Separating characters by blocks

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Abstract

We investigate the problem of finding a set of prime divisors of the order of a finite group, such that no two irreducible characters are in the same p-block for all primes p in the set. Our main focus is on the simple and quasisimple groups. For results on the alternating and symmetric groups and their double covers, some combinatorial results on the cores of partitions are proved, which may be of independent interest. We also study the problem for groups of Lie type. The sporadic groups (and their relatives) are checked using GAP.

1 Introduction. Block separation of characters

Let G be a finite group, $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G, and $\pi(G)$ the set of primes dividing the order $|G|$ of G. If $p \in \pi(G)$, we call two characters in $\mathrm{Irr}(G)$ *p-separated* if they are contained in different p-blocks of G. It is known by a result of Osima [20] that p-separated characters satisfy refinements of the orthogonality relations: They are orthogonal already across the so-called p-sections of G. (See the first sections of [11] for more general results in this direction.)

If $\pi \subseteq \pi(G)$ and $I \subseteq \mathrm{Irr}(G)$, we call I *separable by* π if any two characters in I are p-separated for some prime $p \in \pi$; I is called *separable* if it is separable by $\pi(G)$. Also we call an irreducible character *isolated by* π if it is contained in a p-block of defect 0 for some $p \in \pi$; it is *isolated* if it is isolated by $\pi(G)$. Clearly a set of π-isolated characters is separable by π.

The set $\mathrm{Irr}(G)$ of irreducible characters of G is not always separable. If for instance G is a group of p-power order, then G has only one p-block and there is no separation at all of the irreducible characters. Also, by [6, Lemma IV.4.25] we have that if $p, q \in \pi(G)$, $p \neq q$, and if G has no element of order pq then $\mathrm{Irr}(G)$ is not separated by $\{p, q\}$. In particular, if G is a nonabelian group of order pq where $q > p$ are primes, then $\mathrm{Irr}(G)$ is

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not separable. (Such a group G of course has a unique q-block and several p-blocks.)

Consider a direct product $G \times H$ of non-trivial groups G and H. Then clearly $G \times H$ is separable if G and H are both separable.
If we assume that $\pi(G) \cap \pi(H) = \emptyset$, then $G \times H$ is separable by $\{p, q\}$, where $p \in \pi(G)$, $q \in \pi(H)$. In particular, for a nilpotent group G with $|\pi(G)| \geq 2$, $\mathrm{Irr}(G)$ is separable by $\pi(G)$ and in fact by any subset of $\pi(G)$ with at least two elements.

In this paper we study mainly separability of irreducible characters of finite symmetric groups and related groups and of finite quasisimple groups of Lie type.

In Section 2 we show that for $n \in \mathbb{N}$ (apart from exceptions for small n) a set π of three odd primes $\leq n$ can be found such that the irreducible characters of the symmetric and alternating groups of degree n are separated by π. In this connection it should also be remarked that two primes are not enough.

A similar result holds for their double covering groups as shown in Section 3, except that associate spin characters cannot always be separated.

We also discuss separation in the groups of Lie type in Section 4. In Theorem 4.1 we classify the cases where characters of quasi-simple groups of Lie type are not separated. In the case of exceptional groups of Lie type, we prove a stronger result. With some explicitly listed exceptions we have: In a finite quasi-simple group G with $G/Z(G)$ of exceptional Lie type there is a set π of four primes, such that the set $\mathrm{Irr}(G)$ is π-isolated.

Using GAP [22], one can also easily check the sporadic simple groups to see that except for a pair of irreducible characters of M_{12} any two irreducible characters can always be separated (three primes do not always suffice).
Also, the irreducible characters of the 3-fold cover $3.A_7$ can be separated by the primes 2,5,7. For $3.A_6$, $6.A_6$ and $6.A_7$ there are sets of irreducible characters which cannot be separated. For more results on sporadic groups, their automorphism groups and cyclic central extensions, see Section 5.

Finally, we would like to mention that motivated by a question of the authors, Navarro, Turull and Wolf [17] have shown that for any set π of n primes (where $n \geq 2$) there is a solvable π-group G such that $\mathrm{Irr}(G)$ is π-separable but is not separated by any proper subset of π.

2 The symmetric groups

We refer to [10] or [19] for facts about hooks and cores of partitions.

Consider a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of the integer n. For a given prime $p \in \mathbb{N}$, we denote by $w_p(\lambda)$ the *p-weight* of λ, i.e., $w_p(\lambda)$ is the maximal number of p-hooks that can successively be removed from λ. The resulting partition after removing this maximal number of p-hooks is then the *p-core* $\lambda_{(p)}$ *of* λ. We want to show that partitions are essentially uniquely determined by their set of p-cores for all (odd) primes $p \leq n$. There are only a few exceptions for small values of n. This is related to the question of separation of characters in symmetric groups by the so-called Nakayama conjecture [10, Theorem 6.1.21]. It states that two irreducible characters of S_n are in the same p-block if and only if the partitions labelling them have the same p-core.

As pointed out in the introduction, two primes are not enough to guarantee separation; more precisely, we have:

Proposition 2.1 *Let $n \in \mathbb{N}$, $p, q \leq n$ two different primes. Then there is a partition $\lambda \neq (n)$ of n with the same p-core and the same q-core as (n).*

Proof. Write $n = sp + a$, with $0 \leq a < p$ and $n = tq + b$, with $0 \leq b < q$. If $a = b$, then $n - a$ is divisible by pq, and hence (n) and $\lambda = (a, 1^{n-a})$ have the same p-core and the same q-core.
Now suppose $a > b$. Then $b + 1 \leq a < p$. Note that $b + 1 \leq n - p$ since otherwise $b \geq n - p \geq n - sp = a$, a contradiction.
Hence we can place a hook of length p with arm length b beneath a part $n - p$. The resulting partition $\lambda = (n - p, b + 1, 1^{p-b-1})$ has hook lengths $h_{11} = n - b = tq$ and $h_{21} = p$, and thus λ and (n) have the same p-core (a) and the same q-core (b). $\diamond$

Our first main result is the following:

Theorem 2.2 *Let λ, μ be a partitions of n. Let $p_1, \dots, p_k \leq n$ be different odd primes, satisfying $p_1 > \frac{n}{2}$, $\prod_{i=2}^{k} p_i > n$. If*

$$\lambda_{(p_i)} = \mu_{(p_i)},\ \textit{for } i = 1, \dots, k,$$

then $\lambda = \mu$.

Proof. We assume $\lambda \neq \mu$ and

$$\lambda_{(p_i)} = \mu_{(p_i)},\ i = 1, \dots, k\ ,$$

and we seek a contradiction. Set $p = p_1$. If λ or μ is a p-core we get a contradiction to the assumption $\lambda_{(p)} = \mu_{(p)}, \lambda \neq \mu$. Thus we may assume that $w_p(\lambda) = w_p(\mu) = 1$, since $2p > n$. Let $\kappa \vdash n - p < \frac{n}{2}$ be the common p-core of λ and μ. Let Y be the set of first column hook lengths of κ and let X be the set which is obtained from Y in the following way: It is the union of the set $\{0, 1, \cdots, p-1\}$ and the set obtained from Y by adding p to all its elements. (In the notation of [19] we have $X = Y^{+p}$.) Thus X is a β-set for κ, and its maximum is at most n. We may obtain β-sets X_λ for λ and X_μ for μ by adding p to an element of X. More specifically there exist *different* elements $a_1, a_2 \in X$ such that $a_1 + p, a_2 + p \notin X$ and such that

$$X_\lambda = X \cup \{a_1 + p\} \setminus \{a_1\}$$

$$X_\mu = X \cup \{a_2 + p\} \setminus \{a_2\}\,.$$

For any prime number s the s-cores of λ and μ depend only on the integral vectors $v^\lambda(s) = (v_0^\lambda, v_1^\lambda, \cdots, v_{s-1}^\lambda)$ and $v^\mu(s) = (v_0^\mu, v_1^\mu, \cdots, v_{s-1}^\mu)$, where

$$v_i^\lambda = |\{b \in X_\lambda |\ b \equiv i\ (mod\ s)\}|$$

$$v_i^\mu = |\{b \in X_\mu |\ b \equiv i\ (mod\ s)\}|\,.$$

(See p. 78-79 in [10] or [19, Proposition (3.2)].) Clearly $0 < |a_1 - a_2| \leq n$, since $a_1, a_2 \in X$ are distinct. Since $\prod_{i=2}^k p_i > n$, at least one of the primes $p_2, \ldots, p_k$ does not divide $a_1 - a_2$, say $q = p_j \nmid a_1 - a_2$. Thus $a_1 \not\equiv a_2 \mod q$ and $a_1 + p \not\equiv a_2 + p \mod q$. Also for $i = 1, 2$ $a_i \not\equiv a_i + p \mod q$. Since by assumption $v^\lambda(q) = v^\mu(q)$, we must have $a_1 \equiv a_2 + p \mod q$ and $a_2 \equiv a_1 + p \mod q$. This forces $q \mid 2p$, a final contradiction. ⋄

This result has the following consequence:

Corollary 2.3 *Let λ, μ be partitions of n.*
(1) If λ, μ have the same p-core for all odd primes $p \leq n$, then $\lambda = \mu$, unless $n \leq 6$ and λ, μ are both partitions in one of the following sets:

$\{(2), (1^2)\}$, $\{(3), (2,1), (1^3)\}$, $\{(4), (1^4), (2^2)\}$,
$\{(5), (2, 1^3)\}$, $\{(4,1), (1^5)\}$, $\{(6), (3,2,1), (1^6)\}$.

(2) If λ, μ have the same p-core for all primes $p \leq n$, then $\lambda = \mu$, unless $n \leq 6$ and λ, μ are both partitions in one of the following sets:

$\{(2), (1^2)\}$, $\{(3), (1^3)\}$, $\{(4), (1^4), (2^2)\}$, $\{(6), (1^6)\}$.

Proof. By [21] there are for all $n \geq 11$ at least two distinct primes p_1, p_2 with $\frac{n}{2} < p_1, p_2 \leq n$. Choosing $p_3 = 3$ we get a triple of primes satisfying the conditions of the theorem. Thus we are done in this case. For $n = 7, \ldots, 10$ we choose $(p_1, p_2, p_3) = (7, 5, 3)$. For $n \leq 6$ no suitable set of primes exists, and the corollary is checked directly. ⋄

Remarks 2.4 (i) In fact, the proof of the Corollary shows that for $n \geq 7$, $n \neq 10$, any two distinct partitions may be distinguished by their cores taken at two primes $> \frac{n}{2}$ and the prime 3, for $n = 10$ we take the primes $7, 5, 3$.
(ii) The exceptions in Corollary 2.3(2) have already turned up in [2] where we have classified partitions which are of maximal p-weight for all primes p. In the language here, this means that in particular we had determined all irreducible characters which cannot be separated from the trivial character.

The following corollary on character separation is an immediate consequence of Theorem 2.2 and the Nakayama conjecture [10, Theorem 6.1.21].

Corollary 2.5 *Let $n \in \mathbb{N}$, $n \geq 7$. There exists a set π of three odd primes $\leq n$, such that* $\mathrm{Irr}(S_n)$ *is π-separated.*

Note that by Remark 2.4(i) above, we can provide an explicit set π of three odd primes separating all irreducible characters of S_n, for $n \geq 7$.

Most of the exceptions for $n \leq 6$ above are no longer exceptions for the alternating groups A_n; on the other hand, we have to be slightly careful with symmetric partitions. For a partition λ we denote by λ' the conjugate partition.

Lemma 2.6 *Let $\lambda = \lambda'$ be a partition of $n \geq 3$. Then λ is a p-core for some odd prime $p \leq n$, unless λ is one of the partitions* $(2, 1), (2^2), (3, 2, 1), (4, 3, 2, 1)$. *The partition λ is a p-core for some prime $p \leq n$, unless $\lambda = (2^2)$.*

Proof. Let $p \leq n$ be a prime with $p > \frac{n}{2}$. If $\lambda \neq \lambda_{(p)}$, then λ must have a hook of length p. As $\lambda = \lambda'$, it is easy to see that this must be the hook length at position $(1, 1)$ in the Young diagram of λ. Hence if there is a second prime $q \leq n$ with $q > \frac{n}{2}$, then λ must be a q-core. Hence λ is a core at some odd prime $\leq n$ for $n = 5, 7, 8, 9$ and all $n \geq 11$ (see [21]). For $n = 3, 4, 6, 10$ the argument leads to the exceptions listed in the statement of the Lemma. Apart from the partition (2^2), all the other exceptional partitions are 2-cores. $\diamond$

The p-blocks of A_n are closely related to those of S_n; see [18] or [19] for details. Using this we obtain

Corollary 2.7 *Let $\chi \neq \psi$ be irreducible characters of A_n, $n \geq 3$. Then χ, ψ can be separated by an odd prime $p \leq n$, unless $n \leq 10$ and the character pair is in one of the sets*

$\{\{3\},\{2,1\}_+,\{2,1\}_-\},\{\{4\},\{2^2\}_+,\{2^2\}_-\},\{\{5\},\{4,1\}\},$
$\{\{6\},\{3,2,1\}_+,\{3,2,1\}_-\},\{\{4,3,2,1\}_+,\{4,3,2,1\}_-\}\ .$

The characters χ,ψ are not p-separable, for all primes $p\le n$, if and only if the character pair is in the set $\{\{4\},\{2^2\}_+,\{2^2\}_-\}$.

Remark 2.8 Again, we note that the exceptions in Corollary 2.7 above have already turned up in [2]. In the language here, we had classified in particular those irreducible characters of A_n which cannot be separated from the trivial character by any prime p.

3 The double covering groups

For background on the theory of spin characters of $\tilde{S}_n$ and $\tilde{A}_n$ we refer the reader to [8].
The associate classes of spin characters of a double covering group $\tilde{S}_n$ of S_n are labelled canonically by the partitions λ of n into distinct parts, i.e. $\lambda=(a_1,a_2,\cdots,a_m)$, $a_1>a_2>\cdots>a_m>0$, $a_1+\cdots+a_m=n$.
We also refer to such a partition as a *bar partition*. In analogy with partitions and hooks, for an odd prime p one defines p-bars in a bar partition λ, a corresponding p-bar weight $w_{\bar{p}}(\lambda)$ and a p-bar core $\lambda_{(\bar{p})}$. A detailed description of this may for instance be found in [19]. It is easily seen that the p-bar core of a bar partition is determined by the p-residues of the parts of λ. The set of bar partitions of n having the same p-bar core κ forms the *p-bar block* to κ of p-bar weight $\frac{1}{p}(n-|\kappa|)$.
In the following, we denote the partition obtained by ordering the parts of a composition α by α^o.

As before in Proposition 2.1, two odd primes are not sufficient to separate bar partitions:

Proposition 3.1 *Let $n\in\mathbb{N}$, $p,q\le n$ two different odd primes. Then there is a bar partition $\lambda\neq(n)$ of n with the same p-bar core and the same q-bar core as the bar partition (n).*

Proof. Again, write $n=sp+a$, with $0\le a<p$ and $n=tq+b$, with $0\le b<q$.
If $a=b\neq 0$, then we may take $\lambda=(n-a,a)$ (note that $n>2a$).
If $a=b=0$, then $n\ge pq>2p$ and we may take $\lambda=(n-p,p)$.
We may now assume $a>b$. If $n-p\neq p-b$ then we may choose $\lambda=(n-p,p-b,b)^o$ (if $b=0$, this part does not occur). But if $n-p=p-b$,

then $a+b=p$ and $n=p+a$, and hence we must then have $n-q \neq q-a$ and $n-q \neq a$, and thus we may take $\lambda=(n-q, q-a, a)^o$. $\diamond$

An analogue of Theorem 2.2 for bar partitions is the following:

Theorem 3.2 *Let λ, μ be bar partitions of n. Let $p_1, p_2, p_3 \leq n$ be odd primes with $p_1 > p_2 > p_3 \geq \frac{n}{2}$. If*

$$\lambda_{(\bar{p}_i)} = \mu_{(\bar{p}_i)},\ \textit{for } i=1,2,3,$$

then $\lambda=\mu$.

Proof. If λ or μ is of p_i-bar weight 0 for some $i \in \{1,2,3\}$, then we clearly have $\lambda=\mu$; so we may assume now that both bar partitions are in p_i-bar blocks of positive weight.
Set $p=p_1$, $q=p_2$, $r=p_3$. By assumption, $\lambda_{(\bar{p})}=\mu_{(\bar{p})}$; denote this bar core by κ. As $p > \frac{n}{2}$, λ, μ belong to a p-bar block of weight 1. Observe that κ is a bar partition of $n-p<p$, and hence all parts of κ lie in $\{1,\ldots,p-1\}$.
There are three types of bar partitions in this block:
(I) $\kappa \cup \{p\}$.
$(II)_a$ $\kappa \setminus \{a\} \cup \{a+p\}$, for some $a \in \kappa$.
$(III)_i$ $\kappa \cup \{i, p-i\}$, for some $i \in \{1, ..., \frac{p-1}{2}\}$, where $i, p-i \notin \kappa$.
Assume first that one of the partitions, say λ, is of type (I).
Case (a): μ is of type $(II)_a$.
As $\lambda_{(\bar{q})}=\mu_{(\bar{q})}$, also the partitions (p,a) and $(p+a)$ must have the same $\bar{q}$-core. But $q<p<a+p \leq n<2q$, and hence $a+p \not\equiv 0, a, p \mod q$; this gives a contradiction.
Case (b): μ is of type $(III)_i$.
In this case, the partitions (p) and $(p-i,i)$ must have the same $\bar{q}$-core. As $i \leq \frac{p-1}{2} < \frac{n}{2} < q$, this is only possible if $p-i \equiv 0 \mod q$. As $p-i<p \leq n<2q$, we must then have $p-i=q$, i.e., $i=p-q$. As $p-q \leq n-q < \frac{n}{2}$, we have by assumption that $r \nmid p-q$. But then $p, p-q, q \not\equiv 0 \mod r$, and hence the $\bar{r}$-cores of (p) and $(p-i,i)=(p-q,q)^o$ are different, giving a contradiction.

Thus we may now assume that λ, μ are not of type (I).
Next we consider the case that at least one of the partitions is of type (II), say λ is of type $(II)_a$.
Case (a): μ is of type $(II)_b$; we may assume $b>a$. Note that we have $q<p<a+p<b+p \leq n<2q$, so $a+p, b+p$ are not divisible by q.
As $(b+p,a)_{(\bar{q})}=(a+p,b)_{(\bar{q})}$ and clearly $a \not\equiv a+p \mod q$, we now have the following possibilities.
(i) $a \equiv 0 \mod q$. As $0<a<2q$, this implies $q=a<b<2q$, hence $b \not\equiv 0$

mod q, giving a contradiction.
(ii) $a+b+p \equiv 0 \mod q$. As $a+b \leq |\kappa| = n-p$, $a+b+p \leq n < 2q$, hence $a+b+p = q$, leading to the contradiction $p < q$.
(iii) If $a \equiv b \mod q$, then q divides $b-a$, but $b-a < n-p < \frac{n}{2} < q$, a contradiction.

Case (b): μ is of type $(III)_i$. Note that $a \notin \{i, p-i\}$, $a \leq n-p < \frac{n}{2} < q$ and $q < a+p \leq n < 2q$, so both $a, a+p \not\equiv 0 \mod q$.
By assumption, we have in this case $(a+p)_{(\bar{q})} = (a, i, p-i)^o_{(\bar{q})}$. Hence we must have $a+i \equiv 0 \mod q$ or $a+p-i \equiv 0 \mod q$. As $a+i \leq n-p+\frac{p-1}{2} < n < 2q$ and $a+p-i \leq n-i < n < 2q$, we obtain $a+i = q$ and $a+p-i = q$, respectively, in the two cases. Thus $(a, i, p-i)^o = (a, q-a, p-q+a)^o$ in both cases.
Now by assumption, we also have $(a+p)_{(\bar{r})} = (a, q-a, p-q+a)^o_{(\bar{r})}$.
Then r must divide $p-q+2a$; as r is odd, we even obtain $r \mid \frac{p-q}{2} + a$. But as $\frac{p-q}{2} + a \leq \frac{p-q}{2} + n - p < \frac{n}{2}$, this gives a contradiction.

Finally, we consider the case where both partitions are of type (III), say λ is of type $(III)_i$ and μ is of type $(III)_j$, where we may assume that $i < j$. Note that we thus have $0 < i < j < p-j < p-i < p$.
By assumption on the $\bar{q}$-cores of λ, μ, we know that $(p-i, i)_{(\bar{q})} = (p-j, j)_{(\bar{q})}$. As $0 < i, j \leq \frac{p-1}{2} < \frac{n}{2} < q$, $i, j \not\equiv 0 \mod q$, and also $i \not\equiv j \mod q$. Hence we must have $i \equiv p-j \mod q$, and thus $q \mid p-(i+j) \leq n < 2q$, so $q = p-(i+j)$. Note that $2i < i+j = p-q$, so $i < \frac{p-q}{2}$.
As $p-q < \frac{n}{2} < r$, we have $r \nmid p-q$.
By assumption on the cores, $(p-i, i)_{(\bar{r})} = (q+i, p-q-i)_{(\bar{r})}$. Clearly $i \not\equiv 0 \mod r$. If $p-i \equiv 0 \mod r$, then we must also have $q+i \equiv 0 \mod r$. Hence $r \mid p+q$, and as r is odd, thus $r \mid \frac{p+q}{2}$. As $\frac{p+q}{2} < p \leq n \leq 2r$, we obtain $r = \frac{p+q}{2}$. Thus $q < \frac{p+q}{2} = r \mid q+i < 2q$, so $\frac{p+q}{2} = q+i$. But then $i = \frac{p-q}{2}$, giving a contradiction.
It remains to consider the case where $i \equiv p-q-i \mod r$. Here, $r \mid \frac{p-q}{2} - i < \frac{n}{2}$, immediately giving a contradiction. $\diamond$

Corollary 3.3 *Let λ, μ be bar partitions of n with the same p-bar core, for all odd primes $p \leq n$. Then $\lambda = \mu$, unless $n \leq 10$ and λ, μ are partitions in one of the following sets:*

$\{(3),(2,1)\}$, $\{(4),(3,1)\}$, $\{(5),(3,2)\}$, $\{(6),(5,1),(3,2,1)\}$,
$\{(9),(4,3,2)\}$, $\{(10),(7,3)\}$, $\{(9,1),(4,3,2,1)\}$.

Proof. Using [21], we find that there are primes satisfying the condition of Theorem 3.2 for $n = 13, 14$ and all $n \geq 17$. For the remaining values of n, the proof above helps to reduce the number of bar partitions to be checked, and one finds the list of exceptions as stated in the Corollary. $\diamond$

Correspondingly, we also have a result for block separation of spin characters; this requires the analogue of the Nakayama Conjecture for spin p-blocks of $\tilde{S}_n$ (where p is an odd prime), the Morris Conjecture [16], proved by Cabanes [4] and Humphreys [9] (see also [19]). Again, this also implies a corresponding result for the spin characters of $\tilde{A}_n$ (see [19]).

Corollary 3.4 *Let $n \in \mathbb{N}$, $n \geq 4$. Let χ, ψ be irreducible non-associate spin characters of $\tilde{S}_n$ (or $\tilde{A}_n$), not labelled by an exceptional pair of bar partitions as in Corollary 3.3. Then χ, ψ can be separated by some odd prime $p \leq n$.*

In [1] also the 2-block distribution of spin characters was determined; this needs the notion of the *doubling* of a bar partition: for a bar partition λ its doubling $\mathrm{dbl}(\lambda)$ is the partition obtained from λ by replacing each odd part $2t+1$ by $t+1, t$ and each even part $2t$ by t, t. Then by [1, Theorem 4.1] (see also [19, Theorem 13.20]), two irreducible spin characters of $\tilde{S}_n$ labelled by λ and μ belong to the same 2-block if and only if $\mathrm{dbl}(\lambda)_{(2)} = \mathrm{dbl}(\mu)_{(2)}$. Thus, checking the exceptions in Corollary 3.3, we obtain immediately

Corollary 3.5 *Let $n \in \mathbb{N}$, $n \geq 4$. Let χ, ψ be irreducible non-associate spin characters of $\tilde{S}_n$ (or $\tilde{A}_n$). Then χ, ψ are not p-separable, for all primes $p \leq n$, if and only if their partition label pair is one of $(4), (3,1)$ or $(6), (3,2,1)$.*

Remark 3.6 In most cases we can also separate associate spin characters. The only problem is that there are bar partitions of n which are not p-bar cores for any odd prime $p \leq n$. In fact, when n is even, or when n is odd and $n-2$ or $n-4$ are not primes, then there is at least one pair of associate spin characters of $\tilde{S}_n$ which can not be separated (namely the pairs labelled by (n), $(n-1,1)$ or $(n-2,2)$, respectively). If n is odd and both $n-2$, $n-4$ are primes then also all associate spin characters of $\tilde{S}_n$ can be separated. This is easily seen as most spin characters are then of weight 0 for one of the large primes $n-2$,$n-4$; the only exceptions are (n) and $(n-6,4,2)^o$ (the latter for $n \geq 7$), but these are of even type as n is odd, and hence belong to selfassociate spin characters. Thus in this situation all irreducible spin characters of $\tilde{S}_n$ can be separated.
In the case of $\tilde{A}_n$, there is always a pair of associate spin characters which can not be separated; for odd n, the pair labelled by (n), for even n, the pair labelled by $(n-1,1)$ is not separable.

4 Groups of Lie type

In this section we study quasi-simple groups of Lie type. The main tool will be Lusztig's classification of character degrees and the existence of Zsigmondy primes. Very similar arguments were already used in Hiss and Szczepański [7] as well as in Malle [14] and Malle, Navarro and Olsson [15]. For a prime power q we denote by $l(n,q)$ a Zsigmondy prime divisor of q^n-1, that is, an odd prime p such that $p|q^n-1$ but $p\not| q^i-1$ for all $1 \le i \le n-1$. By Zsigmondy's theorem such primes exist unless $n=1$ and $q-1$ is a power of 2, or $n=2$ and $q+1$ is a power of 2, or $(n,q)=(6,2)$.
Furthermore, we write ${}^{\epsilon}\mathrm{L}_3(q)$ for $\mathrm{L}_3(q)$ if $\epsilon=1$, respectively for $\mathrm{U}_3(q)$ if $\epsilon=-1$.

Theorem 4.1 *Let G be a finite quasi-simple group with $G/Z(G)$ a simple group of Lie type. We also assume that $G/Z(G) \notin \{A_5, A_6, A_8\}$, thus excluding $G = \mathrm{L}_2(4)$,(S)$\mathrm{L}_2(5)$,(S)$\mathrm{L}_2(9), \mathrm{S}_4(2)', (2.)\mathrm{L}_4(2)$. If there exist irreducible characters $\chi_1 \neq \chi_2$ of G faithful on $Z(G)$ which are not separable, then one of the following holds:*

(a) $G = 2.\mathrm{L}_2(q)$ *with* $7 \le q = 2^f - \epsilon$ $(\epsilon = \pm 1)$ *and* $\chi_{1/2}(1) = (q+\epsilon)/2$,

(b) $G = {}^{\epsilon}\mathrm{L}_3(q)$ *with* $q = 2^f - \epsilon$ $(\epsilon = \pm 1)$ *and* $\{\chi_1(1), \chi_2(1)\} = \{1, q(q+\epsilon)\}$,

(c) $G = \mathrm{S}_4(q)$ *with* $5 \le q = 2^f - \epsilon$ $(\epsilon = \pm 1)$ *and* $\{\chi_1(1), \chi_2(1)\} = \{1, q(q+\epsilon)^2/2\}$,

(d) $G = 2.\mathrm{S}_4(q)$ *with q odd and* $\chi_{1/2}(1) \in \{(q^2-1)/2, q^2(q^2-1)/2\}$,

(e) $G = 4_1.\mathrm{L}_3(4)$ *with* $\chi_{1/2}(1) = 8$,

(f) $G = 12_1.\mathrm{L}_3(4)$ *with* $\chi_{1/2}(1) \in \{24, 48\}$,

(g) $G = 12_2.\mathrm{L}_3(4)$ *with* $\chi_{1/2}(1) = 48$,

(h) $G = 12_2.\mathrm{U}_4(3)$ *with* $\chi_{1/2}(1) \in \{36, 216\}$,

(i) $G = \mathrm{U}_4(2) \cong \mathrm{S}_4(3)$ *with* $\{\chi_1(1), \chi_2(1)\} \subset \{1, 6, 24\}$,

(j) $G = 2.\mathrm{S}_6(2)$ *with* $\chi_{1/2}(1) \in \{48, 64\}$,

(k) $G = 3.G_2(3)$ *with* $\chi_{1/2}(1) = 27$.

Proof. Except for $G = 6.{}^2E_6(2)$ the character tables for all exceptional covering groups of finite simple groups of Lie type are contained in the Atlas [5], as well as for the Tits group ${}^2F_4(2)'$, while the table for $6.{}^2E_6(2)$ has recently been computed by F. Lübeck [12]. The claim can be checked using GAP for example. This leads to cases (e)–(k) in the statement. Thus we may assume that G is a central quotient of a finite group of Lie type of simply connected type. In particular, the degrees of the complex irreducible characters of G are known by the work of Lusztig [13]: the set $\mathrm{Irr}(G)$ can be partitioned into so-called Lusztig series $\mathcal{E}(G, s)$ indexed by conjugacy classes of semisimple elements s in the dual group of G. Furthermore, Broué and Michel [3] give a first rough subdivision of the set of irreducible characters of G into unions of blocks. By that result, if $\chi_i \in \mathcal{E}(G, s_i)$, $i = 1, 2$, are two characters lying in the same p-block for some prime p (different from the defining characteristic) then the p-parts of s_1 and s_2 must be conjugate. We will make use of this important result to deal with groups of types B_n and C_n, with n even. Also note that the Steinberg character is always of defect zero for the defining characteristic of G.

First assume that $G/Z(G)$ is of exceptional Lie type. Here we make use of our results in Malle [14, Prop. 4.1] and Malle–Navarro–Olsson [15]. For G of type E_6, ${}^2E_6$, E_7 E_8 it was shown in the proof of [14, Prop. 4.1] that all non-trivial irreducible characters are of p-defect zero for at least one of three suitable Zsigmondy primes p, except for the Steinberg character. Also, from that same proof it follows that the same is true for G of type ${}^2B_2$ or ${}^2G_2$. In the proof of [15, Lemma 5.9] we noted that the same holds for G of type ${}^3D_4$, ${}^2F_4$ and F_4. For G of type G_2 it was shown in [14, Prop. 4.1] that a non-trivial irreducible character different from the Steinberg character is of defect zero for one of two Zsigmondy primes, or it is unipotent of degree $q(q^2-1)^2/3$. For odd q, the latter are of 2-defect zero, for even q they are of defect zero for any prime $p \neq 3$ dividing (q^2-1) (such primes exist for $q > 2$). This completes the investigation of exceptional groups of Lie type.

For G of classical Lie type we again make use of results in [14] on Zsigmondy primes. Let first G be of type A_{n-1}, hence a central quotient of a special linear group. By [14, Sect. 3B] two Zsigmondy primes $l(n, q)$ and $l(n-1, q)$ separate all character different from the trivial and the Steinberg character, or $n \leq 3$, or $G = \mathrm{L}_6(2)$, $\mathrm{L}_7(2)$. The latter two groups can be treated using the Atlas. The characters and blocks of $\mathrm{SL}_2(q)$ are well known and lead to the exceptions in (a). For $\mathrm{SL}_3(q)$ a Zsigmondy prime $l(3, q)$ always exists, while $l(2, q)$ fails to exist when $q + 1$ is a power of 2. The characters not of

$l(3,q)$-defect zero have degrees

$$\{1, q(q+1), q^3, (q-1)^2(q+1)/d, (q-1)^2(q+1)\}$$

with $d = \gcd(3, q-1)$. Now note that for odd q, the last two character degrees contain the full 2-power of $|G|$, thus we arrive at case (b).
For G of type ${}^2A_{n-1}$, a unitary group, again by [14, Sect. 3C] two Zsigmondy primes separate characters unless $n = 3$. For $\mathrm{SU}_3(q)$ a Zsigmondy prime $l(6,q)$ still exists (note that $\mathrm{SU}_3(2)$ is solvable), while $l(1,q)$ fails to exist for $q-1$ a power of 2. In tha case, the character degrees not of $l(6,q)$-defect zero are

$$\{1, q(q-1), q^3, (q-1)(q+1)^2/d, (q-1)(q+1)^2\}$$

with $d = \gcd(3, q+1)$. Again, the last two characters are of 2-defect zero, and we arrive at case (b).
For G of odd orthogonal type B_n, $n \geq 2$, it was shown in [14, Sect. 3D] that two Zsigmondy primes $l(2n,q)$ and $l(n,q)$ separate characters if n is odd, while $l(2n,q)$ and $l(2n-2,q)$ only leave two unipotent characters which are of defect zero for a Zsigmondy prime $l(n-1,q)$ for n even, or $n = 2$, or G is one of $\mathrm{S}_6(2)$, $\mathrm{S}_8(2)$. The latter two are contained in the Atlas. For $G = \mathrm{Sp}_4(q)$ a Zsigmondy prime $l(4,q)$ exists. If $q+1$ is not a power of 2, then $l(2,q)$ exists and we are left with characters in two distinct Lusztig series, of degrees

$$\{1, q(q-1)^2/2, q^4\}, \qquad \{(q^2-1)/2, q^2(q^2-1)/2\}.$$

If moreover $q-1$ is not a power of 2, any odd prime divisor of $q-1$ separates these except for those of degrees $(q^2-1)/2, q^2(q^2-1)/2$ in the Lusztig series of an isolated element of order 2 (which only exists for odd q). This case is dealt with in Proposition 4.2. (The case that $q-1$ is a power of 2 leads to case (c)). On the other hand, if $q = 2^f - 1$ then the remaining characters not of defect zero for odd prime divisors of $q-1$ lie in two Lusztig series and have degrees

$$\{1, q(q+1)^2/2, q^4\}, \qquad \{(q^2-1)/2, q^2(q^2-1)/2\}.$$

Again this leads to cases (c) and (d) with Proposition 4.2. Finally, if both $q-1$ and $q+1$ are powers of 2 then $q = 3$, and this case can be handled using the Atlas, leading to case (i).
For G of symplectic type C_n, $n \geq 3$, q odd, it was shown in [14, Sect. 3E] that two Zsigmondy primes $l(2n,q)$ and $l(n,q)$ separate characters if n is odd. For n even $l(2n,q)$ and $l(2n-2,q)$ only leave two unipotent characters of defect

zero for $l(n-1,q)$ and characters of degrees $(q^n-1)/2$ and $q^{n(n-1)}(q^n-1)/2$. These lie in the two Lusztig series determined by an involution in the dual group with disconnected centralizer of type ${}^2D_n$. By Proposition 4.2 there exists a prime separating the two Lusztig series. Now for even $n \geq 4$, inside each series the two characters lie in different blocks for $l(n-2,q)$, again by [23].
For G of non-split orthogonal type ${}^2D_n$, $n \geq 4$, it was shown in [14, Sect. 3F] that two Zsigmondy primes $l(2n,q)$ and $l(2n-2,q)$ separate characters, or G is the Atlas group $G = \mathrm{O}_8^-(2)$.
For G of split orthogonal type D_n, $n \geq 4$, it was shown in [14, Sect. 3G] that two Zsigmondy primes $l(2n-2,q)$ and $l(n,q)$ separate characters when n is odd, while for n even $l(2n-2,q)$ and $l(n-1,q)$ only leave two unipotent characters of $l(2n-4,q)$-defect zero, or G is the Atlas group $G = \mathrm{O}_8^+(2)$
This completes the proof of the theorem. $\diamond$

We note that it can easily be checked that the cases (a)–(k) in the Theorem really lead to exceptions, using the Atlas for cases (e)–(k), and the known results on blocks for cases (a)–(d).

Proposition 4.2 *Let $G = \mathrm{Sp}_{2n}(q)$, $n \geq 2$, q odd, and $s \in G^* = \mathrm{SO}_{2n+1}(q)$ be an involution with disconnected centralizer $\mathrm{GO}_{2n}^-(q)$. Let $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathrm{Irr}(G)$ be the two Lusztig series of G corresponding to the element s. Then there exists a prime divisor p of $|G|$ such that no p-block of G intersects both $\mathcal{E}_1, \mathcal{E}_2$, or $(n,q) = (2,3)$.*

Proof. If n is odd, all characters in $\mathcal{E}_1, \mathcal{E}_2$ are of defect zero for a Zsigmondy prime $l(n,q)$, so the statement clearly holds. For n even this is a consequence of a result of Srinivasan [23]: First assume that $n = 2$. Let p be an odd prime dividing q^2-1 (this exists if $q \neq 3$), with $p|(q-\epsilon)$, $\epsilon = \pm 1$. Then in the sense of $(q-\epsilon)$-Harish-Chandra theory the two Lusztig series $\mathcal{E}_1, \mathcal{E}_2$ lie above the two characters of degree $(q-\epsilon)/2$ of the Levi subgroup $(q-\epsilon)\mathrm{SL}_2(q)$ of G. These are of p-defect zero for $\mathrm{SL}_2(q)$, hence lie in different p-blocks. Then, by [23, Thm. 3.9] there exists no p-block of G intersecting both $\mathcal{E}_1, \mathcal{E}_2$. See also White [24] for this case.
Now let $n \geq 4$ and write $n = 2m$. Then in (q^m+1)-Harish-Chandra theory $\mathcal{E}_1, \mathcal{E}_2$ lie above the characters of degree $(q^m+1)/2$ of the Levi subgroup $(q^m+1)\mathrm{Sp}_{2m}(q)$. Again, these are of p-defect zero for a Zsigmondy prime $l(n,q)$, and we may conclude with [23, Thm. 3.9] as above. $\diamond$

In the proof of Theorem 4.1 we also verified the following more precise version, and extension to quasi-simple groups, of [15, Lemmas 5.7 and 5.9]:

Corollary 4.3 *Let G be finite quasi-simple with $G/Z(G)$ of exceptional Lie type. Then there exists a set π of at most four prime divisors of $|G|$ such that each faithful irreducible character χ is π-isolated, except for*

(a) $G = 2.{}^2B_2(8)$ *with* $\chi(1) = 2^6$,

(b) $G = 3.G_2(3)$ *with* $\chi(1) \in \{27, 3^6\}$,

(c) $G = 2.G_2(4)$ *with* $\chi(1) \in \{12, 2^{12}\}$,

(d) $G = 2.F_4(2)$ *with* $\chi(1) = 2^{24}$,

(e) $G = 2.{}^2E_6(2)$ *with* $\chi(1) = 2^{36}$.

Note that all exceptions come from exceptional multipliers. Also note that the statement becomes false if we replace 'faithful' by 'non-trivial': for example, the Steinberg character of $G = 3.G_2(3)$ is not faithful, and doesn't have 3-defect zero for G.
We have also proved the following (this extends considerably a result of [7]):

Corollary 4.4 *Let χ be a faithful irreducible character of a finite simple group G of Lie type. Assume that χ is not separable from the principal character. Then one of:*

(a) $G = {}^{\epsilon}\mathrm{L}_3(q)$ *with* $q = 2^f - \epsilon$ $(\epsilon = \pm 1)$ *and* $\chi(1) = q(q+\epsilon)$,

(b) $G = \mathrm{S}_4(q)$ *with* $q = 2^f - \epsilon$ $(\epsilon = \pm 1)$ *and* $\chi(1) = q(q+\epsilon)^2/2$,

Note that case (b) gives two character degrees for $\mathrm{S}_4(3)$.
The exceptions in Theorem 4.1 and Corollary 4.4 all come from Fermat- and Mersenne primes:

Lemma 4.5 *Let p be an odd prime, a a positive integer.*

(a) If $p^a = 2^f + 1$ *then either* $a = 1$ *or* $p^a = 9$.

(b) If $p^a = 2^f - 1$ *then* $a = 1$.

Thus, while clearly Theorem 4.1(d) describes infinitely many cases, this is not known at present for Theorem 4.1(a)–(c).

5 Separation for characters of the sporadic groups and related groups

In the table below, for each cyclic upward and downward extension of a sporadic simple group, we give a minimal set of primes separating all irreducible characters; these data have been computed using GAP [22]. Among the sporadic groups, only M_{12} does not have such a set as there are two characters of degree 16 which are not separable; note that apart from this exception, for J_1 two primes suffice, for He, HN and Th four primes are needed, and all others require just three primes. Furthermore, we also give the data for the automorphism groups and the cyclic central extension groups of the sporadic groups. Here, there are some more exceptions. Again, $2.M_{12}$ has two characters of degree 16 which are not separable, and we also find two such pairs of degree 6 and 64, respectively, for $2.J_2$, and two pairs both of degree 18 for $2.J_3$, as well as a pair of characters of degree 8192 for $2.Ru$. In all other cases we find three or four primes separating the set $\mathrm{Irr}(G)$.

Group	minimal set	Group	minimal set	Group	minimal set
M_{11}	$2, 5, 11$				
M_{12}	$--$	$2.M_{12}$	$--$	$M_{12}.2$	$3, 5, 11$
M_{22}	$3, 5, 7$	$2.M_{22}$	$3, 5, 7$	$M_{22}.2$	$3, 5, 7$
		$3.M_{22}$	$3, 5, 7$		
		$4.M_{22}$	$3, 5, 7$		
		$6.M_{22}$	$3, 5, 11$		
		$12.M_{22}$	$3, 5, 7, 11$		
M_{23}	$2, 5, 11$				
M_{24}	$3, 5, 11$				
J_1	$3, 5$				
J_2	$3, 5, 7$	$2.J_2$	$3, 5, 7$	$J_2.2$	$--$
J_3	$3, 5, 17$	$3.J_3$	$--$	$J_3.2$	$3, 5, 17$
J_4	$2, 31, 37$				
Co_1	$3, 11, 23$	$2.Co_1$	$5, 11, 23$		
Co_2	$3, 7, 11$				
Co_3	$5, 7, 11$				
Fi_{22}	$3, 7, 13$	$2.Fi_{22}$	$5, 11, 13$	$Fi_{22}.2$	$3, 7, 13$
		$3.Fi_{22}$	$3, 7, 13$		
		$6.Fi_{22}$	$5, 11, 13$		
Fi_{23}	$2, 13, 23$				
Fi'_{24}	$5, 11, 23$			Fi_{24}	$5, 7, 17$
HS	$2, 7, 11$	$2.HS$	$2, 3, 7, 11$	$HS.2$	$2, 7, 11$
McL	$5, 7, 11$	$3.McL$	$5, 7, 11$	$McL.2$	$5, 7, 11$
Suz	$2, 7, 13$	$2.Suz$	$2, 7, 13$	$Suz.2$	$2, 7, 13$
		$3.Suz$	$5, 7, 13$		
		$6.Suz$	$5, 7, 13$		
Ru	$3, 7, 29$	$2.Ru$	$--$		
He	$2, 3, 7, 17$			$He.2$	$2, 3, 5, 17$
Ly	$2, 31, 37$				
ON	$2, 19, 31$	$3.ON$	$5, 7, 19$	$ON.2$	$2, 5, 11$
HN	$2, 3, 7, 11$			$HN.2$	$2, 3, 7, 11$
Th	$2, 5, 13, 31$				
B	$3, 19, 23$	$2.B$	$7, 13, 19$		
M	$11, 17, 59$				

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