

THE MONTHLY PROBLEM 11356, APRIL 2008.

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MAY 28, 2008

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Proposed by Michael Poghosyan, Yerevan State University, Yerevan, Armenia.  
Prove that for any positive integer,  $n$ ,

$$\sum_{k=0}^n \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$

Proof:

Introducing the notation of a descending factorial with specified stepsize, in casu 1 or 2:

$$[x, d]_n := \begin{cases} \prod_{j=0}^{n-1} (x - jd) & n \in \mathbb{N} \\ 1 & n = 0 \\ \prod_{j=1}^{-n} \frac{1}{x + jd} & -n \in \mathbb{N}, -x \notin \{d, 2d, \dots, -nd\} \end{cases}$$

we proceed from the left side writing the binomial coefficients out as factorials

$$\sum_{k=0}^n \binom{n}{k} \frac{n!(2k)!(2n-2k)!}{k!(n-k)!(2n)!(2k+1)}$$

Splitting the factorials with a factor 2 in two taking every second factor to get factorials with stepsize 2 we get

$$\sum_{k=0}^n \binom{n}{k} \frac{n![2k, 2]_k [2k-1, 2]_k [2n-2k, 2]_{n-k} [2n-2k-1, 2]_{n-k}}{k!(n-k)![2n, 2]_n [2n-1, 2]_n (2k+1)}$$

Further we change the factorials to stepsize 1 by dividing with the appropriate powers of 2:

$$\sum_{k=0}^n \binom{n}{k} \frac{n!k!2^k [k - \frac{1}{2}, 1]_k 2^k (n-k)! 2^{n-k} [n-k - \frac{1}{2}, 1]_{n-k} 2^{n-k}}{k!(n-k)!n!2^n [n - \frac{1}{2}, 1]_n 2^n (2k+1)}$$

Cancelling common factors we get

$$\sum_{k=0}^n \binom{n}{k} \frac{[k - \frac{1}{2}, 1]_k [n-k - \frac{1}{2}, 1]_{n-k}}{[n - \frac{1}{2}, 1]_n (2k+1)} = \frac{1}{[n - \frac{1}{2}, 1]_n} \sum_{k=0}^n \binom{n}{k} \frac{[k - \frac{1}{2}, 1]_k [n-k - \frac{1}{2}, 1]_{n-k}}{2k+1}$$

To get rid of the denominator we introduce the factorial

$$[n + \frac{1}{2}, 1]_n \cdot \frac{1}{2} = [n + \frac{1}{2}, 1]_{n+1} = [n + \frac{1}{2}, 1]_{n-k} (k + \frac{1}{2}) [k - \frac{1}{2}, 1]_k$$

to get

$$\frac{1}{2k+1} = \frac{[n + \frac{1}{2}, 1]_{n-k} [k - \frac{1}{2}, 1]_k}{[n + \frac{1}{2}, 1]_n}$$

Then we may write

$$\frac{1}{[n - \frac{1}{2}, 1]_n [n + \frac{1}{2}, 1]_n} \sum_{k=0}^n \binom{n}{k} [k - \frac{1}{2}, 1]_k [n - k - \frac{1}{2}, 1]_{n-k} [n + \frac{1}{2}, 1]_{n-k} [k - \frac{1}{2}, 1]_k$$

Now we change the signs of all factors in the factorials containing the variable  $k$ :

$$\frac{1}{[n - \frac{1}{2}, 1]_n [n + \frac{1}{2}, 1]_n} \cdot \sum_{k=0}^n \binom{n}{k} [-\frac{1}{2}, 1]_k (-1)^k [-\frac{1}{2}, 1]_{n-k} (-1)^{n-k} [n + \frac{1}{2}, 1]_{n-k} [-\frac{1}{2}, 1]_k (-1)^k$$

Organized nicely to

$$\frac{(-1)^n}{[n - \frac{1}{2}, 1]_n [n + \frac{1}{2}, 1]_n} \sum_{k=0}^n \binom{n}{k} [-\frac{1}{2}, 1]_k^2 [-\frac{1}{2}, 1]_{n-k} [n + \frac{1}{2}, 1]_{n-k} (-1)^k$$

This may be recognized as the Pfaff-Saalschütz formula, (9.1), in my recent textbook, *Summa Summarum*, A K Peters 2007:

**Theorem 9.1.** *If the numbers satisfy  $a_1 + a_2 + b_1 + b_2 = n - 1$  we have the Pfaff-Saalschütz formula*

$$\sum_{k=0}^n \binom{n}{k} [a_1, 1]_k [a_2, 1]_k [b_1, 1]_{n-k} [b_2, 1]_{n-k} (-1)^k = [b_1 + a_1, 1]_n [b_1 + a_2, 1]_n (-1)^n$$

So we obtain:

$$\begin{aligned} \frac{1}{[n - \frac{1}{2}, 1]_n [n + \frac{1}{2}, 1]_n} [-1, 1]_n^2 &= \frac{n!^2}{[n - \frac{1}{2}, 1]_n [n + \frac{1}{2}, 1]_n} = \\ &= \frac{2^{2n} n!^2}{[2n - 1, 2]_n [2n + 1, 2]_n} = \frac{2^{4n} n!^4}{[2n - 1, 2]_n [2n, 2]_n^2 [2n + 1, 2]_n} = \frac{2^{4n} (n!)^4}{(2n)! (2n + 1)!} \end{aligned}$$