THE MONTHLY PROBLEM

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September 28, 2005

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Dear Peter.

In Monthly 112,7 2005, Heinz–Jürgen Seiffert, Berlin, Germany as problem no. 11172 (b) asks for proof of two identities. Both show up to be Chu–Vandermonde in disguise. Let $n, m > 0$. The identities are:

(1)
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \frac{\binom{n}{2k}\binom{2k}{k}}{\binom{k+m}{m}} = \frac{\binom{2n+2m}{n+m}}{\binom{n+2m}{m}}, \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n}{2k}\binom{2k}{k}}{\binom{2k+m}{k}\binom{2k+2m+1}{m+1}} = \frac{2^n}{\binom{2n+2m+1}{m+1}}.$$

The **factorial** $[x, d]_n$ is defined for any **number,** $x \in \mathbb{C}$, any **stepsize,** $d \in \mathbb{C}$, and any **length,** $n \in \mathbb{Z}$, except for $-x \in \{d, 2d, \cdots, -nd\}$, by

$$[x, d]_n := \begin{cases} \prod_{j=0}^{n-1} (x - jd) & n \in \mathbb{N} \\ 1 & n = 0 \\ \prod_{j=1}^{-n} \frac{1}{x + jd} & -n \in \mathbb{N}, \ -x \notin \{d, 2d, \cdots, -nd\} \end{cases} \tag{2}$$

The traditional Chu–Vandermonde formula looks like

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n} \tag{3}$$

Multiplication with $n!$ gives the form:

$$\sum_{k=0}^{n} t_k = \sum_{k=0}^{n} \binom{n}{k} [x]_k [y]_{n-k} = [x+y]_n \tag{4}$$

This has quotient

$$q_k = \frac{t_{k+1}}{t_k} = \frac{(n-k)(x-k)}{(-1-k)(n-1-y-k)} \tag{5}$$

Sums with the same quotients have the invariant corrected sum

$$\frac{1}{t_0}\sum_{k=0}^{n} t_k \tag{6}$$

Now consider the quotient of the sum in the first problem:

$$\frac{2^{n-2k-2}\binom{n}{2k+2}\binom{2k+2}{k+1}\binom{k+m}{m}}{2^{n-2k}\binom{n}{2k}\binom{2k}{k}\binom{k+1+m}{m}} = \frac{\left(\frac{n}{2}-k\right)\left(\frac{n-1}{2}-k\right)}{(-1-k)(-m-1-k)} = \frac{\left(\lfloor\frac{n}{2}\rfloor-k\right)\left(\lceil\frac{n}{2}\rceil-\frac{1}{2}-k\right)}{(-1-k)(-m-1-k)} \tag{7}$$

So the sum is the 0 term times the corrected Chu–Vandemonde sum:

$$\begin{aligned}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} t_k &= t_0\frac{\left[\lceil\frac{n}{2}\rceil-\frac{1}{2}+\lfloor\frac{n}{2}\rfloor+m\right]_{\lfloor\frac{n}{2}\rfloor}}{\left[\lfloor\frac{n}{2}\rfloor+m\right]_{\lfloor\frac{n}{2}\rfloor}} = 2^n\frac{\left[n+m-\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}}{\left[\lfloor\frac{n}{2}\rfloor+m\right]_{\lfloor\frac{n}{2}\rfloor}}\\ &= 2^{\lceil\frac{n}{2}\rceil}\frac{[2n+2m-1,2]_{\lfloor\frac{n}{2}\rfloor}[n+m]_{\lceil\frac{n}{2}\rceil}}{[n+m]_{\lceil\frac{n}{2}\rceil}\left[\lfloor\frac{n}{2}\rfloor+m\right]_{\lfloor\frac{n}{2}\rfloor}} = \frac{[2n+2m-1,2]_{\lfloor\frac{n}{2}\rfloor}[2n+2m,2]_{\lceil\frac{n}{2}\rceil}}{[n+m]_n}\\ &= \frac{[2n+2m]_n[n+2m]_m}{[n+m]_n[n+2m]_m} = \frac{[2n+2m]_{n+m}}{[n+2m]_{n+m}} = \frac{\binom{2n+2m}{n+m}}{\binom{n+2m}{n+m}} = \frac{\binom{2n+2m}{n+m}}{\binom{n+2m}{m}}\end{aligned} \tag{8}$$

Now consider the quotient of the sum in the second problem:

$$\begin{aligned}&\frac{\binom{n}{2k+2}\binom{2k+2}{k+1}\binom{2k+m}{k}\binom{2k+2m+1}{m+1}}{\binom{n}{2k}\binom{2k}{k}\binom{2k+2+m}{k+1}\binom{2k+2m+3}{m+1}} = \frac{\left(\frac{n}{2}-k\right)\left(\frac{n-1}{2}-k\right)}{(-1-k)\left(-m-\frac{3}{2}-k\right)} =\\ &\frac{\left(\lfloor\frac{n}{2}\rfloor-k\right)\left(\lceil\frac{n}{2}\rceil-\frac{1}{2}-k\right)}{(-1-k)\left(-m-\frac{3}{2}-k\right)}\end{aligned} \tag{9}$$

So the sum is the 0 term times the corrected Chu–Vandemonde sum:

$$\begin{aligned}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} t_k &= t_0\frac{\left[\lceil\frac{n}{2}\rceil-\frac{1}{2}+\lfloor\frac{n}{2}\rfloor+m+\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}}{\left[\lfloor\frac{n}{2}\rfloor+m+\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}} = \frac{1}{\binom{2m+1}{m+1}}\frac{[n+m]_{\lfloor\frac{n}{2}\rfloor}\left[m+\lceil\frac{n}{2}\rceil\right]_{\lceil\frac{n}{2}\rceil}}{\left[\lfloor\frac{n}{2}\rfloor+m+\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}\left[m+\lceil\frac{n}{2}\rceil\right]_{\lceil\frac{n}{2}\rceil}}\\ &= \frac{1}{\binom{2m+1}{m+1}}\frac{[n+m]_n 2^n}{\left[2\lfloor\frac{n}{2}\rfloor+2m+1,2\right]_{\lfloor\frac{n}{2}\rfloor}\left[2m+2\lceil\frac{n}{2}\rceil,2\right]_{\lceil\frac{n}{2}\rceil}} = \frac{1}{\binom{2m+1}{m+1}}\frac{[n+m]_n 2^n}{[n+2m+1]_n}\\ &= \frac{2^n(n+m)!(m+1)!}{(n+2m+1)!} = \frac{2^n}{\binom{2n+2m+1}{m+1}}.\end{aligned} \tag{10}$$