GKP PROBLEM 69.

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Dear Carsten,

Your mentioning of Knuth's problem 6.69 made me think of my remark, that my theorem 18.1 makes the answer "trivial."

Find a closed form of

$$\sum_{k=0}^{n} k^2 H_{n+k}$$

Theorem 18.1 states that the indefinite sums are known:

$$\sum [n+k]_m H_{n+k} \delta k = \frac{[n+k]_{m+1}}{m+1}\left(H_{n+k} - \frac{1}{m+1}\right)$$

So, all one has to do is to write

$$k^2 = [n+k]_2 - (2n-1)[n+k]_1 + n^2$$

yielding

$$\sum k^2 H_{n+k}\delta k = \sum [n+k]_2 H_{n+k}\delta k - (2n-1)\sum [n+k]_1 H_{n+k}\delta k + n^2 \sum [n+k]_0 H_{n+k}\delta k$$

Applying the theorem for $m = 0, 1, 2$ gives the following

$$\begin{aligned}
\sum k^2 H_{n+k}\delta k =& \frac{[n+k]_3}{3}\left(H_{n+k} - \frac{1}{3}\right) \\
& - (2n-1)\frac{[n+k]_2}{2}\left(H_{n+k} - \frac{1}{2}\right) \\
& + n^2 \frac{[n+k]_1}{1}\left(H_{n+k} - \frac{1}{1}\right) \\
=& \left(\frac{[n+k]_3}{3} - (2n-1)\frac{[n+k]_2}{2} + n^2\frac{[n+k]_1}{1}\right) H_{n+k} \\
& - \left(\frac{[n+k]_3}{9} - (2n-1)\frac{[n+k]_2}{4} + n^2\frac{[n+k]_1}{1}\right) \\
=& \frac{1}{6}(n+k)\left(2k^2 - k(2n+3) + (2n+1)(n+1)\right) H_{n+k} \\
-& \frac{1}{36}(n+k)\left(4k^2 - k(10n+3) + 22n^2 + 15n - 1\right)
\end{aligned}$$

Now we may derive the definite sum as

$$
\begin{aligned}
&\sum_{k=0}^{n} k^2 H_{n+k} = \sum\nolimits_0^{n+1} k^2 H_{n+k} \delta k = \\
&\frac{1}{6}(2n+1)(n+1)\left(2(n+1)-(2n+3)+(2n+1)\right) H_{2n+1} \\
-&\frac{1}{36}(2n+1)\left(4(n+1)^2-(n+1)(10n+3)+22n^2+15n-1\right) \\
&-\frac{1}{6}n(2n+1)(n+1)H_n+\frac{1}{36}n\left(22n^2+15n-1\right) = \\
&\frac{1}{3}(2n+1)(n+1)nH_{2n+1}-\frac{1}{18}(2n+1)n(8n+5) \\
&-\frac{1}{6}n(n+1)(2n+1)H_n+\frac{1}{36}n\left(22n^2+15n-1\right)
\end{aligned}
$$

To compare with the solution in GKP we may write $H_{2n+1} = H_{2n} + \frac{1}{2n+1}$. Then we get

$$
\begin{aligned}
&\frac{1}{3}(2n+1)(n+1)n\left(H_{2n}+\frac{1}{2n+1}\right)-\frac{1}{18}(2n+1)n(8n+5) \\
&-\frac{1}{6}n(n+1)(2n+1)H_n+\frac{1}{36}n\left(22n^2+15n-1\right) = \\
&\frac{1}{6}n(n+1)(2n+1)(2H_{2n}-H_n)-\frac{1}{36}n(n+1)(10n-1)
\end{aligned}
$$

which is the form in GKP's solutions.

I think that this derivation using indefinite summation is automatic enough.

Best Regards,

Mogens.