Prime power degree representations of the symmetric and alternating groups

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1 Introduction

In 1998, the second author raised the problem of classifying the irreducible characters of S_n of prime power degree. Zalesskii proposed the analogous problem for quasi-simple groups, and he has, in joint work with Malle, made substantial progress on this latter problem. With the exception of the alternating groups and their double covers, their work provides a complete solution. In this article we first classify all the irreducible characters of S_n of prime power degree (Theorem 2.4), and then we deduce the corresponding classification for the alternating groups (Theorem 5.1), thus providing the answer for one of the two remaining families in Zalesskii's problem. This classification has another application in group theory. With it, we are able to answer, for alternating groups, a question of Huppert: Which simple groups G have the property that there is a prime p for which G has an irreducible character of p-power degree > 1 and all of the irreducible characters of G have degrees that are relatively prime to p or are powers of p?

The paper is organized as follows. In section 2, some results on hook lengths in partitions are proved. These results lead to an algorithm which allows us to show that every irreducible representation of S_n with prime power degree is labelled by a partition having a *large* hook. In section 3, we obtain a new result concerning the prime factors of consecutive integers (Theorem 3.4). In section 4 we prove Theorem 2.4, the main result. To do so, we combine the algorithm above with Theorem 3.4 and work of Rasala on minimal degrees. This implies Theorem 2.4 for large n. To complete the proof, we check that the algorithm terminates appropriately for small n (i.e. those $n \leq 9.25 \cdot 10^8$) with the aid of a computer. In the last section we derive the classification of irreducible characters of A_n of prime power degree, and we solve Huppert's question for alternating groups.

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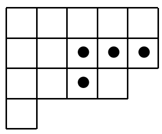
2 An algorithm for hook lengths

We refer to [5], [7] for details about partitions, Young diagrams and hooks. Consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of the integer n. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = n$. We call the λ_i's the parts of λ and m the length of λ. Moreover for $i \geq 1$, $m_i = m_i(\lambda)$ denotes the number of parts equal to i in λ. Thus $m = \sum_{i \geq 1} m_i$. The Young diagram of λ consists of n nodes (boxes) with λ_i nodes in the ith row. We refer to the nodes in matrix notation, i.e. the (i, j)-node is the jth node in the ith row. The (i, j)-hook consists of the nodes in the Young diagram to the right and below the (i, j)-node, and including this node. The number of nodes in this hook is its hooklength, denoted by h_{ij}. Thus



is the Young diagram of $(5^2, 4, 1)$, where we have marked the $(2, 3)$-hook belonging to the third node in the second row, and the corresponding hook length h_{23} is 4.

We put $h_i = h_{i1} = \lambda_i + (m - i)$ for $1 \leq i \leq m$; these are the *first column hook lengths*, abbreviated by *fch*.
The *degree* f_λ of λ is

$$f_\lambda = \frac{n!}{\Pi_{i,j} h_{ij}} \ .$$

It is known that this is the degree of the complex irreducible representation of the symmetric group S_n labelled by λ (see [5], [7]).

Example 2.1 *If* $\lambda = (n - k, 1^k)$, *a partition of* n *with* $0 < k < n$ *then* $f_\lambda = \binom{n-1}{k}$, *a binomial coefficient.*

Binomial coefficients are prime powers only in the "trivial" cases ([3], [11]):

Proposition 2.2 *The binomial coefficient* $\binom{n}{k}$ *is a (nontrivial) power of a prime exactly when* n *is a prime power and* $k = 1$ *or* $k = n - 1$.

This immediately implies the characterization of hook partitions of (non-trivial) prime power degree:

Corollary 2.3 *Suppose that* $\lambda = (n-k, 1^k)$ *with* $0 < k < n-1$. *Then* $f_\lambda = p^r$ *for some prime* p *and integer* $r \geq 1$ *if and only if* $n = p^r + 1$ *and* $k = 1$ *or* $k = p^r - 1$.

The following theorem characterizes those partitions (resp. irreducible characters of symmetric groups) that are of prime power degree.

Theorem 2.4 *Let* λ *be a partition of* n. *Then* $f_\lambda = p^r$ *for some prime* p, $r \geq 1$, *if and only if one of the following occurs:*

$$n = p^r + 1\,,\ \lambda = (p^r, 1) \text{ or } (2, 1^{p^r - 1})\,,\ f_\lambda = p^r$$

or we are in one of the following exceptional cases:

$$\begin{array}{lll}
n = 4: & \lambda = (2^2), & f_\lambda = 2 \\
n = 5: & \lambda = (2^2 1) \text{ or } (3,2)\,, & f_\lambda = 5 \\
n = 6: & \lambda = (4,2) \text{ or } (2^2 1^2)\,, & f_\lambda = 3^2 \\
 & \lambda = (3^2) \text{ or } (2^3)\,, & f_\lambda = 5 \\
 & \lambda = (321)\,, & f_\lambda = 2^4 \\
n = 8: & \lambda = (521) \text{ or } (321^3)\,, & f_\lambda = 2^6 \\
n = 9: & \lambda = (72) \text{ or } (2^2, 1^5)\,, & f_\lambda = 3^3
\end{array}$$

First we state some elementary results about hook lengths. In the following, λ is a partition of n, m_1 is the multiplicity of 1 as a part of λ, and h_{ij}, h_i are the hook lengths as defined above.

The following lemma is elementary.

Lemma 2.5 *If* $h_{i2} \neq 0$ *(i.e.* $\lambda_i \geq 2$*) then*

$$h_{i2} = h_i - m_1 - 1\,.$$

Proposition 2.6 *Let* $1 \leq i, j \leq m$, $i \neq j$. *Then*

$$h_i + h_j - n - 1 \leq m_1\,.$$

Proof. It suffices to prove this for h_1 and h_2. If $\lambda = (n-k, 1^k)$ is a hook partition, then the result is trivially true. If λ is not a hook partition, then $h_{22} \neq 0$ so that $h_{22} = h_2 - m_1 - 1$ by Lemma 2.5. Since $h_{11} = h_1$ and since $h_{11} + h_{22} \leq n$ the result follows. $\diamond$

Lemma 2.7 *Suppose that* $s = h_{ik}$ *and* $t = h_{j\ell}$ *where* $(i,k) \neq (j,\ell)$.

(1) If $i \neq j$ and $k \neq \ell$, then $s + t \leq n$.

(2) If $s + t > n$, then either $i = j = 1$ (both hooks in the first row) or $k = \ell = 1$ (both hooks in the first column).

Proof. (1) By assumption, the two hooks have at most one node in common. If they have a node in common, none of the hooks is the $(1, 1)$-hook. Thus, the hooks plus possibly the $(1, 1)$-node comprise $s + t$ nodes, whence $s + t \leq n$.

(2) By (1), we know that $i = j$ or $k = \ell$. Assume the former so that $k \neq \ell$. If $i > 1$, then $h_{1k} > h_{ik}$ whence $h_{1k} + h_{j\ell} > n$, contradicting (1) applied to $(1, k)$ and (j, ℓ). The case $k = \ell$ is similar. $\diamond$

Corollary 2.8 *For $i \geq 2$, every hook of length $t > n - h_i = n - h_{i1}$ is in the first column of λ.*

Proof. Assume that $t = h_{j\ell}$. If $(j, \ell) = (i, 1)$, then the result is true. Otherwise, apply (2) of Lemma 2.7 with $(i, k) = (i, 1)$ to get $\ell = 1$. $\diamond$

From now on, assume that f_λ is a power of a prime and that λ is *not* a hook partition. For $n \leq 6$ one easily checks Theorem 2.4 by hand (or by using the tables in [5]). So we assume from now on that $n > 6$. Consequently, it follows that $f_\lambda \geq n + 1$ ([5], Theorem 2.4.10).

Proposition 2.9 *If q is a prime for which $n - m_1 \leq q \leq n$, then*

$$q,\ 2q,\ \dots,\ \left[\frac{n}{q}\right] q$$

are all fch of λ.

Proof. Put $w = \left[\frac{n}{q}\right]$, $n = wq + r$, $0 \leq r < q$. By assumption, we have that $(w - 1)q \leq (w - 1)q + r = n - q \leq m_1$. Since m_1 is the multiplicity of 1 in λ, the numbers $1, 2, \dots, m_1$ are *fch*. In particular, we have that $q, 2q, \dots, (w - 1)q$ are *fch*. If $wq \leq m_1$, then we are done. Assume that $m_1 < wq$. At most w hooks in λ are of lengths divisible by q (see e.g. [7], Proposition (3.6)). If there are only the above $(w - 1)$ hooks in the first column of length divisible by q, then $q | f_\lambda$ since $\prod_{i=1}^{w}(iq) \mid n!$. By assumption, f_λ is then a power of q. We get $f_\lambda = (wq)_q$, the q-part of wq. Thus $f_\lambda | wq \leq n$, whence $f_\lambda \leq n$, a contradiction. Let h_{ij} be the additional hook length divisible by q. Since $\lambda \neq (1^n)$, $m_1 \leq h_2$. If $h_2 > m_1$, then $h_{ij} + h_{21} > q + m_1 \geq n$. By Corollary 2.8 we get $j = 1$. If $h_2 = m_1$, then

$\lambda = (n - m_1, 1^{m_1})$ and since $m_1 < wq$ there has to be a hook of length divisible by q in the first row. Since $n - m_1 \leq q$ it has to be the $(1,1)$-hook. Thus $h_{11} = wq$. $\diamond$

Corollary 2.10 *Let $1 \leq i < j \leq m$. If $h \leq n$ has a prime divisor q satisfying $2n - h_i - h_j < q$, then h is a fch of λ.*

Proof. By Proposition 2.6, $n - m_1 \leq 2n + 1 - h_i - h_j$. By assumption $2n + 1 - h_i - h_j \leq q \leq h \leq n$, whence $n - m_1 \leq q \leq n$. By Proposition 2.9, any multiple of q less or equal to n is a *fch*. In particular h is a *fch* of λ. $\diamond$

Lemma 2.11 *If q is a prime, $\frac{n}{2} < q \leq n$, then λ has a hook of length q.*

Proof. This follows immediately from the degree formula and the fact that $f_\lambda \geq n + 1$. $\diamond$

We are now going to strengthen our assumption on n and λ slightly. According to Table 3 of [2] there are, for all $n \geq 12$, at least two distinct primes p, q with $\frac{n}{2} < p, q \leq n$. By Lemma 2.11 there are hooks of length p and q in λ. We will assume that such primes p, q exist for n and that p and q are *fch*. This is not a restriction, see Lemma 2.7(2) (if necessary we may replace λ by its conjugate partition λ^0 as we have $f_\lambda = f_{\lambda^0}$). Now the above assumption forces *any* prime between $\frac{n}{2}$ and n to be a *fch* of λ.

Proposition 2.12 *Suppose we have sequences of integers $s_1 < s_2 < \cdots < s_r \leq n$, $t_1 < t_2 < \cdots < t_r \leq n$ satisfying*

(i) $s_i < t_i$ for all i;

(ii) s_1 and t_1 are primes $> \frac{n}{2}$;

(iii) For $1 \leq i \leq r - 1$, s_{i+1} and t_{i+1} contain prime factors exceeding $2n - s_i - t_i$.

Then $s_1, \ldots, s_r, t_1, \ldots, t_r$ are all fch of λ.

Proof. We use induction on i to show that s_i and t_i are *fch* for λ. For $i = 1$ this is true by our assumption. If s_i and t_i are *fch*, then Corollary 2.10 shows that s_{i+1} and t_{i+1} are *fch* of λ. $\diamond$

We get an algorithm from Proposition 2.12 which shows that h_1 is large and thus λ is "almost" a hook: Start with two large primes $s_1 < t_1$ close to n.

Then $2n - s_1 - t_1$ is small. Choose if possible two integers s_2 and t_2 with $s_2 < t_2$, $s_1 < s_2 \le n$, $t_1 < t_2 \le n$ each having a prime divisor exceeding $2n-s_1-t_1$. Then $2n-s_2-t_2 < 2n-s_1-t_1$. Choose if possible two integers s_3 and t_3 with $s_3 < t_3$, $s_2 < s_3 \le n$, $t_2 < t_3 \le n$ each having a prime divisor exceeding $2n - s_2 - t_2$ and so on. If this process reaches s_r, t_r, then $t_r \le h_1$ by Proposition 2.12.

Example 2.13 $n = 189 = 3^3 \cdot 7$. *Choose* $s_1 = 179$, $t_1 = 181$. *Then* $2n-s_1-t_1 = 18$. *Now choose* $t_2 = 188 = 4 \cdot 47$ *and* $s_2 = 186 = 2 \cdot 3 \cdot 31$ *which have prime factors exceeding* 18. *Then* $2n - s_2 - t_2 = 4$. *Choose* $t_3 = 189$, $s_3 = 188$. *Thus if* f_λ *is a prime power then* $h_1 = 189$ *and* λ *is a hook partition, contradicting Corollary 2.3. Thus none of the* 1.527.273.599.625 *partitions of* $n = 189$ *is of prime power degree greater than one.*

3 Prime factors in consecutive integers and good sequences

In this section we show, for sufficiently large n, that there are suitable sequences as in Proposition 2.12 that end with numbers close to n. Thus, for a partition λ of prime power degree the algorithm described in the previous section shows that λ differs from a hook partition only by a small amount.

Suppose that $n \ge 3$ is a positive integer. Consider two finite increasing sequences of integers $\{A_i\}$ and $\{B_i\}$ which satisfy the following properties:

(i) $A_1 < B_1 \le n$ are two "large" primes not exceeding n.

(ii) For every i, we have that

$$A_i < B_i \le n.$$

(iii) If $B_i < n$, then $A_{i+1} < B_{i+1}$ are integers not exceeding n each with a prime factor exceeding $2n - A_i - B_i$.

Then denote by $A(n)$ (resp. $B(n)$) the largest integer in such a sequence $\{A_i\}$ (resp. $\{B_i\}$).
We want to show that there are such sequences with $n - B(n)$ "small".

More precisely, we prove the following theorem:

Theorem 3.1 *If* $n > 3.06 \cdot 10^8$, *then there is a pair of sequences* $\{A_i\}$ *and* $\{B_i\}$ *as above for which*

$$n - B(n) \le 225.$$

We note that the 225 in the theorem above can be reduced to 2 for sufficiently large n. However, this result is of no use in the present paper.

We first review some facts about the distribution of primes, and we prove a theorem on the prime divisors of a product of consecutive integers. Using this, we then prove Theorem 3.1.

Throughout this section p shall denote a prime. Now we recall three relevant functions. If $X > 0$, then define $\pi(X)$ and $\theta(X)$ by

$$\pi(X) := \#\{p \leq X\}, \qquad (1)$$
$$\theta(X) := \sum_{p\leq X} \log p. \qquad (2)$$

Moreover, recall that von Mangoldt's function $\Lambda(n)$ is defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^{\alpha} \text{ with } \alpha \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

Rosser and Schoenfeld [9], [10] proved the following unconditional inequalities. These inequalities will be important in the proof of Theorem 3.1.

Theorem 3.2 *(Rosser-Schoenfeld)*

(1) If $X > 1$, then

$$\pi(X) \leq 1.25506 \cdot \frac{X}{\log X}.$$

(2) If $X > 1$, then

$$\sum_{1\leq n\leq X} \frac{\Lambda(n)}{n} \leq \log X.$$

(3) If $X \geq 1319007$, then

$$0.998684 \cdot X < \theta(X) < 1.001102 \cdot X.$$

Now we prove the following crucial result about the prime factors of consecutive integers.

Lemma 3.3 *If $1 \leq m \leq k \leq y < n$ are integers for which*

$$\left(\frac{n}{k}\right)^k \geq (n+k)^{\pi(y)+m-1}, \qquad (4)$$

then at least m of the integers $n+1, n+2, \ldots, n+k$ have a prime factor exceeding y.

Proof. If L is defined by

$$L := \prod_{j=1}^{k}(n+j) = \prod_{p} p^{\alpha_p},$$

then let R and S be the unique integers for which $L = RS$ and

$$\begin{array}{rcll} R & = & \prod_{p>y} p^{\alpha_p}, & \quad (5) \\ S & = & \prod_{p\le y} p^{\alpha_p}. & \quad (6) \end{array}$$

Since $n+k$ is the largest factor defining L, if

$$R > (n+k)^{m-1}, \qquad (7)$$

then at least m of the numbers $n+1, n+2, \ldots, n+k$ have a prime factor exceeding y. Therefore, it suffices to prove (7). Since $L = RS > n^k$, we trivially have that

$$R > \frac{n^k}{S},$$

and so by (7) it suffices to prove that

$$\frac{n^k}{S} > (n+k)^{m-1}. \qquad (8)$$

Now we derive an upper bound for S. By definition, we have that

$$\log S = \log \prod_{p\le y} p^{\alpha_p} = \sum_{p\le y} \alpha_p \log p = \sum_{p\le y} \log p \sum_{\alpha=1}^{\infty} \left(\sum_{1\le j\le k\,,\, p^{\alpha}|n+j} 1 \right) .$$

Now, the innermost sum clearly has the upper bound $\left[\frac{k}{p^{\alpha}}\right] + 1$ Moreover, since this bound equals 1 whenever $p^{\alpha} > k$, by Theorem 3.2(2) we find that

$$\begin{array}{rcll} \log S & \le & \sum_{p^{\alpha}\le k} \log p \left[\frac{k}{p^{\alpha}}\right] + \sum_{p\le y} \log p \sum_{p^{\alpha}\le n+k} 1 & \\ & \le & k \sum_{d=1}^{k} \frac{\Lambda(d)}{d} + \sum_{p\le y} \log p \left[\frac{\log(n+k)}{\log p}\right] & \\ & \le & k\log k + \pi(y)\log(n+k). & \qquad (9) \end{array}$$

Therefore, we have that

$$S \le k^k (n+k)^{\pi(y)},$$

and so

$$\frac{n^k}{S} \ge \left(\frac{n}{k}\right)^k (n+k)^{-\pi(y)}.$$

However, since $\left(\frac{n}{k}\right)^k \geq (n+k)^{\pi(y)+m-1}$ by (4) we have that

$$\frac{n^k}{S} \geq (n+k)^{m-1}$$

which is (8). $\diamond$

As a consequence of Theorem 3.2 and Lemma 3.3, we obtain the following crucial result.

Theorem 3.4 *If $n > 3.06 \cdot 10^8$ is an integer and k is a positive integer satisfying*

$$168 \leq k \leq \frac{n}{4},$$

then at least three of the integers $n+1, n+2, \ldots, n+k$ have a prime factor exceeding $4k$.

Proof. By Theorem 3.2 (3), we have that

$$\theta(n+k) - \theta(n) > 0.998684(n+k) - 1.001102n = 0.998684k - 0.002418n.$$

So, if $n/400 < k \leq n/4$ and $n \geq 1319007$, then

$$\theta(n+k) - \theta(n) > 0.0000787n > 2\log(5n/4) \geq 2\log(n+k). \qquad (10)$$

Since

$$\theta(n+k) - \theta(n) = \sum_{n<p\leq n+k} \log p \leq (\pi(n+k) - \pi(n))\log(n+k),$$

(10) implies that there are at least three primes among the numbers $n+1, n+2, \ldots, n+k$ provided that $n/400 < k \leq n/4$ and $n \geq 1319007$. Moreover, by hypothesis these primes are $\geq n+1 > 4k$.
Next we consider the cases where $100 \leq k \leq n/400$ and $n \geq 1.8 \cdot 10^{14}$. By Lemma 3.3 it suffices to verify that

$$\left(\frac{n}{k}\right)^k \geq (n+k)^{\pi(4k)+2} \qquad (11)$$

whenever $100 \leq k \leq n/400$ and $n > 1.8 \cdot 10^{14}$.
By Theorem 3.2(1), if $k \geq 100$, then

$$\pi(4k) + 2 \leq \frac{5.02024k}{\log(4k)} + 2 < \frac{5.15k}{\log(4k)}.$$

Therefore, (11) holds as soon as

$$\left(\frac{n}{k}\right)^k \geq ((1+\frac{1}{400})n)^{\frac{5.15k}{\log(4k)}} \geq (n+k)^{\frac{4.95k}{\log(4k)}}.$$

By taking logarithms, the first inequality is equivalent to

$$\log(4k)\log(n/k) \geq 5.15 \log\left(\left(1+\frac{1}{400}\right)n\right). \qquad (12)$$

However, for a fixed value of n the function on the left hand side of this inequality is an increasing function in k in the interval $[1, \sqrt{n}/2]$ and is decreasing for larger k thus taking the minimal value $\log 400 \cdot \log(n/100)$ at the endpoints. It is easy to verify that (12) holds for all k in the interval $[100, n/400]$ provided that $n > 1.8 \cdot 10^{14}$. Similarly, we can show that (11) holds for all $500 \leq k \leq n/2000$ and $n \geq 3 \cdot 10^8$.

To complete the proof in the remaining cases we use Maple on a PC. On one hand, one can, using the Nextprime function, check that for $j = 0, \dots, 6$ there is always a prime in any interval of type $(m \cdot 10^{4+j}, (m+1) \cdot 10^{4+j}]$ where $10^4 \leq m \leq 10^5$. This implies immediately that there are at least three primes in any interval of type $(n, n + n/2000]$ where $10^8 \leq n \leq 10^{15}$. On the other hand using $\log(n+k) < \log n + k/n$, (11) follows from

$$\log n \geq \frac{k \log k + (\pi(4k)+2)k/n}{k - \pi(4k) - 2}.$$

This is easily verified (using only a table of primes below 2000) for $168 \leq k \leq 500$ and $n > 3.06 \cdot 10^8$. All cases are considered. $\diamond$

Proof of Theorem 3.1.
Suppose that $n > 3.06 \cdot 10^8$ and pick two primes $0.8n < A_1 < B_1 \leq 0.9n$ which is allowable by Theorem 3.2 (3). Note that $B_1 - A_1 \leq n - B_1$ by hypothesis. Now suppose that when constructing the sequences we have

$$B_i - A_i \leq n - B_i.$$

Now we seek new integers $A_{i+1} < B_{i+1} < n$ for which

$$\begin{aligned} B_{i+1} - A_{i+1} &\leq n - B_{i+1}, \qquad (13)\\ B_i < A_{i+1} &< B_{i+1} \leq n. \qquad (14) \end{aligned}$$

and each with a prime factor exceeding $3(n - B_i)$.
Now we apply Theorem 3.4 with $n = B_i$ and $k = \lceil \frac{3}{4}(n - B_i) \rceil$ (Note. $\lceil x \rceil$ denotes the smallest integer $\geq x$). Obviously, we have $4k \geq 3(n - B_i) > 2n - A_i - B_i$. As long as $k \geq 168$ we can find three integers $B_i < a < b < c \leq B_i + k$ each with a prime factor $> 4k$.
We now show that either the pair a and b or the pair b and c satisfies (13) and (14). If neither does, then

$$c - b > n - c \quad \text{and} \quad b - a > n - b$$

which are equivalent to

$$c - b \geq n - c + 1 \quad \text{and} \quad b - a \geq n - b + 1.$$

These imply that

$$\begin{aligned} a \leq 2b - n - 1 \quad & \leq 2(2c - n - 1) - n - 1 = 4c - 3n - 3 \\ & \leq 4(B_i + k) - 3n - 3 \\ & \leq 4B_i + 4\left(\tfrac{3}{4}(n - B_i) + \tfrac{3}{4}\right) - 3n - 3 = B_i. \end{aligned}$$

Since $B_i < a$ we see that we can always choose such an A_{i+1} and B_{i+1} provided that $k \geq 168$. Suppose that A_{i+1} and B_{i+1} are the last terms which are found this way, then we have $n - B_{i+1} \leq 225$. This proves Theorem 3.1. $\diamond$

Corollary 3.5 *Let λ be a partition of n with largest hook length h_1. If λ is of prime power degree and $n > 3.06 \cdot 10^8$, then $n - h_1 \leq 225$.*

Proof. This follows immediately from Theorem 3.1 and Proposition 2.12. $\diamond$

4 Proof of the classification result for S_n

For dealing with the situation where $c = n - h_1$ is small, we provide a good upper bound for the p-powers in the character degrees for S_n. This is similar to the case of binomial coefficients (i.e. the case of hook partitions).

Proposition 4.1 *Let λ be a partition of n, and set $c = n - h_1$. Let p be a prime, and l the integer with $p^l \leq n < p^{l+1}$.*
Then

$$\nu_p(f_\lambda) \leq \nu_p((2c+2)!) + 2l \ .$$

In particular, a bound for the p-part of f_λ is given by

$$(f_\lambda)_p \leq n^2 \cdot ((2c+2)!)_p \ .$$

Proof. Let $k = m_1$ be the multiplicity of 1 in λ. By looking at the Young diagram we see that

$$\lambda_1 - \lambda_2 \geq h_1 - k - (c+2) = n - 2c - k - 2 \ .$$

Let A denote the set of nodes other than the final k nodes in the leg and the final $n - 2c - k - 2$ nodes in the arm of the $(1,1)$-hook; for a node $y \in A$

let h_y denote the corresponding hook length. Then from the degree formula we obtain

$$\nu_p(f_\lambda) = \sum_{i=1}^{l} \left(\left[\frac{n}{p^i}\right] - \left[\frac{k}{p^i}\right] - \left[\frac{n-2c-k-2}{p^i}\right] \right) - \sum_{y \in A} \nu_p(h_y)$$

For fixed i, the ith summand in the first sum gives a contribution of at most $\left[\frac{2c+2}{p^i}\right] + 2$, hence

$$\nu_p(f_\lambda) \leq \sum_{i=1}^{l} \left(\left[\frac{2c+2}{p^i}\right] + 2 \right) \leq \nu_p((2c+2)!) + 2l \; .$$

The second inequality follows immediately from this. $\diamond$

We first consider the case of small n. Assume again that λ is a partition of n of prime power degree but not a hook.

Using the available tables (or with the aid of MAPLE), it is easy to check the main theorem for $n < 43$. In other words, the only partitions having prime power degree are of the form $(n-1, 1)$ together with their conjugates, or are on the short list of exceptions given in the theorem.

For a midsized n (i.e. $43 \leq n \leq 9.25 \cdot 10^8$), we use the following number theoretic condition.

Fix a number b. Given a number n, let p_1, p_2 be the two largest primes below n. Then check whether there is a prime divisor q of $n(n-1)(n-2)\cdots(n-b)$ with $p_1 + p_2 + q > 2n$.

A computer program (written in C++ using the LiDIA number theory package, run on a super computer with 32 nodes, running time 2 hours) was used to check that this condition is satisfied for all n from 29 to $9.25 \cdot 10^8$ for $b = 4$.

We now want to use this in the situation where λ is a partition of n of prime power degree, $43 \leq n \leq 9.25 \cdot 10^8$. Since the two largest primes p_1, p_2 are *fch* of λ, if q is a prime divisor as in the condition above (with $b = 4$), then by Corollary 2.10 one of the numbers $n, n-1, \ldots n-4$ is a *fch* of λ. Hence $n - h_1 \leq 4$.

From Proposition 4.1 we know that if λ is of prime power degree, then

$$f_\lambda \leq o(c) \cdot n^2$$

where $c = n - h_1$ and $o(c) = \max\{(2c+2)!_p \mid p \text{ prime }\}$.
In our situation we have $c \leq 4$, so $o(c) \leq o(4) = 256$, and hence we know

$$f_\lambda \leq 256 \cdot n^2$$

Now we use information on the minimal degrees of S_n-representations. Burnside's theorem on the minimal degree > 1 for S_n was greatly generalized by Rasala [8], giving in a suitable sense the list of the minimal degrees for sufficiently large n (depending on the requested length of the list). We use the notation from [8].

For any k, the list of minimal degrees for S_n starts with the degrees of partitions of n coming from partitions of numbers $d \leq k$ by adding on a part $n - d$, if $n \geq B_k$, a bound which is provided explicitly in [8]. The degree polynomial $\varphi_\mu(n)$ for any partition μ of k is also given explicitly.

For $k = 5$, one has $B_5 = 43$. One easily checks that for $n \geq 43$

$$256 \cdot n^2 \leq \varphi_5(n) = \frac{1}{5!}(n-9)\prod_{i=0}^{3}(n-i)$$

Hence f_λ is among the minimal degrees for S_n. But the list of minimal degrees is easily computed [8], and none of these is a prime power except possibly $n-1$ which occurs only for $(n-1, 1)$ and its conjugate (use that the degree formula in [8], Theorem A, gives a factorized expression in which consecutive numbers appear). Hence we do not get any further partitions of prime power degree.

Now we deal with the case of large n (i.e. $n > 9.25 \cdot 10^8$). By Corollary 3.5 we know that $n - h_1 \leq 225$. We want to use the bound for the degree and the minimal degree argument again in this situation. From [8] we get the numbers B_k from which on the minimal degrees all come from partitions of weight at most k.

Since $B_{18} = 310.390.100 < n$, we thus know a long list of minimal degrees. The maximal entry on this list comes from the partition $\mu = (64321^3)$ which is of maximal degree 16.336.320 among the partitions of 18.
For $c = 225$ we have $o(c) = 2^{448}$ and one checks (for example, using MAPLE) that for $n \geq 9.25 \cdot 10^8$ one has

$$f_\lambda \leq 2^{448} \cdot n^2 \leq \varphi_\mu(n) = \frac{[\mu](1)}{18!}\prod_{i=1}^{18}(n - \mu_i - 18 + i)$$

where $\mu = (\mu_1, \ldots, \mu_{18})$, extending μ by parts 0 if necessary.

Hence f_λ is among the minimal degrees, but as before, one can check that all the minimal degrees on this list are not prime powers, except possibly $n-1$. This completes the proof. $\diamond$

5 Alternating groups and a question of Huppert

The purpose of this section is to prove the analogue of Theorem 2.4 for the alternating groups. Also we answer a question of B. Huppert about character degrees in alternating groups.
If λ is a partition of n, then the irreducible representation of S_n labelled by λ remains irreducible when restricted to A_n if and only if $\lambda \neq \lambda^0$, the conjugate (associated) partition of λ. If $\lambda = \lambda^0$ the restriction is a sum of two irreducible representations of the same degree. This leads us to the following definition providing the character degrees in A_n ([5], 2.5).
Let λ be a partition of n. Then

$$\tilde{f}_\lambda := \begin{cases} f_\lambda & \text{if} \quad \lambda \neq \lambda^0 \\ \frac{1}{2} f_\lambda & \text{if} \quad \lambda = \lambda^0 . \end{cases}$$

Theorem 5.1 *Let λ be a partition of n. Then $\tilde{f}_\lambda = p^r$ for some prime p, $r \geq 1$, if and only if one of the following occurs:*

$$n = p^r + 1 > 3\,,\ \lambda = (p^r, 1) \text{ or } (2, 1^{p^r-1})\,,\ \tilde{f}_\lambda = p^r$$

or we are in one of the following exceptional cases

$$\begin{array}{lll} n=5: & \lambda = (2^2 1) \text{ or } (3,2)\,, & \tilde{f}_\lambda = 5 \\ & \lambda = (31^2)\,, & \tilde{f}_\lambda = 3 \\ n=6: & \lambda = (4,2) \text{ or } (2^2 1^2)\,, & \tilde{f}_\lambda = 3^2 \\ & \lambda = (3^2) \text{ or } (2^3)\,, & \tilde{f}_\lambda = 5 \\ & \lambda = (321)\,, & \tilde{f}_\lambda = 2^3 \\ n=8: & \lambda = (521) \text{ or } (321^3)\,, & \tilde{f}_\lambda = 2^6 \\ n=9: & \lambda = (72) \text{ or } (2^2, 1^5)\,, & \tilde{f}_\lambda = 3^3 \end{array}$$

Proof. If $\lambda \neq \lambda^0$ we may apply Theorem 2.4. Suppose $\lambda = \lambda^0$. The only self conjugate partitions occurring in the list of Theorem 2.4 are $(2,1)$ and (2^2) and here $\tilde{f}_\lambda = 1$. Using the character tables of [1] there are no further occurrences of self conjugate partitions with $\tilde{f}_\lambda = p^r$ for $n \leq 13$. When $n > 13$ we may always find two primes p_1, p_2 satisfying $\frac{n+1}{2} < p_1 < p_2 \leq n$, (using [2], Table 3). Then $2p_1$ and $2p_2$ are not character degrees of S_n. If $\lambda = \lambda^0$ and $f_\lambda = 2p^r$ then $p \neq p_1$. Thus λ has to contain hooks of lengths p_1 and p_2. Since λ contains only one hook of length p_1 and p_2 respectively, both of them have to be in the diagonal, i.e. $p_1 = h_{ii}$, $p_2 = h_{jj}$ for some i, j.

This contradicts Lemma 2.7. $\diamond$

In recent work on the characterization of the finite simple groups $PSL(n,q)$ by character degree properties [4], B. Huppert needs the following:

Corollary 5.2 *Suppose that for some simple alternating group A_n there is a prime p, such that all irreducible character degrees of A_n are either prime to p or powers of p. Assume that some power of p is a degree for A_n. Then $n = 5$ and $p = 2, 3$ or 5, or $n = 6$ and $p = 3$.*

Proof. Using the Atlas [1] we may assume $n > 13$. By Theorem 5.1, A_n only has a prime power character degree p^r when $n = p^r + 1$. But then

$$\tilde{f}_{(n-2,1^2)} = \frac{(n-1)(n-2)}{2} = p^r(p^r - 1)/2$$

is a character degree which is divisible by p and not a power of p.

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