible characters of the symmetric group S_n are contained in the same (modular) p-block, if and only if the partitions labelling them have the same p-core. See [3], 6.2.21. The hook structure of a partition is conveniently determined by its *first column hook lengths*, or more generally any of its *β-sets* ([6], section 1).

Generally, a *β-set* is a finite subset X of $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. For $i \geq 0$ let X^{+i}, the *i'th shift* of X, be the β-set which is obtained from X in the following way: It is the union of the set $\{0, 1, \cdots, i-1\}$ and the set obtained from X by adding i to all its elements. In particuar $X^{+0} = X$. The β-set $\{0, 2, 3, 6, 7\}$ equals $\{1, 2, 5, 6\}^{+1}$. Let λ be a partition. Let $\beta(\lambda)$ be the β-set consisting of all first column hook lengths of λ. Thus if $\lambda = (3, 3, 1, 1)$ then $\beta(\lambda) = \{1, 2, 5, 6\}$. A β-set on the form $\beta(\lambda)^{+i}$ is called a β-set for λ. Any β-set is a β-set for a unique partition.

In this note we want to illustrate the usefulness of the (s,t)-abacus by showing the following result:

Theorem 1: *Let s,t be relatively prime positive integers. Suppose that ρ is a t-core. Then the s-core of ρ is again a t-core.*

It should be mentioned that examples show that when you remove s-hooks from the partition ρ of the theorem you may have to go through arbitrarily long series of intermediate partitions which cannot be chosen as t-cores. The overall behaviour may in fact appear to be rather chaotic. Yet the final result turns out to be again a t-core. The proof of Theorem 1 is surprisingly simple, once you understand how to use the (s,t)-abacus. The Theorem plays an important role in two forthcoming papers ([4] and some joint work with A. Berkovic.)

As we shall see below there is an analogous result for bar partitions and bar cores (Theorem 4) under the additional assumption that s and t are both odd. The proof in this case is somewhat more delicate.

A partition which is at the same time an s- and t-core is called an *(s,t)-core*. There are only finitely many such partitions and the maximal one has cardinality $m_{s,t} = (s^2-1)(t^2-1)/24$ ([1], [5]). Therefore our result implies the following:

Corollary 2: *A t-core of n has s-weight at least $w = \lfloor (n - m_{s,t})/s \rfloor$. Thus it contains at least w hooks of length divisible by s.*

As an example, $m_{3,5} = 8$ so that the 3-core $(8,6,4,2^2,1^2)$ of 24 must contain at least 3 hooks of length divisible by 5. It actually contains 4 such hooks.

There are a few obvious questions you might ask after having seen the theorem, but unfortunately they seem to have negative answers:

If λ is a partition define

$$\lambda_{(s,t)} = (\lambda_{(s)})_{(t)}.$$

Thus $\lambda_{(s,t)}$ is the t-core of the s-core of λ. By our theorem $\lambda_{(s,t)}$ is actually an (s,t)-core and you may call it the (s,t)-core of λ. But *generally* $\lambda_{(s,t)} \neq \lambda_{(t,s)}$. Indeed, if for example $\lambda = (3), s = 2, t = 3$ then $\lambda_{(2,3)} = (1)$ whereas $\lambda_{(3,2)}$ is the empty partition.

Also you may define a (s,t)-block of n as the set of partitions with the same (s,t)-core. Obviously it is a union of s-blocks of n. But it is not necessarily a union of t-blocks. Indeed the partitions of 5 with empty $(2,3)$-core (0) are $(4,1)$ and $(2,1^3)$. They form a 2-block (of weight 1) but obviously not a 3-block.

Finally, it is known that the number of partitions of n with a given s-core only depends on the s-weight, ([3], 2.7.17). But it is not true, that the number of t-cores of n with a given s-core only depends on the s-weight. Indeed here are some examples:

Weight 1: The number of 5-cores of 10 with 7-core (3) is 2 and the number of 5-cores of 10 with 7-core (2,1) is 4.

Weight 3: The number of 5-cores of 10 with 3-core (1) is 8. The number of 5-cores of 11 with 3-core (2) is 3.

Theorem 1 is proved below. As an application we get the following result. Define the *principal* s-block of n to be the s-block containing the partition (n).

Corollary 3: *Let s and t be relatively prime. Let κ be an s-core which is not a t-core. Then any s-block with s-core κ does not contain any t-core. In particular, when the residue of n mod s is at least t, then the principal s-block of n contains no t-core.*

Proof: If ρ is a t-core, then by Theorem 1 its s-core is also a t-core, hence it cannot be contained in an s-block with s-core κ. For the second assertion, let r be the s-residue of n and take $\kappa = (r)$. $\diamond$

There is an analogous result to Theorem 1 for bar partitions under the additional assumption that s and t are odd. A *bar partition* is a partition into distinct parts. For these partitions there is for odd integers a theory of *bars* corresponding to the theory of hooks in arbitrary partitions. See e.g. [6], Section 4 for details. In particular each bar partition has for any given odd integer s a unique $\bar{s}$-core obtained by removing a series of s-bars from the partition. We have

Theorem 4: *Let s, t be relatively prime odd positive integers. Suppose that ρ is a $\bar{t}$-core. Then the $\bar{s}$-core of ρ is again a $\bar{t}$-core.*

There is an analogue of Corollary 2 for bar partitions, but the statement is less precise. Also the first statement of Corollary 3 has an analogue.

We assume now that $s, t \in \mathbb{N}$ are relatively prime positive integers.

The s-abacus was introduced by G. James. Its relation to the study of s-cores and s-quotients of partitions is explained in detail in [3], Section 2.7. (Or see Section 3 in [6].) The s-abacus has s infinite runners, numbered $0, ..., s-1$ going from north to south. The i'th runner contains the nonnegative integers which are congruent to i modulo s in increasing order. Here is part of the 7-abacus:

Runner:	0	1	2	3	4	5	6
	0	1	2	3	4	5	6
	7	8	9	10	11	12	13
	14	15	16	17	18	19	20
	21	22	23	24	25	26	27
	28	29	30	31	32	33	34
	....						

Generally you may arrange the first column hook lengths of the maximal (s, t)-core $\kappa_{s,t}$ in a diagram, which is called the (s, t)-diagram [1],[5].

Start with the largest entry $st - s - t$ in the lower left hand corner and subtract multiples of s along the rows and multiples of t along the columns as long as possible. Then the first column hooklengths of any (s, t)-core must be among the numbers of this diagram. The reason is, that $st - s - t$ is the largest integer which cannot be written in the form $as + bt$ where a, b are non-negative integers. More details may for example be found in [1].

Here is the (5,7)-*diagram.*

2				
9	4			
16	11	6	1	
23	18	13	8	3

Note that the numbers in the columns (read from north to south) are part of some runners on the t-abacus. In the example you have runners 1,2,3,4,6 of the 7-abacus represented. The order of the runners is changed and some runners are missing. The missing runners may be represented by extending the diagram to the south like this:

2						
9	4					
16	11	6	1			
23	18	13	8	3		
30	*25*	*20*	*15*	*10*	*5*	*0*

We have just added part of the 0'th 5-runner to the south written in *italics*. In this diagram also the missing 7-runners numbered 0 and 5 are represented.

If you continue adding rows to the south you get the t-abacus with runners in a different order, with numbers increasing from north to south. This is referred to as the (s,t)-*abacus*.

The (s,t)-abacus and a numbering of its rows is illustrated by this example $(s = 5,\ t = 7)$:

Runner:	2	4	6	1	3	5	0
Row -4	2						
Row -3	9	4					
Row -2	16	11	6	1			
Row -1	23	18	13	8	3		
Row 0	30	25	20	15	10	5	0
Row 1	37	32	27	22	17	12	7
Row 2	44	39	34	29	24	19	14
Row 3	51	46	41	36	31	26	21
Row 4	58	53	48	43	38	33	28
Row 5	65	60	55	50	45	40	35
Row 6	72	67	62	57	52	47	42
Row 7	79	74	69	64	59	54	49
Row 8	86	81	76	71	66	61	56
....							

The rows in the (s,t)-diagram are numbered -1,-2,..., starting from the bottom. The rows below the (s,t)-diagram are numbered 0,1,2... starting from the top as indicated in the example. Thus the i-th row contains a decreasing sequence of numbers which are congruent modulo s. The difference between neighbouring numbers is s and the eastmost number in the row (on the runner 0) is $t \cdot i$. Clearly any non-negative integer is represented uniquely by a position on the (s,t)-abacus. The runners of the s-abacus are visible in the rows of the (s,t)-abacus. Rows whose numbers differ by a multiple of s (like rows -2,3 and 8 in the example) contain numbers from the same runner of the s-abacus. Thus the s-runners are broken into pieces.

This means that (as already mentioned) the (s,t)-abacus is useful for studies involving the relations between s-cores and t-cores. Take a β-set for a given partition and represent its numbers as beads on the (s,t)-abacus. This means that you place a bead in the position numbered i on the abacus for all i in the β-set. Adding/removing s-hooks from partitions are reflected by horizontal moves of the beads where a "horizontal move" could include a shift of s rows, corresponding to the breakup of the s-runners just described. Adding/removing t-hooks are reflected by vertical moves of the beads on the (s,t)-abacus.

We are now in the position to prove Theorem 1. Of course a t-core need not be an s-core. We show that the s-core of a t-core is an (s,t)-core.

Proof of Theorem 1: Suppose that ρ is a t-core. Let $X = \beta(\rho)$ be the set of first column hook lengths of ρ. If X does not already contain all the numbers of the (s,t)-diagram, you extend it to a larger β-set $Y = X^{+i}$ for ρ in such a way that it contains all the numbers in the (s,t)-diagram. You then represent the numbers in Y as beads on the (s,t)-abacus. The result is a diagram where there is no empty space to the north of a bead (since ρ is a t-core) and the part consisting of the (s,t)-diagram is filled with beads. Each runner contains a number of beads outside the (s,t)-diagram.

The removal of an s-hook from ρ is reflected by moving a bead to an empty space next to it to the east or (if it is on the eastmost runner) to an empty space at the westmost runner s rows above. You have reached a bead configuration for the s-core of ρ when no more moves of this kind are possible.

Notice that the parts represented by any s consecutive beads on a runner have different residue classes modulo s and thus they do not influence the s-core of ρ.

We reach a bead configuration for the s- core of ρ in two steps.

Step 1: Remove for as long as possible series of s consecutive beads on runners, starting from below, leaving all the beads in the (s,t)-diagram. The new diagram has no beads in rows with numbers s or higher. You still have a t-core with the same s-core as ρ.

Step 2: Move all beads as far to the east as possible in their respective rows. Then the number of beads outside the (s,t)-diagram on the runners is decreasing, when you move from the west to the east. Moreover the beads still represent a t-core.

After Step 2 no more horizontal moves are possible, also not to an empty space at the westmost runner s rows above. This is because Step 1 left you with at most $s-1$ beads on each runner south of the (s,t)-diagram. Thus Step 2 leaves you with a bead configuration for the s-core of ρ. Since each step results in a t-core, the result follows. $\diamond$

Here is an example illustrating the steps of the proof ($s = 5, t = 7$): The numbers in boldface in the first diagram are the ones in the β-set Y, as described in the proof. The initial 7-core is

$$\rho = (42, 36, 30, 24, 18, 12, 11, 7, 6, 2^4, 1),$$

a partition of 195. The set $X = \beta(\rho)$ of first column hook lengths of ρ is

$$X = \{1, 3, 4, 5, 6, 11, 13, 18, 20, 27, 34, 41, 48, 55\}$$

and $Y = X^{+12}$.

You apply first Step 1 and then Step 2. Rows without beads are omitted. The 5-core of ρ is $\kappa = (5, 4, 2, 1^4)$, another 7-core (of 15), and the 5-weight of ρ is 36.

Runner:	2	4	6	1	3	5	0
Row -4	**2**						
Row -3	**9**	**4**					
Row -2	**16**	**11**	**6**	**1**			
Row -1	**23**	**18**	**13**	**8**	**3**		
Row 0	**30**	**25**	20	**15**	**10**	**5**	**0**
Row 1	37	**32**	27	22	**17**	12	**7**
Row 2	44	**39**	34	29	24	19	14
Row 3	51	**46**	41	36	31	26	21
Row 4	58	**53**	48	43	38	33	28
Row 5	65	**60**	55	50	45	40	35
Row 6	72	**67**	62	57	52	47	42

Runner:	2	4	6	1	3	5	0
Row -4	**2**						
Row -3	**9**	**4**					
Row -2	**16**	**11**	**6**	**1**			
Row -1	**23**	**18**	**13**	**8**	**3**		
Row 0	**30**	**25**	20	**15**	**10**	**5**	**0**
Row 1	37	**32**	27	22	**17**	12	**7**

Runner:	2	4	6	1	3	5	0
Row -4	**2**						
Row -3	**9**	**4**					
Row -2	**16**	**11**	**6**	**1**			
Row -1	**23**	**18**	**13**	**8**	**3**		
Row 0	30	**25**	**20**	**15**	**10**	**5**	**0**
Row 1	37	32	27	22	**17**	**12**	**7**

This represents the β-set $\{1, 2, 3, 4, 6, 9, 11\}^{+14} = \beta(\kappa)^{+14}$.

We now turn to the case of *bar partitions* and the proof of Theorem 4. Let s, t be relatively prime odd positive integers. As in [2] the (s, t)-diagram is divided into 3 parts. Let $u = \frac{s-1}{2}, v = \frac{t-1}{2}$. There is a rectangular subdiagram with u rows and v columns with the number $st - s - t$ in its lower left hand corner and the number $(s+t)/2$ in its upper right hand corner. We refer to this as the *mixed part*. Outside of this there are two disjoint subdiagrams. We refer to the upper one as the *Yin part* and to the lower one as the *Yang part*. In the example below the Yin part is with numbers in **bold** and the Yang part with numbers in *italics*. That two runners are *conjugate* w.r.t. t means that the sum of any number on one runner and any number on the second runner is divisible by t. Similarly you define conjugate runners w.r.t. s.

The divided (5,7)-*diagram*:

2				
9	**4**			
16	11	6	*1*	
23	18	13	*8*	*3*

In addition you divide the rows of (s,t)-abacus, which are relevant for the proof, into 4 parts: Part A consists of the rows with numbers $k < -u$. Part B is the rows $-u$ to -1. Part C is the rows 1 to u. Part D is the rows $u+1$ to $2u$. It should be noted, that Part A is the Yin part of the (s,t)-diagram and that Part B contains the Yang part of the (s,t)-diagram.

The rows numbered by the pairs of integers in the following lists are called *paired*:

$(u+1-j, u+j), j = 1, ..., u$ are (C,D)-paired (placed symmetrically around a line between rows u and $u+1$)

$(-j, j), j = 1, ..., u$ are (B,C)-paired (placed symmetrically around row 0)

$(-u-j, -(u+1)+j), j = 1, ..., u$ are (A,B)-paired (placed symmetrically around a line between rows $-u$ and $-(u+1)$.)

Proof of Theorem 4: You represent the parts of ρ as beads on the (s,t)-abacus. Since ρ is a $\bar{t}$-core all beads are in the top positions on their runners and one of each pair of conjugate runners w.r.t. t is empty. Suppose that runner i contains $m_i \geq 0$ beads. Notice that the parts represented by any s consecutive beads on a runner have different residue classes modulo s and thus they do not influence the $\bar{s}$-core of ρ.

Step 1: Remove series of s consecutive beads on runners, starting from below. You still have a $\bar{t}$-core with the same $\bar{s}$-core as ρ.

After this you assume without loss of generality that $m_i \leq s-1$.

You decompose $m_i = m_i(A) + m_i(B) + m_i(C) + m_i(D) + e$ according to the number of beads in the parts A, B, C, D respectively, where $e = 0, 1$ accounts for a possible bead in row 0.

Step 2a: Remove all beads in row 0. Then consider those i for which $m_i(D) > 0$. Do the following: Remove $m_i(D)$ pairs of beads from the i'th runner, where each pair of beads is on (C,D)-paired rows. Modify the m_i's accordingly. You are then in the situation that $m_i(D) = 0$ for all i and you still have a $\bar{t}$-core with the same $\bar{s}$-core as ρ.

Step 2b: Consider those i for which $m_i(C) > 0$. Do the following: Remove $\min\{m_i(C), m_i(B)\}$ pair of beads from the i'th runner where each pair of beads is on (B, C)-paired rows. Modify the m_i's accordingly. You are then in the situation that for each i either $m_i(B) = 0$ or $m_i(C) = 0$. You still have a bar partition with the same $\bar{s}$-core as ρ.

Step 2c: Consider those i for which $m_i(B) > 0$. Do the following: Remove $\min\{m_i(A), m_i(B)\}$ pair of beads from the i'th runner where each pair of beads is on (A, B)-paired rows. Modify the m_i's accordingly. You are then in the situation that for each i at most one of $m_i(A), m_i(B), m_i(C)$ is nonzero and $m_i(D) = 0$. You still have a bar partition with the same $\bar{s}$-core as ρ.

Step 3: Any beads left on part C of a runner are moved to part A of the (empty) conjugate runner. The row number in which a bead is placed is reduced by s. Thus the beads are moved into the Yin part. Any beads left on part B of a runner are moved to part B of the (empty) conjugate runner, if it is to the right. Thus the beads are moved into the Yang part. Otherwise they are not moved. The row number is unchanged. Any beads left on part A of a runner are left where they are.

After Step 3 you have reached the bead configuration of a $\bar{t}$-core with the same $\bar{s}$-core as ρ. The only beads left are in the Yin and Yang parts.

Step 4: If possible remove for as long as possible pairs of beads in (A, B)-paired rows, one bead in Yin, the other in Yang, moving west to east in both rows. Then move the remaining beads as far to the east as possible in their rows. After Step 4 you have reached the bead configuration of the $\bar{s}$-core of ρ.

This is also a $\bar{t}$-core. All beads are on the top positions on their runners, due to the fact that all beads are moved as far east as possible. Moreover t-conjugate runners cannot possibly both contain beads. Indeed, such a bead configuration could only occur, when at least one of the beads has been moved. But the t-conjugate runners are placed symmetrically around a vertical line in the middle of the (s, t)-diagram. Since you start Step 4 in a situation where at least one of each pair of t-conjugate runners is empty, this is not possible. $\diamond$

Here is an example illustrating the steps of the proof.
(C, D)-paired rows: (2,3) and (1,4)
(B, C)-paired rows: (-1,1) and (-2,2)
(A, B)-paired rows: (-3,-2) and (-4,-1)

Runner:		2	4	6	1	3	5	0
Part A:	Row -4	2						
	Row -3	9	**4**					
Part B	Row -2	16	**11**	6	**1**			
	Row -1	23	**18**	13	**8**	3		
	Row 0	30	25	20	**15**	10	**5**	0
Part C	Row 1	37	32	27	**22**	17	**12**	7
	Row 2	44	39	34	29	24	**19**	14
Part D	Row 3	51	46	41	36	31	**26**	21
	Row 4	58	53	48	43	38	33	28

....

In step 2a you remove **5** and **15** (both in row 0) and the pair **19, 26** on runner 5, rows 2 and 3. In step 2b you remove the pair the pair **8, 22** on runner 1, rows -1 and 1. In step 2c you remove the pair **4, 11** on runner 4, rows -3,-2. You are left with:

Runner:		2	4	6	1	3	5	0
Part A:	Row -4	2						
	Row -3	9	4					
Part B	Row -2	16	11	6	**1**			
	Row -1	23	**18**	13	8	3		
	Row 0	30	25	20	15	10	5	0
Part C	Row 1	37	32	27	22	17	**12**	7
	Row 2	44	39	34	29	24	19	14
....								

In step 3 **12** in the position row 1, runner 5 (Part C) is moved to **2** in the position row -4 (Part A). Also **18** in row -1 runner 3 is moved to **3** in the same row on the conjugate runner.

Runner:		2	4	6	1	3	5	0
Part A:	Row -4	**2**						
	Row -3	9	4					
Part B	Row -2	16	11	6	**1**			
	Row -1	23	18	13	8	**3**		
	Row 0	30	25	20	15	10	5	0
Part C	Row 1	37	32	27	22	17	12	7
....								

In the final step 4 the pair **2, 3** is removed, leaving just the partition (1), a $\bar{7}$-core.

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