MOGENS ESROM LARSEN: PHYSICAL MATHEMATICS

ASSYRIAN CLAYTABLETS

WE READ TABLES OF PYTHAGOREAN NUMBERTRIPLES,

$$\{(a,b,c) \in \mathbb{Z}^3 | a^2 + b^2 = c^2\}$$

E.G.,

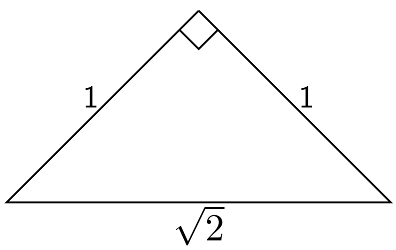
$$(3,4,5), (5,12,13), \cdots, \cdots, (13500, 12709, 18541)$$

THEY FOUND NO EXAMPLE WITH

$$a = b$$

PYTHAGORAS' DISAPPOINTMENT AND DISCOVERY

UNABLE TO FIND THE ISOSCELES RIGHTANGULAR TRIANGLE WITH INTEGER SIDES, HE EXCHANGED THE NUMBERS WITH PHYSICAL MAGNITUDES,
IN THIS CASE HE CONSTRUCTED SUCH A TRIANGLE AND HAD THE IMPOSSIBLE HYPOTHENUS ($\sqrt{2}$):



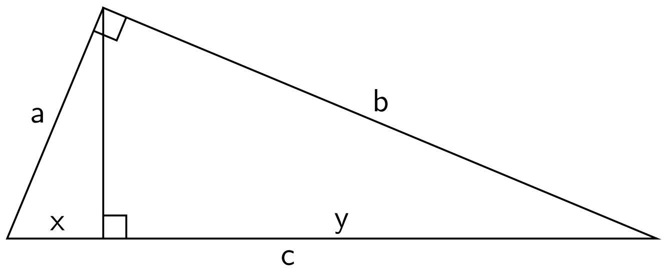
THIS IS EASELY CONSTRUCTED WITH COMPASS AND RULER IN THE EUCLIDEAN WAY.

THE GREEKS ABANDOMED NUMBERS AND CONSIDERED IN THE GEOMETRY WHATEVER COULD BE CONSTRUCTED.

THEY MADE A THEORY OF MAGNITUDES OF GEOMERICAL NATURE.

PROOF OF PYTHAGORAS' THEOREM

THE ASSYRIANS KNEW THIS THEOREM, – NOT SO SURPRICING. IT FOLLOWS READILY FROM CONSIDERING THREE SIMILAR TRIANGLES:

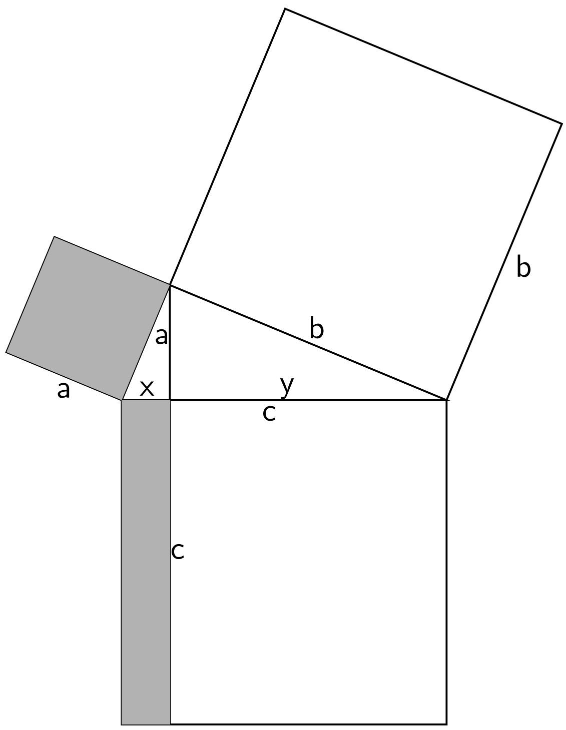


THE ALTITUDE DIVIDES THE TRIANGLE IN TWO SMALLER, BOTH SIMILAR TO THE ORIGINAL. THIS GIVES THE RELATIONS:

$$\frac{x}{a} = \frac{a}{c} \qquad \frac{y}{b} = \frac{b}{c}$$

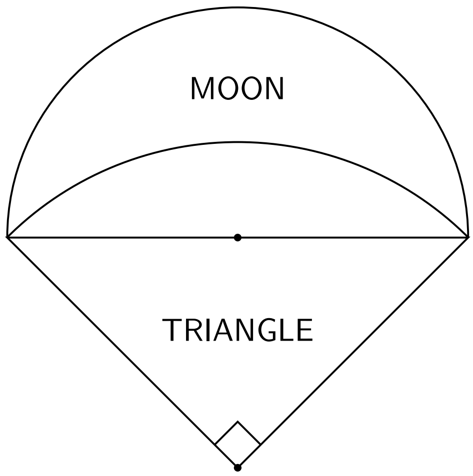
THESE EQUALITIES MAY BE UNDERSTOOD AS BETWEEN SOME SQUARES AND RECHTANGLES

$$a^2 = c \times x \qquad b^2 = c \times y$$

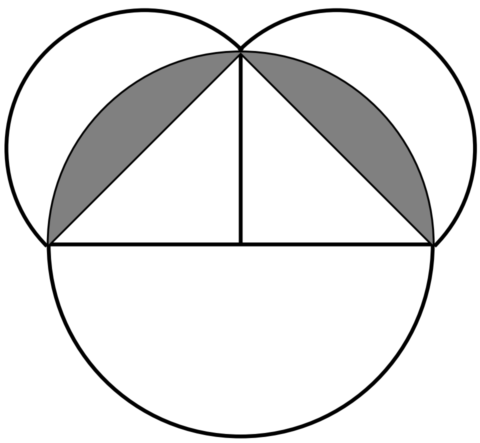


HIPPOKRATES' MOON

A BEAUTIFUL EXAMPLE OF THEIR TECHNIQUE IS THE THEOREM OF HIPPOKRATES:
THE MOON IS EQUAL TO THE TRIANGLE:
THE MOON IS BETWEEN THE ARCS OF CENTERS IN THE RIGHT ANGLE AND THE MIDPOINT OF THE HYPOTHENUS OF AN ISOSCELES RIGHTANGULAR TRIANGLE.



PROOF:
IN THE FIGURE BELOW THE BIG HALFDISC IS EQUAL TO THE SUM OF OF THE TWO SMALL HALFSDISCS, BECAUSE THEY ARE PROPORTIONAL TO THE SQUARES OF PYTHAGORAS.



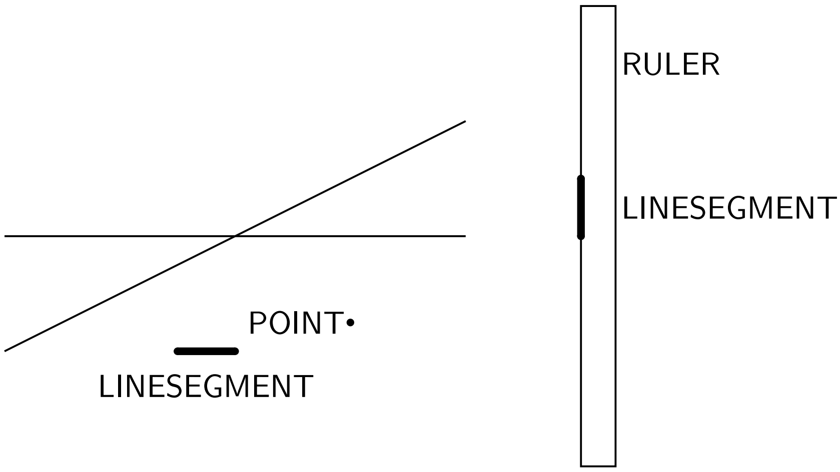
NOW THE TRIANGLE IS EQUAL TO THE BIG HALF DISC MINUS THE TWO SHADOWED AREAS, BUT SO IS THE THE SUM OF THE TWO MOONS, AS THEY ARE THE SUM OF THE TWO SMALL HALFDISCS MINUS THE TWO SHADOWED AREAS.
BY DIVIDING THE FIGURE IN HALF WE ARE PROVIDED WITH THE RESULT.

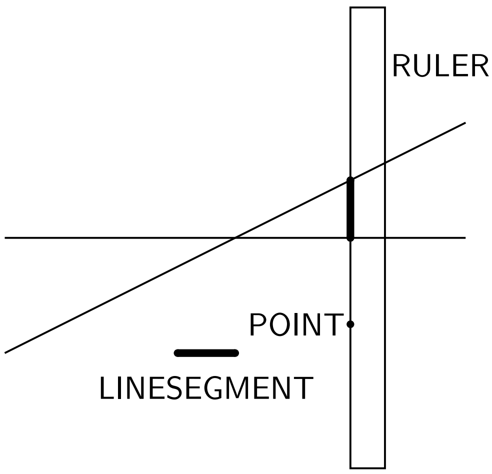
TRISECTING THE ANGLE

THE CONSTRUCIONS MAY BE MADE WITH THE PHYSICAL TOOLS, COMPASS AND RULER. BUT THE USE OF THE RULES IS AMBIGUES: EUCLID RESTRICTED THE USE TO DRAWING A LINE BETWEEN TWO POINTS. BUT WITHOUT THIS RESTRICTION WE MAY EVEN SOLVE AN ARBITRARY EQUATION OF THIRD DEGREE!

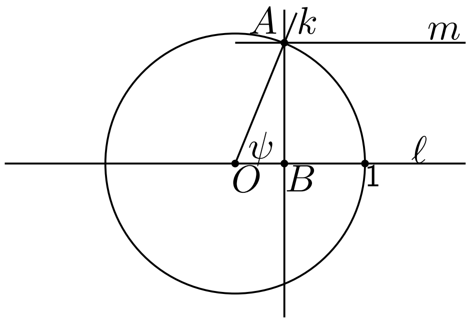
INSERTION

GIVEN TWO CUTTING LINES, A POINT AND A LINESEGMENT, WE MAY DRAW A LINE THROUGH THE POINT SUCH THAT THE GIVEN LINES CUTS OFF A LINESEGMENT OF THE CONSTRUCTED LINE HAVING THE PRESCRIBED LENGTH.





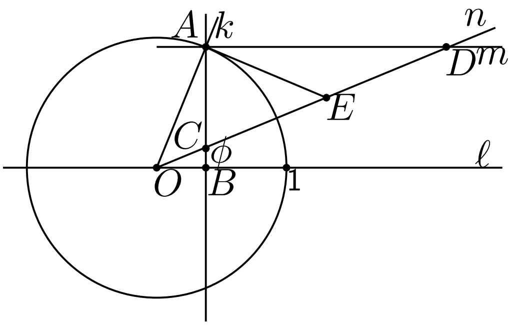
TRISECTING THE ANGLE



WE WANT TO TRISECT THE ANGLE $\angle\psi$ BETWEEN THE LINES ℓ AND k. WE DRAW THE CIRCLE WITH CENTER O AND RADIUS 1 (A POINT ON ℓ), IT CUTS THE LINE k IN THE POINT A. THEN WE DRAW AB RIGHTANGULAR TO ℓ. EVENTUALLY WE DRAW THE LINE m THROUGHT A PARALLEL TO ℓ.

CONSTRUCTION

WE INSERT A LINE, n, THROUGH O CUTTING AB IN C AND m IN D, SUCH THAT $CD = 2$.



E IS NOW THE MIDPOINT OF THE LINESEGMENT CD. BECAUSE THE TRIANGLE $\triangle ACD$ IS RIGHTANGULAR WITH $\angle A$ AS THE RIGHT ANGLE, WE CONCLUDE THAT A CIRCLE WITH CENTER IN E AND RADIUS 1 MUST GO THROUGHT A, C og D. HENCE $AE = CE = DE = 1$ AND THE TRIANGLE $\triangle AED$ IS ISOSCELES, SO WE CONCLUDE THAT $\angle DAE = \angle ADE = \angle COB = \phi$. BECAUSE THE TRIANGLE $\triangle OAE$ IS ISOSCELES TOO, WE HAVE $\angle AOC = \angle AEC = \angle EAD + \angle ADE = 2\phi$.

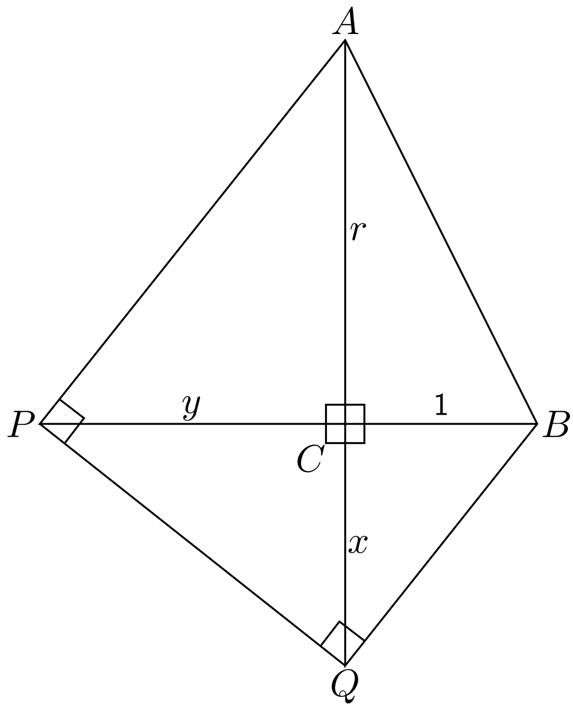
WE HAVE DIVIDED THE ANGLE $\angle AOB = \psi$ IN THREE EQUAL PARTS.

THIS PROCEDURE WAS KNOWN SINCE THE 5. CENTURY BC.

DOUBLING A DICE

AS WE HAVE EASELY CONSTRUCTED A LINE SEGMENT OF LENGTH $\sqrt{2}$, I.E., THE SIDE OF A SQUARE OF DOUBLE SIZE, IT IS TEMPTING TO ASK FOR THE DOUBLING AF A DICE, I.E., TO CONSTRUCT A LINE SEGMENT OF SIZE $\sqrt[3]{2}$. THIS IS IMPOSSIBLE IN GENERAL IN THE EUCLIDEAN GEOMETRY, BUT WITH PHYSICAL INSTRUMENTS IT IS EASY. WE NEED THE FOLLOWING CONSTRUCTION,

GENERALIZING THE NUMBER 2 TO ANY REAL NUMBER – ALREADY CONSTRUCTED – r:



GIVEN A RIGHTANGULAR TRIANGEL, $\triangle ABC$ WITH CATHESI 1 AND r, WE NEED TO FIND P AND Q ON THE PROLONGATION AF THE CATHESI IN A WAY, SUCH THAT THE TWO ANGLES ARE RIGHT: $\angle APQ = \angle PQB = \frac{\pi}{2}$. THEN THE THREE NEW TRIANGLES ARE SIMILAR,

$$\triangle ACP \sim \triangle PCQ \sim \triangle QCB$$

HENCE WE HAVE

$$\frac{1}{x} = \frac{x}{y} = \frac{y}{r}$$

FROM THIS WE COMPUTE

$$x^2 = y \quad \wedge \quad y^2 = rx \quad \Rightarrow \quad x^4 = y^2 = rx$$

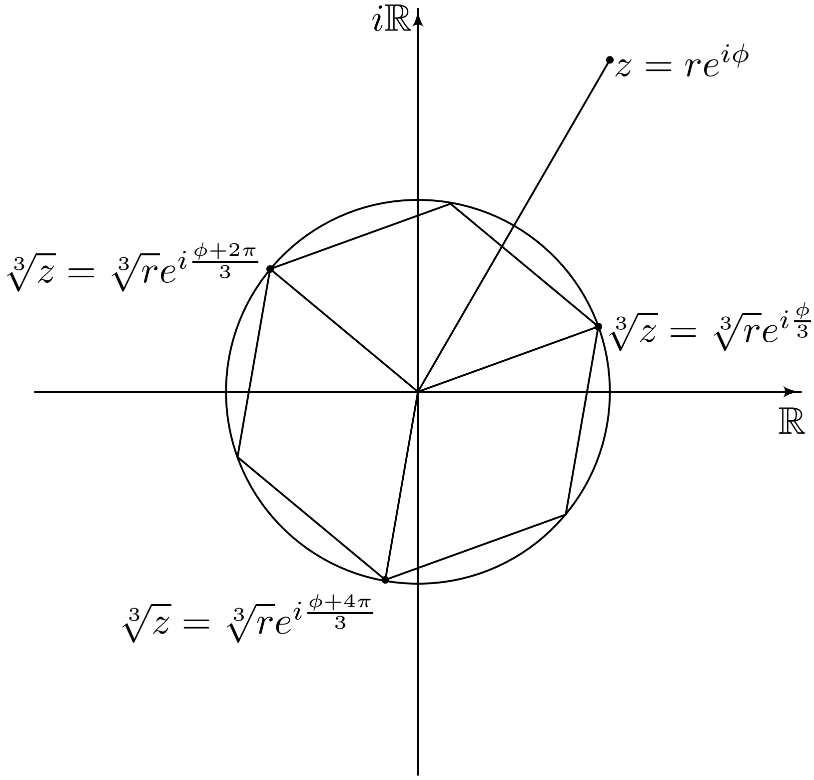
SHOWING THAT THE LINESEGMENT x SOLVES THE PROBLEM.
TO DRAW THE FIGURE WE MAY APPLY A COUPLE AF CARPENTER'S SQUARES TO FIT THE TWO RIGHT ANGLES ON THE APPROPRIATE LINES.

GENERAL CUBIC ROOTS

THESE TWO CONSTUCTIONS ACTUALLY ALLOW US TO CONSTRUCT THE CUBIC ROOT OF AN ARBITRARY COMPLEX NUMBER, REPRESENTED AS A VECTOR IN THE PLANE. IF THIS VECTOR HAS THE REPRESENTATION

$$re^{i\phi}$$

THEN WE MAY CONSTRUCT THE TWO NECESSARY MAGNITUDES, $\sqrt[3]{r}$ AND $\frac{\phi}{3}$. TO ADD THE ANGLES $\frac{2\pi}{3}$ AND $\frac{4\pi}{3}$ IS EASY:



CUBIC EQUATIONS

WE MAY EVEN SOLVE ANY CUBIC EQUATION CONSTRUCTIVELY AS SEEN BY FRANCIS VIETA (1540–1603). SUPPOSE WE HAVE A CUBIC EQUATION (I.E., $A \neq 0$)

$$Ax^3 + Bx^2 + Cx + D = 0$$

THEN WE MAY DIVIDE BY A AND WRITE

$$x^3 + bx^2 + cx + d = 0$$

WE MAY ALSO CHANGE THE VARIABLE TO $y = x - \frac{b}{3}$ TO GET RID OF THE SQUARE TERM

$$y^3 + \left(c - \frac{b^2}{3}\right) y + d + \frac{2b^3}{27} - \frac{bc}{3} = 0$$

OR SIMPLY

$$y^3 + ey + f = 0$$

NOW WE TRY TO WRITE $y = p + q$, p AND q TO BE CHOSEN. WE GET

$$p^3 + q^3 + 3pq(p+q) + e(p+q) + f = 0$$

IT IS SEEN, THAT IF WE CHOOSE p AND q SATISFYING THE EQUATIONS

$$3pg + e = 0$$
$$p^3 + q^3 + f = 0$$

THEN WE HAVE A SOLUTION $y = p + q$. BUT WE MAY CONSIDER p^3 and q^3 AS SATISFYING

$$p^3 g^3 = -\frac{e^3}{27}$$
$$p^3 + q^3 = -f$$

HENCE THEY ARE THE SOLUTIONS TO A QUADRATIC EQUATION, I.E., THEY ARE

$$\frac{-f \pm \sqrt{f^2 + 4\frac{e^3}{27}}}{2}$$

ALL OF WHICH ARE CONSTRUCTABLE MAGNITUDES, POSSIBLY AS COMPLEX VECTORS.

ALL WE NOW NEED IS TO CONSTUCT THEIR CUBIC ROOTS WITH THE METHODS ABOVE. REMEMBER TO CONSTRUCT ALL THREE COMPLEX ROOTS, ALSO OF A REAL VALUE. AND WITH THE 9 POSSIBLE SUMS OF $p + q$, IT IS NECESSARY TO VERIFY THE TRUE VALUES EVENTUALLY!

EUCLID'S 5. BOOK – THE THEORY OF MAGNITUDES

THE RELATION:

$$A : B = C : D \quad \Leftrightarrow \quad A : C = B : D$$

SEEMS TRIVIAL, SINCE WE HAVE THE OBVIOUS PROOF:

$$B \cdot C = A \cdot D = C \cdot B$$

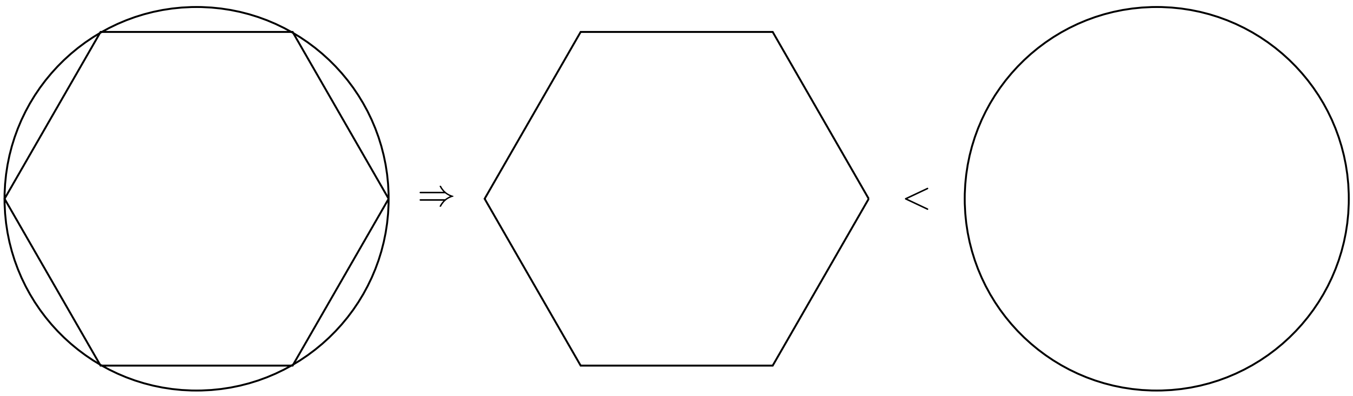
YES, FOR LINESEGMENTS THIS IS AN EQUALITY FOR RECTANGLES.

BUT EUCLID'S THEOREM (16) IS VALID FOR PLANE AND SPACIAL FIGURES AS WELL, AND THE 4– OR MORE–DIMENSIONAL VOLUMES WERE NOT CONSIDERED!

ARCHIMEDES' PHYSICAL IDEA

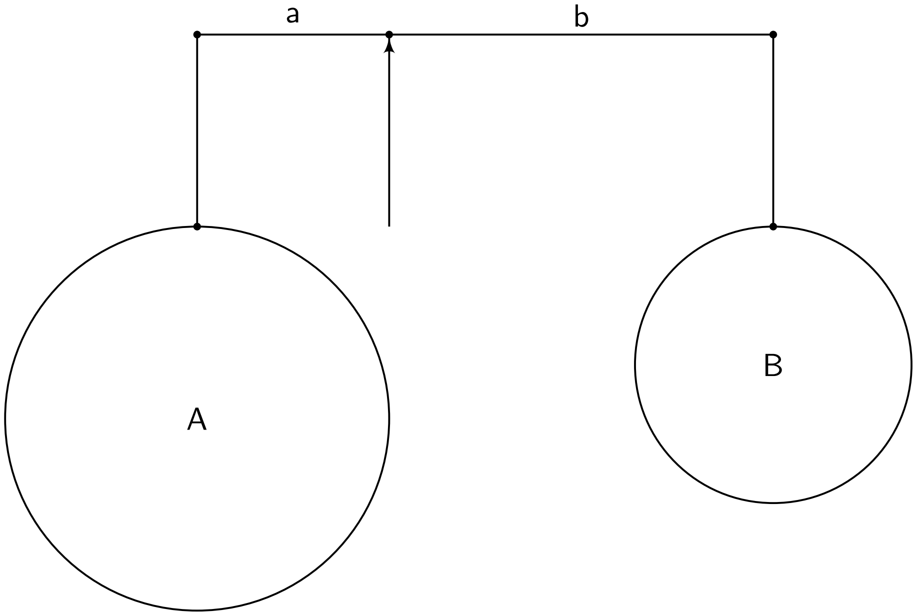
ARCHIMEDES (†212 B.C.) FINDS TWO WAYS TO PROCEDE FROM EUCLID'S THEORY OF MAGNITUDES.

FIRST HE WANTS TO EXPAND THE KINDS OF MATGNITUDES FROM LINESEGMENTS AND PLANE FIGURES TO CURVES AND SURFACES.



BASED ON THE IDEA THAT THE CIRCUMFERENCE OF A CONVEX BODY INSIDE ANOTHER CONVEX BODY IS THE SMALLER, HE EXTENDS THE THEORY.

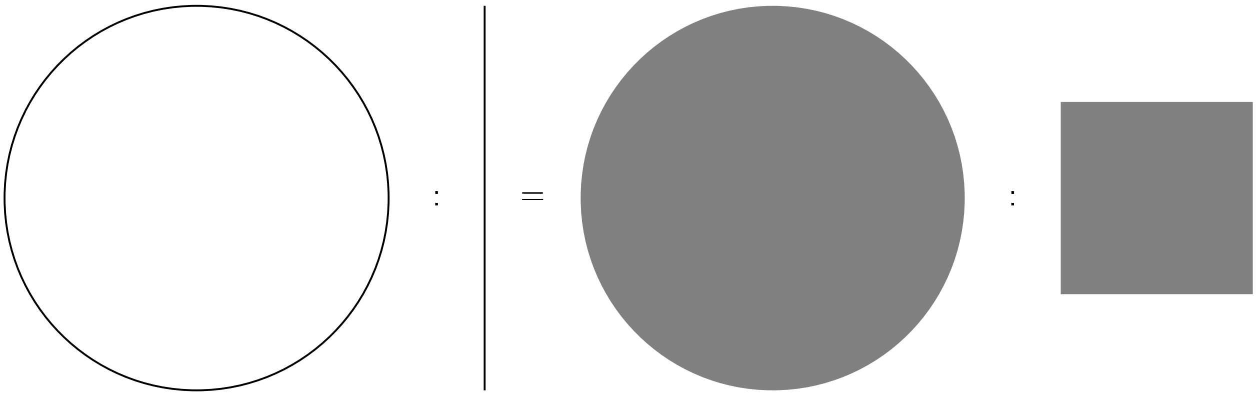
THE SECOND IS HIS PHYSICAL IDEA OF AN IMAGINARY BALANCE WEIGHT:



BALANCE MEANS THAT

$$A : B = b : a$$

IN THIS WAY HE EXTENDS TO COMPARING RELATIONS BETWEEN MAGNITUDES OF DIFFERENT DIMENSIONS, IN WHICH CASE MULTIPLYING ACROSS MIGHT ALSO REQUIRE VOLUMES OF 4–DIMENSIONAL OBJECTS OR HIGHER. THEN HE MAY PROVE THE RELATION–EQUATION:



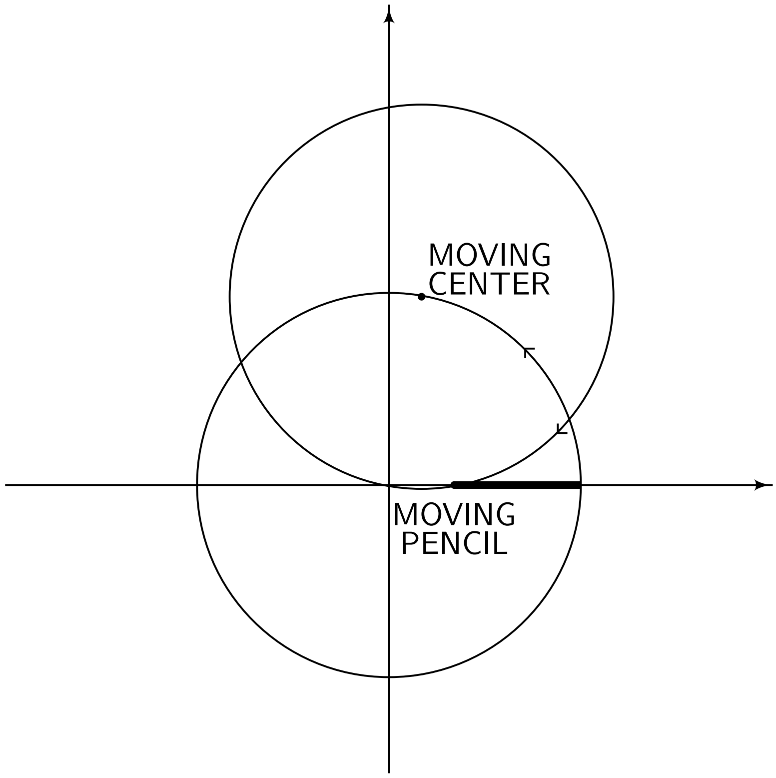
THE CIRCLE RELATES TO ITS DIAMETER AS THE DISC TO THE SQUARE ON ITS RADIUS.
THIS RELATION WAS LATER BAPTISED

$$\pi$$

(BY EULER).

EUCLID AND GEORG MOHR

THE AMBIGUITY OF THE USE OF THE RULER MIGHT HAVE BEEN AVOIDED! GEORG MOHR (1640–97) PROVED IN 1672 THAT ALL POINTS WHICH CAN BE CONSTRUCTED WITH COMPASS AND RULER IN THE EUCLIDEAN WAY, MAY BE CONSTRUCTED WITH THE COMPASS ALONE.
THINKING PHYSICALLY THIS IS NOT SO SURPRICING. IMAGINE A DOUBLE COMPASS, ONE COMPASS IS DRAWING A CIRCLE WITH A CERTAIN SPEED, A SECOND COMPASS HAS ITS NEEDLE FOLLOWING THE PENCIL OF THE FIRST, WHILE THE PENCIL OF THE SECOND IS DRAWING A CURVE MOVING WITH THE SAME SPEED IN THE OPPOSITE DIRECTION, THE TWO CIRCLES HAVING THE SAME RADII.

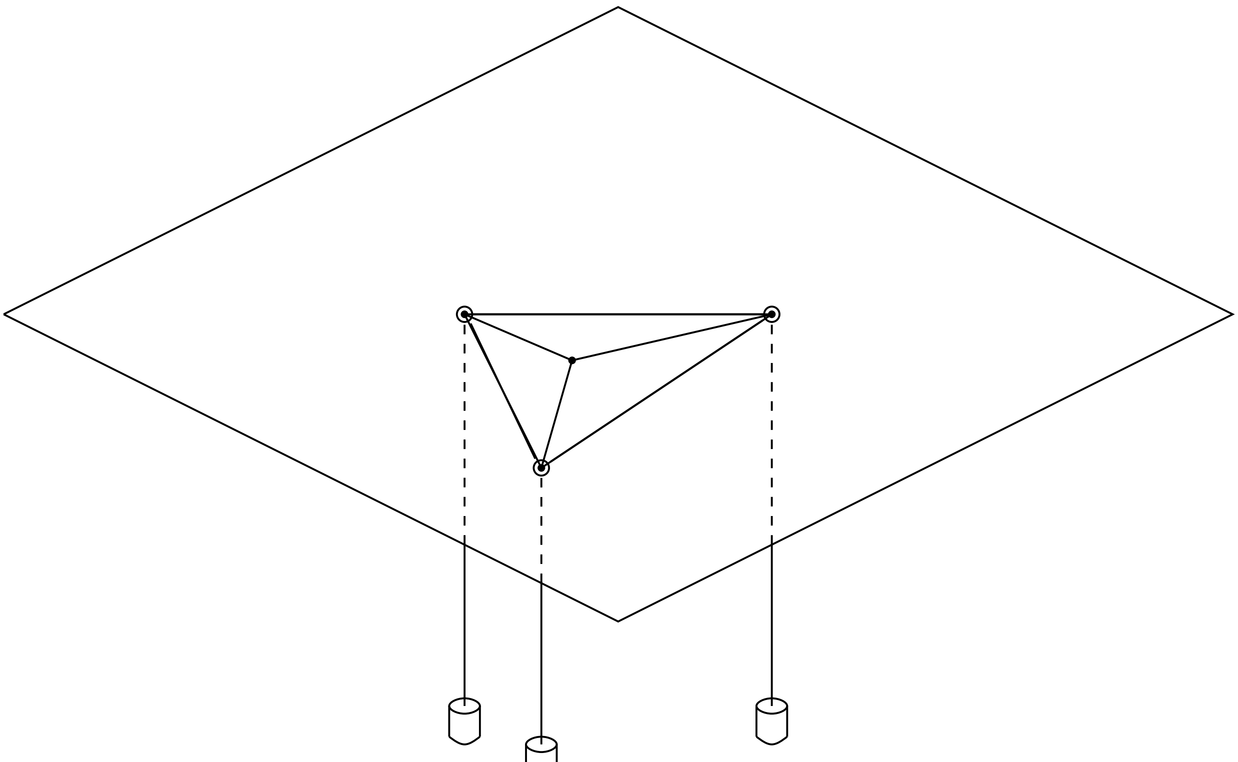


PROOF: IN COMPLEX NOTATION THE PARAMETRICED CURVE MUST BE

$$e^{it} + e^{-it} = 2\cos t \in \mathbb{R}$$

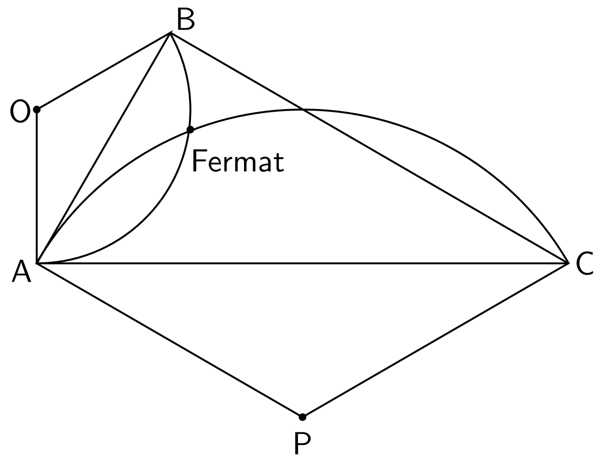
THE POINT OF FERMAT

A BEAUTIFUL EXAMPLE OF MIXING PHYSICS AND MATHEMATICS IS THE SOLUTION TO PIERRE DE FERMAT'S (1601–65) PROBLEM:
GIVEN A TRIANGLE. FIND THE POINT, FROM WHICH THE SUM OF THE DISTANCES TO THE THREE CORNERS IS MINIMAL. NOW WE ANALYZE THE PROBLEM WITH THE HELP OF A PHYSICAL MENTAL EXPERIMENT. ASSUME THE TRIANGLE IS ON A PLATE AND DRILL A HOLE IN EACH CORNER. FROM THE POINT WE DRAW THREE CORDS JOINT IN THE POINT AND GOING THROUGH THE THREE HOLES. IN THE END OF EACH CORD WE PLACE A WEIGHT, THE THREE WEIGHTS HAVING THE SAME SIZE. THIS SYSTEM FINDS AN EQUILIBRIUM WHERE THE POINT OF GRAVITY IS LOWEST WHICH IS THE SAME AS MINIMIZING THE SUM OF THE DISTANCES TO THE CORNERS. AND AS THE POINT RESTS, AND THE POWERS DRAWING IN THE THREE DIRECTIONS ARE OF EQUAL SIZES, THE THREE ANGLES BETWEEN THE CORDS MUST BE EQUAL, I.E. $120°$. (OF COURSE, IT REQUIRES THAT THE ANGLES OF THE ORIGINAL TRIANGLE ARE ALL SMALLER THAN $120°$.)

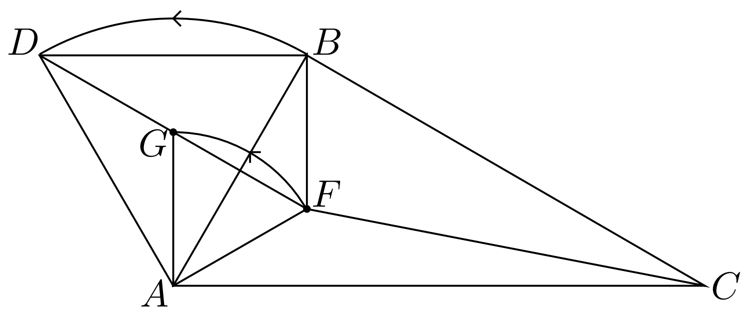


SUCH A POINT EXISTS AND MAY BE CONSTRUCTED BY DRAWING THE ARCS OF THE CIRCUMFERENTIAL ANGLE OVER TWO OF THE SIDES OG THE TRIANGLE. THEY HAVE THE CENTERS O AND P AND ACCIDENTIAL ARC

LENGTH EQUAL TO 120^o. THE ARCS CUT ONE ANOTHER IN THE FERMAT POINT.



A MATHEMATICAL PROOF REFERRING TO THE THEOREM, THAT THE SHORTEST CURVE BETWEEN TWO POINTS IS THE STRAIGHT LINE, IS EQUALLY BEAUTIFUL.
ANALYSIS. ASSUME WE HAVE THE POINT F. IT IS CONNECTED TO THE THREE CORNERS OF THE TRIANGLE. NOW WE TURN THE VECTORS AB AND AF THE ANGLE 60^o AROUND A TO THE VECTORS AD AND AG. THEN $|DG| = |BF|$ AND $|GF| = |AF|$, SO THE SUM OF THE DISTANCES FROM F TO THE CORNERS IS EQUAL TO THE LENGTH OF THE BROKEN LINE FROM D TO C. IT IS SHORTEST IF IT HAPPENS TO BE THE STRAIGHT LINE FROM D TO C. CHOOSING THAT WE GET THE FERMAT-POINT AS ABOVE.



NON–EUCLIDEAN GEOMETRY

THE DIVORCE OF MATHEMATICS FROM PHYSICS STARTED AROUND 1800. UNTIL THEN THE EUCLIDEAN GEOMETRY WAS CONSIDERED AS THE PERFECT MATHEMATICAL MODEL OF THE PHYSICAL SPACE. THE PHILOSOPHER IMMANUEL KANT (1724–1804) IS THE EXPONENT FOR THIS OPINION, BECAUSE HE CONSIDERS THE EUCLIDEAN GEOMETRY AS THE ONLY THEORY WHICH IS BOTH "A PRIORI" – GIVEN IN ADVANCE – AND "ANALYTIC – INDEPENDENT OF EXPERIENCE. BUT THE NEXT GENERATION OF MATHEMATICIANS CAME TO DOUBT THIS OPINION. THE PROBLEM IS WHETHER THE PARALLEL POSTULATE IS A GENOUIN AXIOM OR A THEOREM NOT YET PROVED (AS KANT MIGHT HAVE THOUGHT). IT CLAIMS THAT GIVEN A LINE AND A POINT NOT ON THE LINE, THEN THERE EXISTS EXACTLY ONE LINE THROUGH THIS POINT PARALLEL TO THE GIVEN LINE (OR NOT CUTTING IT). (THIS IS THE GLOBAL AXIOM – ABOUT THE SPACE AS A WHOLE. THE OTHER AXIOMS ARE LOCAL.)FROM THIS AXIOM YOU CONCLUDE THAT THE SUM OF THE ANGLES IN A TRIANGLE IS 180° (EUCLID: "TWO RIGHT ANGLES.") THERE HAS BEEN AN UNCOUNTABLE NUMBERS OF TRIALS TO PROVE THIS AS A THEOREM.

CARL FRIEDRICH GAUß (1777–1855) MUST HAVE HAD HIS DOUBT – OR RATHER BELIEVED IN THE INDEPENDENCE OF THE OTHER AXIOMS. HE THEN ASKED THE QUESTION, WHAT IS THE BEST MODEL OF THE PHYSICAL WORLD? HE MEASURED A LARGE TRIANGLE BETWEEN THE GERMAN MOUNTAINS, BROCKEN, HOHENHAGEN AND INSELBERG, TO SEE WHETHER THE SUM OF THE ANGLES IN PHYSICS WAS 180°. HE FOUND THIS TO BE TRUE WITHIN THE ACCURACY OF MEASURE.

BUT OTHERS, NICOLAI IVANOVITSCH LOBATSCHEVSKIJ (1793–1856) AND JOHANN BOLYAI (1802–60) DEVELOPED NON–EUCLIDEAN GEOMETRIES WITH THE PARALLEL POSTULATE EXCHANGED WITH EITHER THE AXIOM THAT ANY TWO LINES DO CUT ("ELLIPTIC GEOMETRY"), OR THE AXIOM, THAT GIVEN A LINE AND A POINT OUTSIDE, THERE EXIST POSSIBLY SEVERAL LINES NOT CUTTING THE GIVEN LINE ("HYPERBOLIC GEOMETRY").

IN THIS WAY THE MATHEMATICIANS CAME TO STUDY MATHEMATICS WITHOUT REFERENCE TO POSSIBLE APPLICATIONS IN PHYSICS. SIMULTANIOUSLY THEY CONSIDERED SPACES OF ANY NUMBER OF DIMENSIONS.

THAT SUCH PHANTASY–MODELS COULD BE RELEVANT TO PHYSICS HAS HAPPENED BEFORE. THE MOST STRIKING EXAMPLE IS JOHANNES KEPLER'S (1571–1630) ELLIPTIC MODEL OF THE ORBIT OF MARS. HE USES THE THEORY OF CONIC SECTIONS KNOWN FROM EUCLID AND LATER APOLLONIOS (2. CENT. BC).

SET THEORY

AT THE SAME TIME THE MATHEMATICIANS HAVE LEFT THE PHYSICAL CONTENT AND TURNED TO A KIND OF SEMANTIC BUILD ON GEORG CANTOR'S (1845–1918) "SET THEORY," WHICH ONE IS DIFFICULT TO DEAL WITH. THE AXIOMS LEAD IMMIDIATLY TO PARADOXES OR CONTRADICTIONS. THE SET OF SUBSETS OF A SET IS ALWAYS GREATER THAN THE SET ITSELF, WHICH MEANS THAT NO GREATEST SET CAN EXIST, E.G., THE SET OF ALL SETS. THIS IS THE SO-CALLED *RUSSELL PARADOX*, AFTER BERTRAND RUSSELL (1872–1970).

PROOF OF RUSSELL'S PARADOX

A FINITE SET OF N ELEMENTS HAS 2^N SUBSETS, AND $N < 2^N$ HOLDS FOR ALL N.
THE GENERAL PROOF MAY PROCEED: LET A BE A SET AND D THE SET OF ITS SUBSETS. IF THEY HAVE THE SAME SIZE, WE MAY FIND A FUNCTION, $F : A \to D$, SUCH THAT EVERY SUBSET, $D \in D$, IS THE IMAGE OF ONE OF THE ELEMENTS, $A \in A$. LET $F : A \to D$ BE ANY FUNCTION. WE SHALL PROVE THAT THERE MUST EXIST A SUBSET OF A NOT THE IMAGE OF ANY ELEMENT OF A BY THE FUNCTION F. CONSIDER THE SUBSET

$$B = \{X \in A | X \notin F(X)\}$$

ASSUME, THAT $B = F(Y)$. IF $Y \in B$, WE CONCLUDE FROM THE DEFINITION, THAT $Y \notin B = F(Y)$. AND IF $Y \notin B = F(Y)$, THEN WE CONCLUDE FROM THE DEFINITION, THAT $Y \in B$. HENCE, $B \neq F(Y)$ FOR ANY $Y \in A$, I.E., THERE ARE MORE SUBSETS THAN ELEMENTS IN EVERY SET.

THE BANACH–TARSKI PARADOX

THE COUNTER-INTUITIVITY OF SET THEORY IS BEAUTIFULLY STRESSED BY THE FAMOUS BANACH–TARSKI PARADOX, DUE TO FELIX HAUSDORFF (1868–1942), BUT NAMED AFTER STEFAN BANACH (1892–1945) AND ALFRED TARSKI (1901–83), WHO WROTE ABOUT IT IN 1924, WITH REFERENCE TO HAUSDORFF'S BOOK, *GRUNDZÜGE DER MENGENLEHRE* FROM 1914.
IN ALL SIMPLICITY IT SAYS THAT WE MAY DIVIDE ANY SPHERE (WITHOUT CENTER) IN 3 CONGRUENT DISJOINT SUBSETS IN SUCH A WAY, THAT 2 OF THEM MAY BE JOINT TO FILL THE WHOLE SPERE! THIS PHENOMENON CONTRADICTS OUR PHYSICAL INTUITION.
FROM A MATHEMATICAL POINT OF VIEW WE HAVE JUST PROVED, THAT WE MAY NOT DEFINE A MEASURE (NOT IDENTICALLY ZERO) SUCH THAT ALL SUBSETS HAVE ONE. SO, WE MUST INTRODUCE THE CONCEPT OF "MEASURABLE" AMONG SUBSETS.
WE ARE THROWN BACK TO PYTHAGORAS, THE NUMBERS DO NOT SUFFICE TO DESCRIBE THE RELATIONS BETWEEN THE OBJECTS UNDER CONSIDERATION!

A LOGIAL PARADOX

NOT ONLY THE SET THEORY CONTRADICTS OUR PHYSICAL INTUITION. ALSO PURE LOGIC MIGHT DO.
IN A CLOSET WE HAVE TWO DRAWERS. ON THE FIRST ONE IS A TEXT SAYING: "THE RING IS IN THE OTHER DRAWER." ON THE SECOND THE TEXT SAYS: "ONLY ONE OF THE STATEMENTS ON THE DRAWERS IS TRUE."
NOW, THE STATEMENTS MAY BE TRUE OR FALSE. IF THE SECOND STATEMENT IS FALSE, WE CONCLUDE THAT BOTH STATEMENTS ARE FALSE. THIS MEANS THAT THE RING IS EN THE FIRST DRAWER. AND IF THE SECOND STATEMENT IS TRUE, THEN THE FIRST STATEMENT IS FALSE AND THE RING IS IN THE FIRST DRAWER!
BUT IN A PHYSICAL WORLD NOTHING PREVENTS A LOGICAL IGNORANT FROM PLACING THE RING IN THE SECOND DRAWER!

Mogens Esrom Larsen