Spin Representations and Powers of 2

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1 Introduction

The starting point for this article was an observation by Yamada. He had computed the determinants of the (reduced) spin 2-decomposition matrices for the double covers $\widehat{S}_n$ of the symmetric groups, which are available up to $n = 15$ in [Be], and had found them to be 2-powers. After computing the elementary divisors of these matrices, the first author then formulated a conjecture explicitly describing the elementary divisors even blockwise: they are roughly the square roots of the elementary divisors of the Cartan matrix of the corresponding 2-block of S_n. We prove this below as one of our main results in Theorem 4.4.

On the way to this result we have discovered some interesting 2-divisibility properties of spin characters. Generalizing the known result that the minimal 2-power in the degrees of the spin characters of $\widehat{S}_n$ is $2^{[(n-s(n))/2]}$ (where $s(n)$ is the number of summands in the 2-adic sum decomposition of n), we have found the minimal 2-power in the spin character values on all conjugacy classes corresponding to a cycle type with odd parts only. This is given below in Theorem 3.4.

2 Preliminaries

We let $\widehat{S}_n$ denote a covering group of the symmetric group S_n. Thus there is (for $n \geq 2$) a non-split exact sequence of groups

$$1 \to \langle z \rangle \to \widehat{S}_n \xrightarrow{\pi} S_n \to 1$$

where z is central and of order 2.

For more details on the remarks below leading to the first proposition we refer to [MiO], Section 1. If H is a subgroup of S_n we let

$$H^+ = \pi^{-1}(H)\,,\ H^- = \pi^{-1}(H \cap A_n)$$

where A_n is the alternating group. The set, *on which H operates* is defined as

$$\Omega_H = \{1 \leq i \leq n \mid \text{ there exists an } x \in H \text{ s.t. } x(i) \neq i\}$$

the set of non fixed points of H. We say that subgroups $H_1, H_2, \cdots, H_r$ of S_n *operate on disjoint sets* if

$$\Omega_{H_i} \cap \Omega_{H_j} = \varnothing \quad \text{for all} \quad i \neq j \ .$$

In that case $H_1, H_2, \cdots, H_r$ in fact form a direct product as subgroups of S_n and if $H = H_1 \times H_2 \times \cdots \times H_r$ then

$$H^+ = H_1^+ \hat{\times} H_2^+ \hat{\times} \cdots \hat{\times} H_r^+$$

where $\hat{\times}$ denotes a twisted central product as described in [H], [S].

When G is any finite group we let $\mathrm{Irr}(G)$ denote the set of complex irreducible characters of G.

For any subgroup $H \subseteq S_n$, $\mathrm{Irr}(H^+)$ is divided into 2 disjoint subsets according to whether $\langle z \rangle$ is in the kernel of a character or not. The set of characters in H^+ with z in their kernel is identified with $\mathrm{Irr}(H)$. The remaining characters in $\mathrm{Irr}(H^+)$ are called *spin characters* (of H or H^+). This set of characters is denoted $\mathrm{Irr}_s(H^+)$. Thus

$$\mathrm{Irr}(H^+) = \mathrm{Irr}(H) \cup \mathrm{Irr}_s(H^+) \ .$$

Characters in $\mathrm{Irr}(H^+)$ are called associate, if they have the same restriction to H^-. An associate class of characters consists of one (self-associate) character or of two associate characters (associate pair). We associate the sign $\sigma(\chi) = 1$ to a self-associate character and $\sigma(\chi) = -1$ to a non self-associate character. (We are here only considering subgroups H of S_n with $H^+ \neq H^-$, i.e. $H \not\subseteq A_n$. For the general case we refer to [MiO], Section 1.) We have the following result for spin characters of twisted central products (see [H], [MiO]).

Proposition 2.1 *Let $H_1, H_2, \cdots, H_r \subseteq S_n$ operate on disjoint sets and put $H = H_1 \times H_2 \times \cdots \times H_r$. There is a surjective map*

$$\widehat{\otimes} : \mathrm{Irr}_s(H_1^+) \times \cdots \times \mathrm{Irr}_s(H_r^+) \to \mathrm{Irr}_s(H^+)$$
$$(\chi_1, \chi_2, \cdots, \chi_r) \to \chi_1 \widehat{\otimes} \chi_2 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r$$

with the following properties:

Suppose $\chi_i, \varphi_i \in \mathrm{Irr}_s(H_i^+)$, $i = 1, \cdots, r$. Then

(1) $\sigma(\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r) = \sigma(\chi_1) \cdots \sigma(\chi_r)$;

(2) $(\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r)(1) = 2^{[t/2]} \chi_1(1) \cdots \chi_r(1)$
where $t = |\{i \mid \sigma(\chi_i) = -1\}|$;

(3) $\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r$ *and* $\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_r$ *are associate if and only if* χ_i *and* φ_i *are associate for all* i;

(4) $\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r = \varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_r$ *if and only if the following two conditions hold:*

   (i) χ_i *and* φ_i *are associate for all* i;

   (ii) *Either* $\sigma(\chi_1) \cdots \sigma(\chi_r) = 1$ *or*
   $\sigma(\chi_1) \cdots \sigma(\chi_r) = -1$ *and* $|\{i \mid \chi_i \neq \varphi_i\}|$ *is even.*

The associate classes of spin characters of $\widehat{S}_n$ are labelled canonically by the partitions λ of n into distinct parts, i.e. $\lambda = (a_1, a_2, \cdots, a_m)$, $a_1 > a_2 > \cdots > a_m > 0$, $a_1 + \cdots + a_m = n$. We write $|\lambda| = n$ and $\ell(\lambda) = m$, the *cardinality* and *length* of λ. Also the sign of λ is $\sigma(\lambda) = (-1)^{n-m}$. According to the signs the set $\mathcal{D}(n)$ of partitions of n into distinct parts is divided into disjoint subsets $\mathcal{D}^+(n)$ and $\mathcal{D}^-(n)$. The partitions in the set $\mathcal{D}^+(n)$ resp. $\mathcal{D}^-(n)$ are called *even* resp. *odd* partitions.

Then the self-associate spin characters in S_n are labelled by the partitions in $\mathcal{D}^+(n)$ and the associate pairs of spin characters are labelled by the partitions in $\mathcal{D}^-(n)$. Thus if $\langle\lambda\rangle$ is a spin character, then $\sigma(\langle\lambda\rangle) = \sigma(\lambda)$.

The conjugacy classes of elements of odd order in S_n are labelled canonically via their cycle type by the elements in the set $\mathcal{O}(n)$ of partitions of n into odd parts. We will use an 'exponential' notation for partitions $\alpha \in \mathcal{O}(n)$:

$$\alpha = (1^{m_1}, 3^{m_3}, \cdots)$$

Thus $|\alpha| = \sum_{i \text{ odd}} i\, m_i$, $\quad \ell(\alpha) = \sum_{i \text{ odd}} m_i$.

It is well known that $|\mathcal{D}(n)| = |\mathcal{O}(n)|$; we denote this cardinality by $d(n)$. In fact, already in 1883 J.W.L. Glaisher [G] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; so in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. (A generalization of this map to all partitions may be found in [O].) In this situation, Glaisher's map G is defined as follows. Suppose that $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2

$$m_i = \sum_j 2^{a_{ij}} .$$

This is the 2-adic decomposition of m_i. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, of course in descending order.

For any integer $m \geq 0$, $s(m)$ is the number of summands in the 2-adic decomposition of m. With this notation we see that if $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$ then for the length of $G(\alpha)$ we have $\ell(G(\alpha)) = \sum_{i \text{ odd}} s(m_i)$. We define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i)) \ ,$$

and we denote by $\mathcal{O}^+(n)$ resp. $\mathcal{O}^-(n)$ the sets of partitions in $\mathcal{O}(n)$ with k_α even resp. odd. Then, since $n = \sum_{i \text{ odd}} i\, m_i \equiv \sum_{i \text{ odd}} m_i \pmod 2$ we obtain

Lemma 2.2 *For $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ we have*

$$\sigma(G(\alpha)) = (-1)^{k_\alpha} \ .$$

Hence the Glaisher map G induces bijections $\mathcal{O}^\epsilon(n) \to D^\epsilon(n)$, where ϵ is a sign.

The integer k_α also occurs in another connection. For any integer m, we denote by $\nu(m)$ the exponent to which 2 divides m. Thus $2^{\nu(m)}$ is the exact 2-power dividing m. We also write $m_2 := 2^{\nu(m)}$.

Lemma 2.3 *Let $\alpha \in \mathcal{O}(n)$ and let x'_α be an element with cycle type α in S_n. Then*

$$C_{S_n}(x'_\alpha) \cong \prod_{i \ odd} \mathbb{Z}_i \ wr \ S_{m_i} \ ,$$

a direct product of wreath products. The subgroups $\mathbb{Z}_i$ wr S_{m_i} of S_n operate on disjoint sets. Moreover, $\nu(|C_{S_n}(x'_\alpha)|) = \prod_{i \ odd} \nu(m_i!) = k_\alpha$.

Proof. See [JK], 4.1.19 and 1.2.15. Use that for any $m \geq 0$, $\nu(m!) = m - s(m)$. $\diamond$

Let C_α be the conjugacy class of S_n labelled by $\alpha \in \mathcal{O}(n)$. By Lemma 2.3 k_α is the 2-defect of C_α. Then $\pi^{-1}(C(\alpha))$ consists of two conjugacy classes in $\widehat{S}_n$, say $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$. We choose notation such that the elements of $C_\alpha^{(1)}$ have odd order. Then $C_\alpha^{(2)} = z\,C_\alpha^{(1)}$, and the elements in this second conjugacy class have even order. These conjugacy classes have 2-defect $k_\alpha + 1$.

Obviously the values of any spin character of $\widehat{S}_n$ on $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$ differ only by a sign. Moreover associate spin characters have the same value on $C_\alpha^{(1)}$. Also spin characters of $\widehat{S}_n$ vanish on all conjugacy classes of elements $x \in \widehat{S}_n$ where $\pi(x)$ has even order (possibly with one exception which is of no importance here).

Thus the values of all spin characters of $\widehat{S}_n$ on all conjugacy classes may be easily recovered from the *reduced spin character table* $Z_s = Z_s(n)$, a square matrix with rows indexed by $\lambda \in \mathcal{D}(n)$ and columns indexed by $\alpha \in \mathcal{O}(n)$. The entries are $\langle\lambda\rangle(x_\alpha)$, where $\langle\lambda\rangle$ is a spin character labelled by λ and $x_\alpha \in C_\alpha^{(1)}$. Examples of reduced spin character tables may be found in [M], [HH] up to $n = 14$.

In the context of 2-modular representations, we consider the part of the 2-decomposition matrix for $\widehat{S}_n$ corresponding to spin characters. Since the rows corresponding to associate spin characters are equal, this part of the decomposition matrix is determined by the submatrix $D_s = D_s(n)$, where for each $\lambda \in \mathcal{D}(n)$ we keep only one row for each associate class of spin characters. We call D_s the reduced spin 2-decomposition matrix; it is a square matrix of the same size as Z_s.

We prove a rather special result on determinants which is needed in the next sections.

Lemma 2.4 *Let D be a square $\ell \times \ell$-matrix. Let $\widetilde{D}$ be an $(\ell + s) \times \ell$-matrix obtained from D by repeating s different rows from D. Then*

$$\det(\widetilde{D}^t \widetilde{D}) = 2^s (\det D)^2 \ .$$

Proof. We may assume that the ℓ rows of D are pairwise different, since otherwise both sides of the equation are zero.

By the Cauchy-Binet theorem (see e.g. [MM], Thm. 2.6.1) we obtain

$$\det(\widetilde{D}^t \widetilde{D}) = \sum_A \det A^t \det A$$

where the sum runs over all $\ell \times \ell$ submatrices of $\widetilde{D}$.

We only get a contribution from submatrices A with pairwise distinct rows, i.e. out of the s pairs of repeated rows we choose only one of the pair. Thus we have 2^s such submatrices, and each of these is a certain row permutation of the matrix D. Thus for each such submatrix we get a contribution

$(\det D)^2$ to the sum, and hence the claim follows. $\diamond$

We quote some general facts from modular representation theory due to R. Brauer, which we need in section 4. The results are contained in [F], V.10 and [NT], 5.11 but are not stated there explicitly in the form we need them.

If G is a finite group, p a prime, we let $\ell(G)$ be the cardinality of the set $\mathcal{C}l_1(G)$ of p-regular conjugacy classes in G. For each $C \in \mathcal{C}l_1(G)$ we let x_C denote an element in C. A *defect group* of C is a p-Sylow subgroup of $C_G(x)$ for some $x \in C$. We let $\mathrm{IBr}(G)$ denote the set of modular irreducible characters of G, and we set $\Phi_G = (\varphi(x_C))_{\substack{\varphi\in IBr(G)\\ C\in\mathcal{C}l_1(G)}}$, the Brauer character table of G. It is wellknown that the Brauer character table is non-singular modulo p, i.e.

$$\det \Phi_G \not\equiv 0 \pmod{p} .$$

Furthermore, we let C denote the Cartan matrix and $D = (d_{\chi,\varphi})_{\substack{\chi\in \mathrm{Irr}(G)\\ \varphi\in \mathrm{IBr}(G)}}$ denote the p-decomposition matrix for G. Then, let $\mathrm{Bl}(G)$ denote the set of p-blocks of G. For $B \in \mathrm{Bl}(G)$, $\mathrm{Irr}(B)$ is the set of ordinary irreducible characters in B, $\mathrm{IBr}(B)$ is the set of modular irreducible characters in B, $\ell(B) = |\mathrm{IBr}(B)|$, $C(B)$ is the Cartan matrix for B and $D(B) = (d_{\chi,\varphi})_{\substack{\chi\in \mathrm{Irr}(B)\\ \varphi\in \mathrm{IBr}(B)}}$ denotes the p-decomposition matrix for B.

Then

$$\ell(G) = \sum_{B\in\mathrm{Bl}(G)} \ell(B)$$

and C resp. D are the block direct sums of the matrices $C(B)$ resp. $D(B)$, $B \in \mathrm{Bl}(G)$.

The following is proved in [Br], section 5.

Theorem 2.5 *There exists a disjoint decomposition of $\mathcal{C}l_1(G)$*

$$\mathcal{C}l_1(G) = \bigcup_{B\in Bl(G)} \mathcal{C}l_1(B)$$

such that the following conditions are fulfilled

(1) $|\mathcal{C}l_1(B)| = \ell(B)$ *for all B.*

(2) *For* $\Phi_B = (\varphi(x_C))_{\substack{\varphi\in IBr(B)\\ C\in\mathcal{C}l_1(B)}}$, *we have* $\det \Phi_B \not\equiv 0 \pmod{p}$.

(3) *The elementary divisors of the Cartan matrix $C(B)$ of B are exactly the orders of the defect groups of $C \in \mathcal{C}l_1(B)$.*

3 Powers of 2 in spin character values

We use the notation from section 2. In this section we determine the exact power of 2 dividing the value of all spin characters on a given conjugacy class labelled by $\alpha \in \mathcal{O}(n)$.

As a preparation we generalize a result of A. Wagner ([W], Lemma 4.2).

Proposition 3.1 *Let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$ be a Young subgroup of S_n, $n = n_1 + n_2 + \cdots + n_r$. Let P be a 2-Sylow subgroup of H ($\subseteq S_n$). Then any spin character of P^+ has degree $2^{[k/2]}$, where $k = \sum_{i=1}^{r}(n_i - s_i)$, $s_i = s(n_i)$. Let $t = |\{i \mid 1 \leq i \leq r\,,\, n_i - s_i \text{ is odd }\}|$. The group P^+ has at most two spin characters; more precisely, P^+ has exactly one spin character which is self-associate if t is even, and P^+ has exactly two spin characters which are associate to each other if t is odd.*

Proof. For $r = 1$ our statement coincides with Wagner's result. To prove the general case we apply Proposition 2.1. Let

$$P^+ = P_1^+ \hat{\times} P_2^+ \hat{\times} \cdots \hat{\times} P_r^+$$

where P_i is a 2-Sylow subgroup of S_{n_i}. Now $|P_i| = 2^{n_i - s_i}$, since $\nu(n_i!) = n_i - s_i$.

If $n_i - s_i$ is even then P_i^+ has exactly one spin character of degree $2^{(n_i-s_i)/2}$. If $n_i - s_i$ is odd then P_i^+ has exactly 2 spin characters (which are associate) of degree $2^{(n_i-s_i-1)/2}$. Since each P_i^+ has only one associate class of spin characters the same is true for P^+ by Proposition 2.1 (3). As we have seen the degrees of the spin characters of P_i are known, namely $2^{[(n_i-s_i)/2]}$. Therefore the degree of a spin character χ of P may be computed using Proposition 2.1 (2). We get

$$\begin{aligned}
\nu(\chi(1)) &= \left[\frac{t}{2}\right] + \sum_{i=1}^{r}\left[\frac{n_i - s_i}{2}\right] \\
&= \left[\frac{t}{2}\right] + \sum_{\{i|n_i - s_i \text{even}\}} \frac{n_i - s_i}{2} + \sum_{\{i|n_i - s_i \text{odd}\}} \frac{n_i - s_i - 1}{2} \\
&= \left[\sum_{\{i|n_i - s_i \text{even}\}} \frac{n_i - s_i}{2} + \sum_{\{i|n_i - s_i \text{odd}\}} \frac{n_i - s_i - 1}{2} + \frac{t}{2}\right] \\
&= \left[\sum_{i=1}^{r} \frac{n_i - s_i}{2}\right] .
\end{aligned}$$

The statement about the number of spin characters of P follows from Proposition 2.1 (4). $\diamond$

Theorem 3.2 *Let $Z_s = Z_s(n)$ be the reduced spin character table as described in section 2. Let $d^-(n) = |\mathcal{D}^-(n)|$. Then*

$$2^{d^-(n)}(\det Z_s)^2 = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)| \, .$$

Proof. By doubling the rows in Z_s which correspond to $\lambda \in \mathcal{D}^-(n)$ we obtain a complete spin character matrix $\widetilde{Z}$ on the 2-regular classes. Since $|C_{\widehat{S}_n}(x_\alpha)| = 2|C_{S_n}(x'_\alpha)|$ for all $\alpha \in \mathcal{O}(n)$, the column orthogonality relations on the character tables for S_n and $\widehat{S}_n$ imply that $\widetilde{Z}^t\widetilde{Z}$ is a diagonal matrix with $|C_{S_n}(x'_\alpha)|$, $\alpha \in \mathcal{O}(n)$, as its diagonal entries. Using Lemma 2.4 we now obtain

$$2^{d^-(n)}(\det Z_s)^2 = \det(\widetilde{Z}^t\widetilde{Z}) = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)| \, .$$

$\diamond$

Corollary 3.3 *Let Z_s be the reduced spin character table for $\widehat{S}_n$. Then*

$$(\det Z_s)_2 = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]} \, .$$

Proof. By Lemma 2.2, $d^-(n) = |O^-(n)|$. For any integer $k \geq 0$

$$k - 2[k/2] = \begin{cases} 0 & \text{if} \quad k \text{ is even} \\ 1 & \text{if} \quad k \text{ is odd} \end{cases}$$

Hence by the Theorem above and by Lemma 2.3

$$2 \sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2] + d^-(n) = \sum_{\alpha\in\mathcal{O}(n)} k_\alpha = 2\nu(\det Z_s) + d^-(n) \, ,$$

so that

$$\nu(\det Z_s) = \sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2]$$

as claimed. $\diamond$

We are now able to prove the first main result of this paper.

Theorem 3.4 *Let $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Let $k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i))$. Then $2^{[k_\alpha/2]}$ is the maximal power of 2 which divides all spin character values $\langle\lambda\rangle(x_\alpha)$, $\lambda \in \mathcal{D}(n)$.*

Proof. We first prove that $2^{[k_\alpha/2]} \mid \langle\lambda\rangle(x_\alpha)$ for all $\lambda \in \mathcal{D}(n)$. Consider the restriction of $\langle\lambda\rangle$ to $K_\alpha^+ := C_{\hat{S}_n}(x_\alpha)$. If $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$ and $x'_\alpha = \pi(x_\alpha) \in S_n$ has cycle type α, then by Lemma 2.3

$$K_\alpha = C_{S_n}(x'_\alpha) = \prod_{i \text{ odd}} K_\alpha^{(i)}$$

where $K_\alpha^{(i)} \simeq \mathbb{Z}_i \text{ wr } S_{m_i}$, a wreath product. The subgroups $K_\alpha^{(i)}$ operate on disjoint sets whence

$$K_\alpha^+ = K_\alpha^{(1)+} \hat{\times} K_\alpha^{(3)+} \hat{\times} \cdots .$$

We claim that a spin character of K_α^+ has a degree divisible by $2^{[k_\alpha/2]}$. A 2-Sylow subgroup of K_α is isomorphic to a 2-Sylow subgroup of $S_{m_1} \times S_{m_3} \times \cdots$, which is a Young subgroup of S_m, $m = m_1 + m_3 + \cdots$.

If we restrict a spin character of K_α^+ to a 2-Sylow subgroup, then each irreducible constituent has degree exactly $2^{[k_\alpha/2]}$, by Proposition 3.1. The claim about spin character degrees of K_α^+ follows. Let

$$\langle\lambda\rangle_{|K_\alpha^+} = \sum_{\xi \in \mathrm{Irr}_s(K_\alpha^+)} a_\xi \, \xi$$

where $a_\xi \geq 0$. For each $\xi \in \mathrm{Irr}_s(K_\alpha^+)$, $\xi(x_\alpha) = \kappa_\xi \xi(1)$, where κ_ξ is a root of unity, since $x_\alpha \in Z(K_\alpha^+)$. Each ξ has a degree divisible by $2^{[k_\alpha/2]}$ as seen above. Therefore each $\kappa_\xi \xi(1)/2^{[k_\alpha/2]}$ is an algebraic integer for each ξ. Thus

$$\langle\lambda\rangle(x_\alpha)/2^{[k_\alpha/2]} = \sum_\xi a_\xi \kappa_\xi \xi(1)/2^{[k_\alpha/2]}$$

is an algebraic integer. Since $\langle\lambda\rangle(x_\alpha)$ is an integer we get $2^{[k_\alpha/2]} \mid \langle\lambda\rangle(x_\alpha)$, as desired.

Since $2^{[k_\alpha/2]}$ divides all entries in the column of Z_s labelled by α, for any $\alpha \in \mathcal{O}(n)$, Corollary 3.3 forces this to be the maximal power of 2 dividing all the entries in the column, for any $\alpha \in \mathcal{O}(n)$. $\diamond$

4 The elementary divisors of the reduced spin 2-decomposition matrix of a 2-block

In this section we first deal with the 'global' situation for the reduced spin 2-decomposition matrix and afterwards refine this to the block version.

As before, for $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ we set $k_\alpha = \sum\limits_{i \text{ odd}} (m_i - s(m_i))$.

Theorem 4.1 *Let D_s be the reduced spin 2-decomposition matrix for $\widehat{S}_n$. The elementary divisors of this matrix are*

$$2^{[k_\alpha/2]}\,,\ \alpha \in \mathcal{O}(n)\;.$$

Proof. First we show that the determinant of D_s is correct.
By doubling the rows in the matrix D_s which correspond to $\lambda \in \mathcal{D}^-(n)$ we obtain a complete spin 2-decomposition matrix $\widetilde{D}$. Now using [NT], 5.8.11, we have $\widetilde{D}^t\widetilde{D} = C$, the 2-Cartan matrix for S_n. Hence by Lemma 2.4 we have

$$\det C = 2^{d^-(n)}(\det D_s)^2\;.$$

On the other hand, by [NT], 3.6.32 (or use Theorem 2.5(3)) and Lemma 2.3 we have

$$\det C = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)|_2 = \prod_{\alpha\in\mathcal{O}(n)} 2^{k_\alpha}\;.$$

As before we see

$$\nu(\det C) = \sum_{\alpha\in\mathcal{O}(n)} k_\alpha = 2\sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2] + d^-(n)\;,$$

so that

$$|\det D_s| = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]}\;.$$

For the result on the elementary divisors we now study Smith normal forms. For any integral square matrix X we let $S(X)$ denote its Smith normal form, i.e. the diagonal matrix with the invariant factors of X as diagonal elements. We need the following property of the Smith normal form: If X

and Y are square $n \times n$ matrices with relatively prime determinants, then $S(XY) = S(X)S(Y)$. (See e.g. [N], Theorem II.15).

Let Φ be the table of Brauer characters of $\widehat{S}_n$, so that

$$Z_s = D_s \Phi \ .$$

As $|\det D_s|$ is a power of 2, and $\det \Phi$ is odd, we have

$$S(Z_s) = S(D_s)S(\Phi) \tag{1}$$

Let Δ be a diagonal matrix with diagonal elements $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n)$ (in the usual order). By Theorem 3.4 there exists an *integral* matrix W such that

$$Z_s = W\Delta \ .$$

Since $\det \Delta = |\det D_s|$ we get $|\det W| = |\det \Phi|$, so that the determinants of W and Δ are relatively prime. Thus

$$S(Z_s) = S(W)S(\Delta) \ . \tag{2}$$

Since the S-matrices are diagonal we conclude that

$$S(\Delta) = S(D_s) \ .$$

Since Δ itself is a diagonal matrix, the result follows. $\diamond$

Corollary 4.2 *The elementary divisors of D_s are exactly $2^{[c_1/2]}, \cdots, 2^{[c_\ell/2]}$, where $c_1, \cdots, c_\ell$ are the 2-defects of the conjugacy classes of elements of odd order in S_n.*

Next we prove a block refinement of the result on the elementary divisors of the reduced spin 2-decomposition matrix. First, we describe briefly the 2-block structure in $\widehat{S}_n$ (see [BO] for further details).

If $B \in \mathrm{Bl}(S_n)$ then all irreducible characters (ordinary and modular) of B may be considered as characters in a unique block $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$. We write $B \subseteq \widehat{B}$. With this identification $\mathrm{IBr}(B) = \mathrm{IBr}(\widehat{B})$, but note that $\widehat{B}$ contains also some spin characters in addition to the ordinary characters of B. The description of the distribution of spin characters into the 2-blocks of $\widehat{S}_n$ was a main result in [BO].

To the blocks $B \in \mathrm{Bl}(S_n)$ and $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$ (where $B \subseteq \widehat{B}$ as before) is associated a non-negative integer $w = w(B) = w(\widehat{B})$ called the *weight* of the block. Most block theoretic invariants of B and $\widehat{B}$ depend only on w. Thus for example for the number of modular characters we have

$$\ell(B) = \ell(\widehat{B}) = p(w)$$

where p is the partition function [JK] and $w = w(B)$. Moreover, by [BO], Theorem (2.1) the number of associate classes of spin characters in $\widehat{B}$ is also $p(w)$.

We may consider therefore a *reduced spin 2-decomposition matrix* $D_s(\widehat{B})$ of a 2-block $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$. This is a $p(w) \times p(w)$ square matrix with rows labelled by the associate classes of spin characters in $\widehat{B}$ and the columns by the modular irreducible characters in $\widehat{B}$ (i.e. in B). In this section we determine the elementary divisors of $D_s(\widehat{B})$. Since the matrix D_s is a direct sum of the $D_s(\widehat{B})$, $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$ and D_s has determinant a power of 2, the same is true for each $D_s(\widehat{B})$.

By Theorem 2.5 and Lemma 2.3 we have

Lemma 4.3 *There exists a disjoint decomposition*

$$\mathcal{O}(n) = \bigcup_{B \in Bl(S_n)} \mathcal{O}(n, B)$$

such that the following holds

(1) $|\mathcal{O}(n, B)| = \ell(B) = p(w)$.

(2) *Set* $\Phi_B = (\varphi(x'_\alpha))_{\substack{\varphi \in IBr(B) \\ \alpha \in \mathcal{O}(n,B)}}$. *Then* $\det \Phi_B \not\equiv 0 \pmod 2$.

(3) *The elementary divisors of the Cartan matrix $C(B)$ of B are exactly the numbers 2^{k_α}, $\alpha \in \mathcal{O}(n, B)$.*

We may now prove the main result:

Theorem 4.4 *Let $\widehat{B} \in Bl(\widehat{S}_n)$, $B \in Bl(S_n)$, $B \subseteq \widehat{B}$. Suppose that $2^{c_1}, 2^{c_2}, \cdots, 2^{c_\ell}$ are the elementary divisors of $C(B)$. Then the elementary divisors of $D_s(\widehat{B})$ are $2^{[c_1/2]}, 2^{[c_2/2]}, \cdots, 2^{[c_\ell/2]}$.*

Proof. We have to show that $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n, B)$ are the elementary divisors of $D_s(\widehat{B})$. Let $Z_s(\widehat{B})$ be the reduced spin character table for $\widehat{B}$ on the 2-regular classes. It has the same row indices as the reduced spin 2-decomposition matrix $D_s(\widehat{B})$ and the column indices are the $\alpha \in \mathcal{O}(n, B)$.

Also we let Φ_B be the table of modular character values for modular characters $\varphi \in \mathrm{IBr}(B)$ on the conjugacy classes C_α, $\alpha \in \mathcal{O}(n,B)$. Then we have

$$Z_s(\widehat{B}) = D_s(\widehat{B})\Phi_B \ .$$

As a divisor of $\det D_s$, $|\det D_s(\widehat{B})|$ is a 2-power, whereas by Lemma 4.3(2), $\det \Phi_B$ is odd. As before we let $S(X)$ denote the Smith normal form of an integral square matrix X. Then

$$S(Z_s(\widehat{B})) = S(D_s(\widehat{B}))S(\Phi_B) \ .$$

Let Δ_B be a diagonal matrix with diagonal elements $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n,B)$. By Theorem 3.4 there exists an *integral* matrix W, such that

$$Z_s(\widehat{B}) = W \cdot \Delta_B \ .$$

Hence $\det \Delta_B$ divides $\det D_s(\widehat{B})$. By Theorem 2.5 and Theorem 4.1 we have

$$\prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]} = \prod_B \det \Delta_B \ \Big| \ \prod_B |\det D_s(\widehat{B})| = |\det D_s| = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]}$$

Hence we have

$$\prod_{\alpha\in\mathcal{O}(n,B)} 2^{[k_\alpha/2]} = \det \Delta_B = |\det D_s(\widehat{B})|$$

for all blocks B resp. $\widehat{B}$. Thus $|\det W| = |\det \Phi_B|$ is odd, and so

$$S(Z_s(\widehat{B})) = S(W) \cdot S(\Delta_B)$$

and we conclude that $S(\Delta_B) = S(D_s(\widehat{B}))$. Since Δ is itself a diagonal matrix, the result follows. $\diamond$

The elementary divisors of $C(B)$, B a 2-block of S_n, have been determined explicitly in [O]. Unfortunately, the formula was misstated there; we take the opportunity to give the correct formula here. First we have to introduce some notation.

Let p be a prime, λ a partition. For any pair of non-negative integers (m,a) such that $m \geq 1$, $(m,p) = 1$ and $a \geq 0$, let $t_\lambda(m,a)$ be the multiplicity of mp^a as a part of λ. Then the partition λ is characterized by the non-negative integers $t_\lambda(m,a)$, where (m,a) runs through all pairs as above.

We define an integer $e(\lambda)$ as follows:

$$e(\lambda) = \sum_{m,a} t_\lambda(m,a)(1+p+\cdots+p^a) \ .$$

For all $i \geq 0$ let

$$p_0^i(n) = |\{\lambda \vdash n \mid \lambda \ p\text{-regular and } e(\lambda) = i\}| \ .$$

Finally, let $P(x) = \sum_{n\geq 0} p(n)x^n$ be the partition generating function, and define the integers $m(n)$ by

$$P(x)^{p-2}P(x^p) = \sum_{n\geq 0} m(n)x^n \ .$$

Then the following holds (see [O]):

Theorem 4.5 *Let B be a p-block of S_n of weight w, and let $i \geq 0$. Then the multiplicity of p^i as an elementary divisor of the Cartan matrix $C(B)$ is*

$$\sum_{t=0}^{w} m(w-t)p_0^i(t) \ .$$

Acknowledgments. The authors are grateful to the Danish Natural Science Foundation for the support of their collaboration. We would also like to thank H. Yamada for communicating his observation on the determinants to us.

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