PHYSICAL MATHEMATICS?

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Mathematics and Physics divorced in recent times. The subjects are now developing parallel. Mathematics creates models and Physics may apply what models it finds fittest. But in ancient times the subjects were married.

Claytablets and numbersystems

At first there was observation. The oldest surviving mathematical texts from Assyria were written a couple of thousands of years B.C. In contemporary cultures in Egypt and China burnable material were used, while Assyrians pressed small cuneiforms in wet clay. Although a tablet might be reused many times, it occasionaly happened that a house burned down immortalizing the tablets. This was the luck for the archeologists though annoying for the Assyrians.

The climate in Assyria is pretty dry, remember Hammurapis' (1792–50) famous laws about use and abuse of watercanals. The advantage is that one can almost always see the stars and watch their movements. The Assyrians discovered their periods – short and long. And while most stars move rather parallel, there are seven exceptions, the sun, the moon and the five "tramps" – planets. They move back and forth relative to the other stars. The Assyrians considered these seven as veritable gods and there exceptional movements as omens. The planets still keeps their god–names, but now Roman, Venus, Mars, Jove etc.

In ancient time people were not thinking in the pattern "cause–effect" but rather "omen–event." Gods decided what shall happen, but might be influenced by requests.

The periods allowed the learned people to predict certain events like an eclipse of the moon, most impressive for laymen. After such a successful prediction the king would of course ask for more earthy predictions, giving the learned people genouine problems. To refuse was impossible, and in case of wrong predictions a capital punishment might result. So they soon learned to hide their statements in a cloud of reservations.

Among the tablets we find calculations with length, area, volume, weight, prize and salary. Also the multiplication table, but remind you, in the 60–numbersystem it requires all products from 2 to 59, i.e., 1711 products. Our table from 2 to 9

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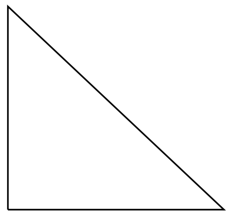
contains only 36. And other tables of Pythagorean triples, 3 4 5, 5 12 13, ..., 12709 13500 18541, etc. They must have known these rightangular triangles.

The last one looks like:

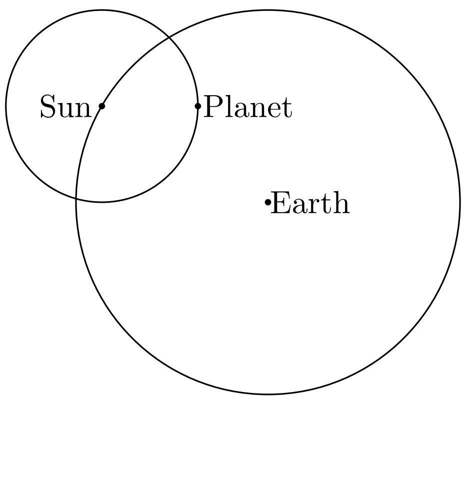


It looks almost isosceles. But we found no example with exactly equal cathesi!

The Assyrians used numbers to denote the positions of stars and planets, they invented the positional system using the base 60. They divided the perimeter in 360 grades and gave the positions with two sexagesimal fractions, so a grade was divided in 60 minutes (small units) and each of them in 60 seconds (the second kind of small units). When these tables came to the knowledge of the Greek astronomers, they preserved the division into the smaller parts, but not using a positional system for signifying numbers, they wrote the numbers of minutes and seconds in the Greek number system. So, each position of a star was given with three whole numbers, the grades, the minutes and the seconds, the latter two as numbers between 0 and 59. Not having a symbol for zero, Ptolemaios (2. century) did chose the meaningless number 70 for the case of no minutes. As the Greeks used the alphabet for the numbers too, α=1, β=2, etc., the symbol for 70 was accidentally *omikron,* o.

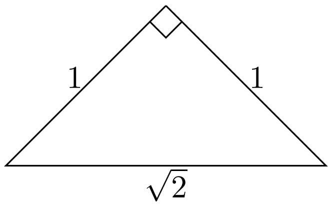
Astronomy

The Greek astronomers were not so satisfied with the idea of the gods' wills. They created a virtual mathematical model of the tramping of the planets. They imagined the earth as a ball, the stars as lights on a sphere far out, and the planets as objects moving on epicycles, i.e., circular orbits with the center moving on another circle with the earth in its center. Aristarchos (3. century), nicknamed "mathematichos" – the knowing all – had the sun in the center of the circles of the planets.



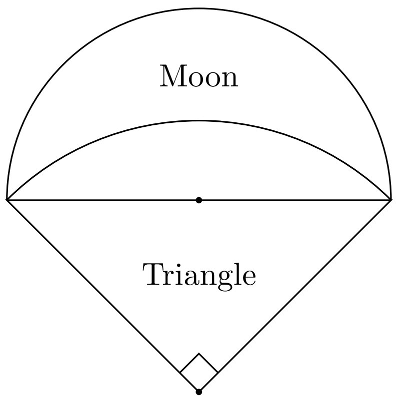
Geometry

Pythagoras (6. century B.C.) was as fund of numbers as was the Assyrians. He noticed that the hypothenus in a rightangular isosceles triangle could not be measured by the other sides with rational numbers. Or, the diagonal in a square with side 1 is $\sqrt{2}$, an irrational number. But it was easy to construct with compass and ruler.



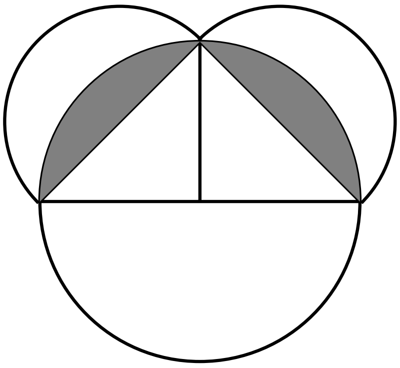
These facts led the Greeks to avoid numbers and develop a geometry as a description of the space in its own fashion, dealing with the figures directly, being of dimensions 1, 2 or 3.

They managed to define concepts of equality and inequality corresponding to the meaning of lengths, areas and volumes without computing the relevant numbers as measures. In this context they even managed to prove that the moon of Hippokrates (5. century B.C.) is equal to a certain rightangular isosceles triangle.



If the triangle has cathesi both equal to 1 and then in our terms hypothenus equal to $\sqrt{2}$ and area equal to $\frac{1}{2}$, then the moon drawn by two circular arcs of radii 1 and $\frac{\sqrt{2}}{2}$ and centers in the rightangular corner and the center of the hypothenus respectively has the area $\frac{1}{2}$ too.

The proof of this identity is surprisingly simple. Look at the figure:



Discs of different radii relates as the corresponing squares on the radii. Hence the Pythagorean theorem applies also to the half-discs on the sides of a rightangular triangle. So referring to the figure, the big half-disc is equal so the sum of the two equal small half-discs. This means that the triangel is equal to the big half-disc minus the grey area, and this equals the sum of the two small half-discs minus the grey area. So the half of the triangle is equal to the moon.

The Greek geometry compared figures with figures and avoided numbers. The Greeks demanded certain requirements – in Greek "axioma" – which should reflect the geometrical praxis of drawing figures with compass and ruler. E.g., given two distinct points, then there exists exactly one line containing both, and exactly one circle with the first point as the center and containg the second.

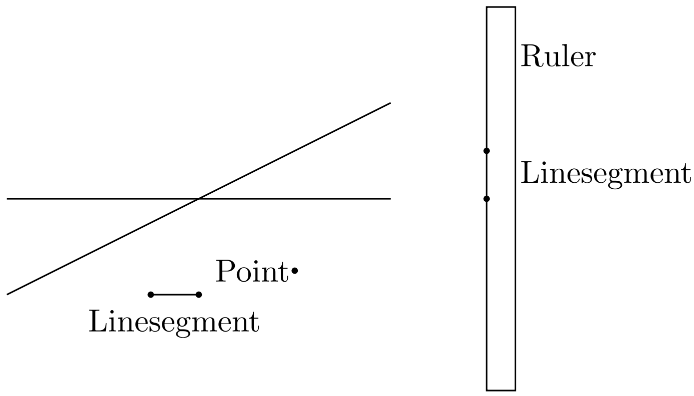
From these prerequisites they derived all the logical consequences they had phantasy to imagine and created in this way the first example of a genuine mathematical model of a physical phenomenon, the space we inhabit.

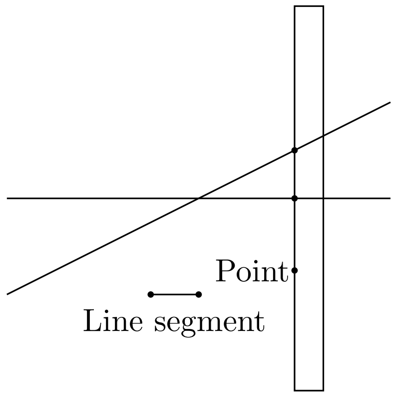
Insertion

When we talk about construction with compass and ruler, it is an exaggeration or disengenousness. We mean using the axioms of Euclid (3. century B.C.). From his axioms we may only construct points which in moderne language have such rational or irrational coordinates, which can be derived by taking a number of square roots. Because we shall solve a second degree equation emerging from the equation describing the circle(s).

Two famous problems considered in the ancient times was the doubling af a cube and the division of an arbitrary angle into three equal parts. Both require the solution of a third degree equation, not to be obtained by the axioms of Euclid.

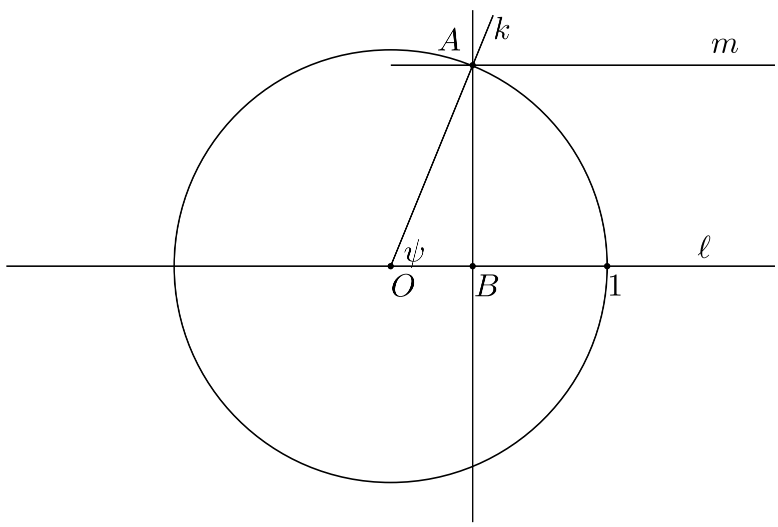
But allready in the 5. century B.C. they managed to solve the problem by physical means. If we consider the ruler as a piece of wood with a straight line, we might mark two points on it with a given distance. Then we may, given two crossing lines and a point anywhere, manage to draw a line through this point and such that the two points on the ruler lie on each one of the given lines.





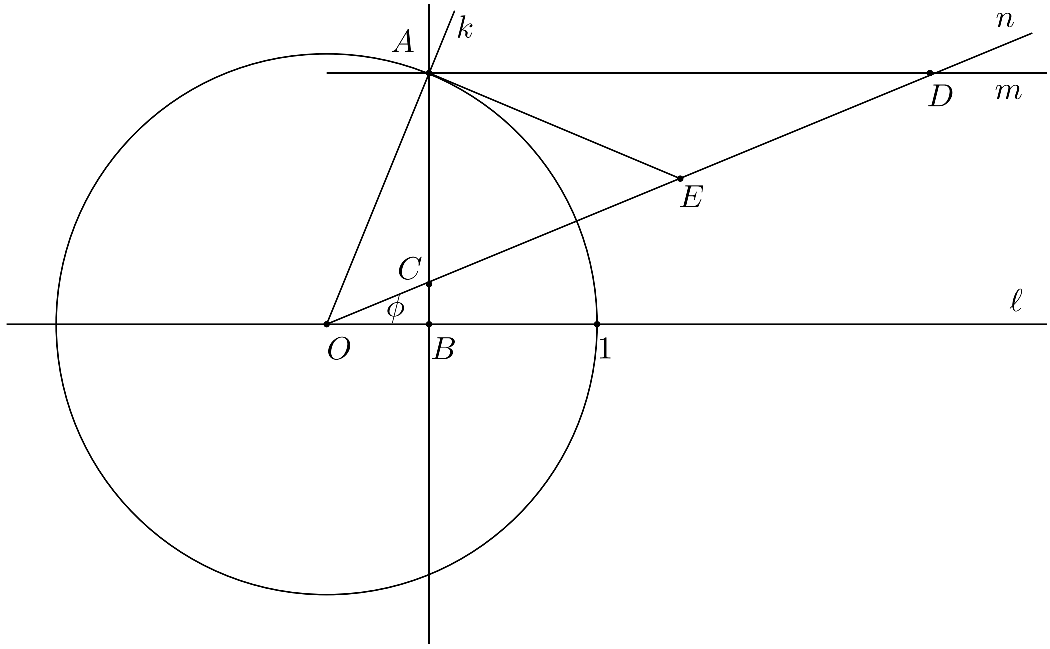
This means that we have added the axiom, that given two line and a point and a line segment, then there exist a line through the given point and such that the two lines cuts a line segment of the given length of the new line. With this addition to the Euclidean axioms we are able to construct third roots and in particular trisect an angle and double a cube.

To show how, we shall at first look at the trisection of the angle.



We want to trisect the angle ψ between the lines ℓ and k. We draw the circle with center in O and radius 1 – a point on ℓ. It cuts the line k in the point A. Then we construct AB rightangular on ℓ. Eventually we draw the line m through A parallel to ℓ.

We now apply the new axiom to insert the line n through O such that the lines AB and m cuts a linesegment of n, CD of length 2:



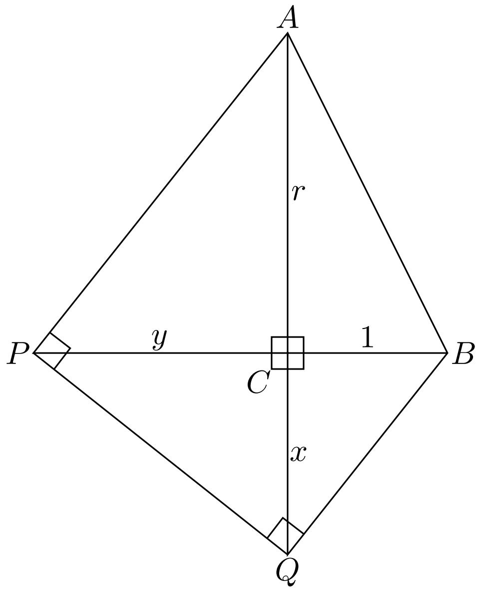
E is constructed as the midpoint of the linesegment CD. As the triangle $\triangle ACD$ is rightangular with $\angle A$ as the right angle, we note that a circle with center E and radius 1 must go through A, C and D. Therefore $AE = CE = DE = 1$ and hence the triangle $\triangle AED$ is isosceles, such that $\angle DAE = \angle ADE = \angle COB = \phi$. As the triangle $\triangle OAE$ also is isosceles, we conclude that $\angle AOC = \angle AEC = \angle EAD + \angle ADE = 2\phi$.

We have trisected the angle $\angle AOB = \psi$ in three equal parts.

Doubling a dice

As we have easely constructed a line segment of length $\sqrt{2}$, i.e., the side of a square of double size, it is tempting to ask for the doubling af a dice, i.e., to construct a line segment of size $\sqrt[3]{2}$.

This is impossible in the Euclidean geometry for the same reason as the impossibility of the trisecting of an angle, but with physical instruments it is easy. We need the following construction, generalizing the number 2 to any real number – already constructed – r:



Given a rightangular triangel, $\triangle ABC$ with cathesi 1 and r, we need to find P and Q on the prolongation af the cathesi in a way, such that the two angles are right: $\angle APQ = \angle PQB = \frac{\pi}{2}$. Then the three new triangles are similar, $\triangle ACP \sim \triangle PCQ \sim \triangle QCB$. Hence we have

$$\frac{1}{x} = \frac{x}{y} = \frac{y}{r}$$

From this we compute

$$x^2 = y \quad \wedge \quad y^2 = rx \quad \Rightarrow \quad x^4 = y^2 = rx$$

showing that the linesegment x solves the problem.

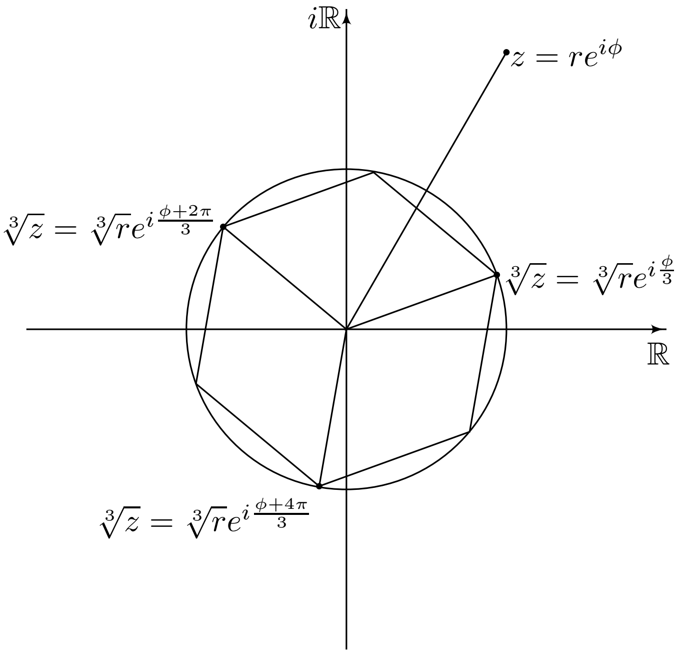
To draw the figure we may apply a couple af carpenter's squares to fit the two right angles on the appropriate lines.

General cubic roots

These two constuctions actually allow us to construct the cubic root of an arbitrary complex number, represented as a vector in the plane. If this vector has the representation

$$re^{i\phi}$$

then we may construct the two necessary magnitudes, $\sqrt[3]{r}$ and and $\frac{\phi}{3}$. To add the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ is easy:



Cubic equations

We may even solve any cubic equation constructively as seen by Francis Vieta (1540–1603). Suppose we have a cubic equation (i.e., $A \neq 0$)

$$Ax^3 + Bx^2 + Cx + D = 0$$

Then we may divide by A and write

$$x^3 + bx^2 + cx + d = 0$$

We may also change the variable to $y = x - \frac{b}{3}$ to get rid of the square term

$$y^3 + \left(c - \frac{b^2}{3}\right) y + d + \frac{2b^3}{27} - \frac{bc}{3} = 0$$

or simply

$$y^3 + ey + f = 0$$

Now we try to write $y = p + q$, p and q to be chosen. We get

$$p^3 + q^3 + 3pq(p + q) + e(p + q) + f = 0$$

It is seen, that if we choose p and q satisfying the equations

$$\begin{aligned} 3pg + e &= 0 \\ p^3 + q^3 + f &= 0 \end{aligned}$$

then we have a solution $y = p + q$. But we may consider p^3 and q^3 as satisfying

$$\begin{aligned} p^3 g^3 &= -\frac{e^3}{27} \\ p^3 + q^3 &= -f \end{aligned}$$

hence they are the solutions to a quadratic equation, i.e., they are

$$\frac{-f \pm \sqrt{f^2 + 4\frac{e^3}{27}}}{2}$$

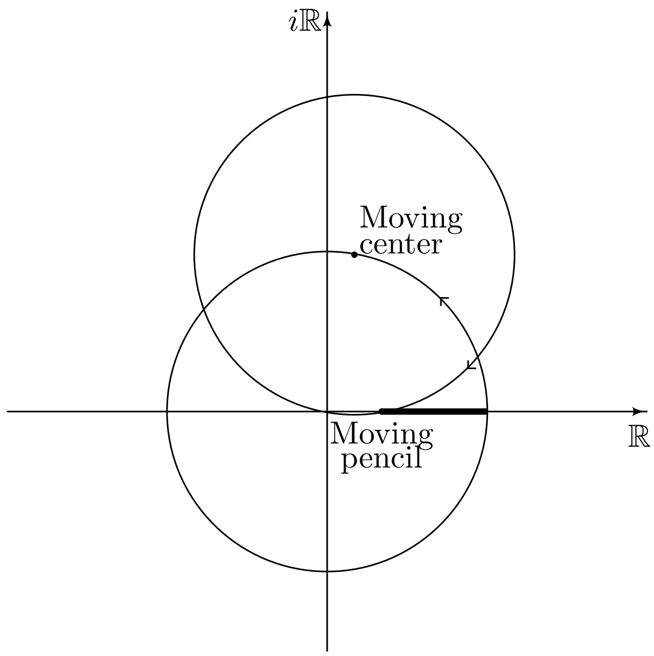
all of which are constructable magnitudes, possibly as complex vectors.

All we now need is to constuct their cubic roots with the methods above. Remember to construct all three complex roots, also of a real value. And with the 9 possible sums of $p + q$, it is necessary to verify the true values eventually!

Euclides Danicus

The ambiguity of the use of the ruler may be completely avoided. The "Danish Euclid," Georg Mohr (1640–97), proved in 1672 that all points, which can be constructed with compass and ruler in the Euclidean sense can be constructed with the compass alone. We do only need a ruler to construct cubic roots!

Thinking physically this is not so surpricing. Imagine a double compass, one compass is drawing a circle with a certain speed, a second compass has its needle following the pencil of the first, while the pencil of the second is drawing a curve moving with the same speed in the opposite direction. The two circles have the same radius.



Proof: In complex notation the parametriced curve must be

$$e^{it} + e^{-it} = 2\cos t \in \mathbb{R}$$

The theory of magnitudes

The most important tool in their reasoning was the so called theory of magnitudes – Euclid's 5. book – which defines the meaning of the relation between four magnitudes of the same kind (i.e. dimension), A, B, C og D, that A relates to B as C to D, written

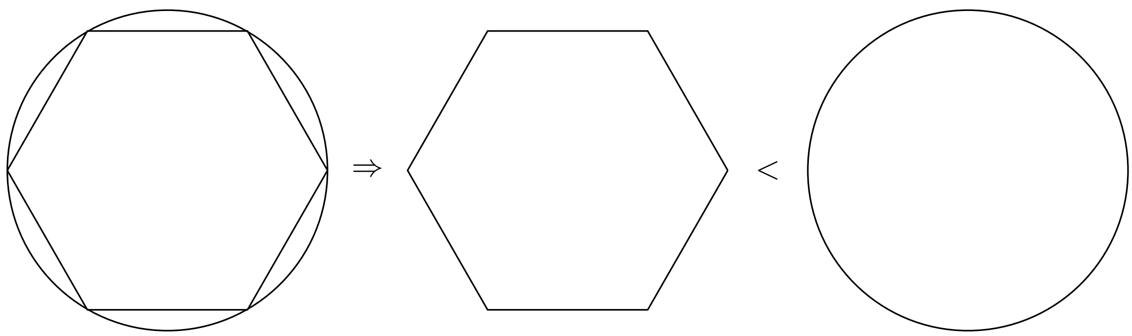
$$A : B = C : D$$

The moon and the triangle above relates as any other pair of equal magnitudes. About $\sqrt{2}$ we might say, that the diagonal in a square relates to the side as the same two in any other square.

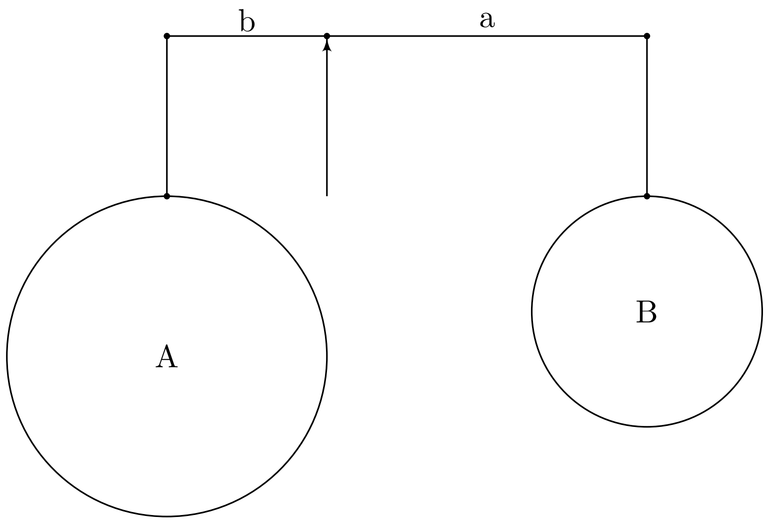
For one–dimensional magnitudes, you might multiply across and say, that the two relations agree, if the two recktangles obtained are equal (in size). But the Greeks were not willing to define a four– or higher–dimensional volume, so the equality of relations between two– or three–dimensional magnitudes was not related to the cross product.

Archimedes

Archimedes (†212 B.C.) finds two ways to go from Euclid's theory of magnitudes. One stroke of genious is to generalize the objects to compare from straight lines to curves and from plane figures to surfaces of spatial figures. He just requires that they surround something convex. E.g., a polygon of cordes to a circle is then assumed to have a shorter circomference than the very circle. Euclid would only compare the areas.



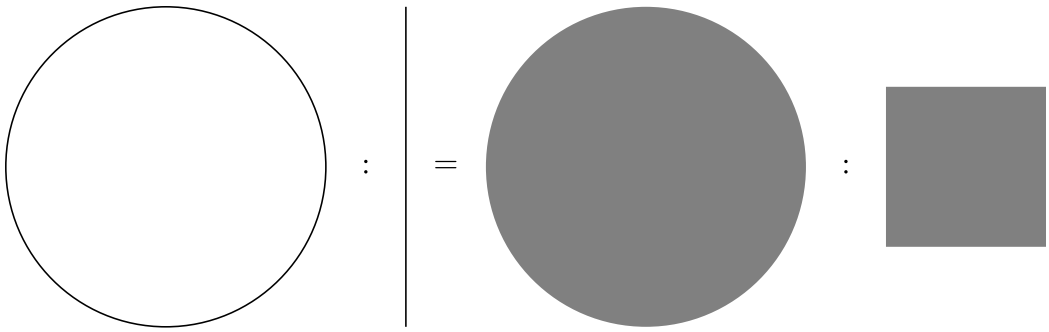
The other is to compare pairs of one dimension to pairs of another. Given two line segments, a and b, and two plane figurs, A and B, such that a relates to b as does A to B, then we might multiply across and say that the two spatial figures $a \times B$ and $b \times A$ have the same volume. But Archimedes has a more physical idea. He imagines a balance weight with arms of length a and b and the two figures hanging in the end of one arm each, B in a and A in b and defines the equality of the relations by the equilibrium of the weight. And this idea may be used for spatial figures too, where the cross product would require a 4–dimensional volume. He uses this way of reasoning to establish the area of a segment of a parabola too.



Balance means that

$$A : B = a : b$$

In this way he may claim that the circumference of the circle relates to the diameter as the disc to the square on the radius. This means, that we only need one π! He also proves that the area of a sphere is equal to the area of the disc having the diameter of the sphere as radius. To this discovery he claims the famous quotation: “It has always been so, but I am the first one to know!”



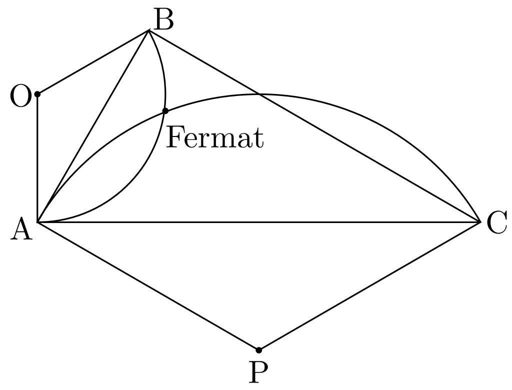
Analysis

The introduction of axioms as foundation of mathematics is not the only Greek contribution to the development of thinking. A method of solving problems called “analysis” has proved most important both inside and outside mathematics. Given a problem, you imagine to have found a solution. Then you deduct what restrictions it must satisfy. If these deduced properties are in conflict, you conclude, that no solution exists. This is what is also called an indirect proof or “deductio ad absurdum.” Are you more lucky, you establish so many restrictions that a possible solution is uniquely determined. This is the typical behavior solving one or several equations. You assume there is a solution, call it x and derives as many consequences as possible hopefully finding its uniqueness. But this does not solve the problem as it might be an incomplete proof of its impossibility. You must verify the validity of the suggested result!

The point of Fermat

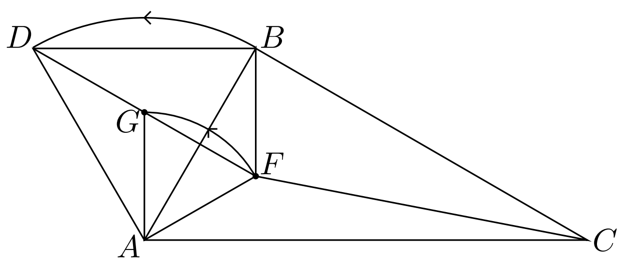
A beautiful example of mixing physics and mathematics is the solution to Pierre de Fermat's (1601–65) problem: Given a triangle. Find the point, from which the sum of the distances to the three corners is minimal. Now we analyze the problem with the help of a physical mental experiment. Assume the triangle is on a plate and drill a hole in each corner. From the point we draw three cords joint in the point and going through the three holes. In the end of each cord we place a weight, the three weights having the same size. This system finds an equilibrium where the point of gravity is lowest which is the same as minimizing the sum of the distances to the corners. And as the point rests, and the powers drawing in the three directions are of equal sizes, the three angles between the cords must be equal, i.e., 120°. (Of course, it requires that the angles of the original triangle are all smaller than 120°.)

Such a point exists and may be constructed by drawing the arcs of the circumferential angle over two of the sides og the triangle. They have the centers O and P and accidential arc length equal to 120^o. The arcs cut one another in the Fermat point.



A mathematical proof referring to the theorem, that the shortest curve between two points is the straight line, is equally beautiful.

Analysis. Assume we have the point F. It is connected to the three corners of the triangle. Now we turn the vectors AB and AF the angle 60^o around A to the vectors AD and AG. Then $|DG| = |BF|$ and $|GF| = |AF|$, so the sum of the distances from F to the corners is equal to the length of the broken line $DGFC$. It is shortest if it happens to be the straight line from D to C. Choosing that we get the Fermat-point as above.



Non–Euclidean geometry

The divorce of mathematics from physics started around 1800. Until then the Euclidean geometry was considered as the perfect mathematical model of the physical space. The philosopher Immanuel Kant (1724–1804) is the exponent for this opinion, because he considers the Euclidean geometry as the only theory which is both “a priori” – given in advance – and “analytic – independent of experience. But the next generation of mathematicians came to doubt this opinion. The problem is whether the parallel postulate is a genouin axiom or a theorem not yet proved (as Kant might have thought). It claims that given a line and a point not on the line, then there exists exactly one line through this point parallel to the given line (or not cutting it). (This is the global axiom – about the space as a whole. The other axioms are local.)From this axiom you conclude that the sum of the angles in a triangle is 180° (Euclid: “Two right angles.”) There has been an uncountable numbers of trials to prove this as a theorem.

Carl Friedrich Gauß (1777–1855) must have had his doubt – or rather believed in the independence of the other axioms. He then asked the question, what is the best model of the physical world? He measured a large triangle between the German mountains, Brocken, Hohenhagen and Inselberg, to see whether the sum of the angles in physics was 180°. He found this to be true within the accuracy of measure.

But others, Nicolai Ivanovitsch Lobatschevskij (1793–1856) and Johann Bolyai (1802–60) developed non–Euclidean geometries with the parallel postulate exchanged with either the axiom that any two lines do cut (“elliptic geometry”), or the axiom, that given a line and a point outside, there exist possibly several lines not cutting the given line (“hyperbolic geometry”).

In this way the mathematicians came to study mathematics without reference to possible applications in physics. Simultaniously they considered spaces of any number of dimensions.

That these phantasy–models could be relevant to physics was imagined by somebody. E.g., William Kingdon Clifford (1845–79) wrote an abstract to a dissertation, “On the Space Theory of Matter” in 1876, a predecesssor for Albert Einstein’s (1879–1955) theory of relativity. The event that physicist suddenly find an old piece of mathematics useful has happened before. The most striking example is Johannes Kepler’s (1571–1630) elliptic model of the orbit of Mars. He uses the theory of conic sections known from Euclid and later Apollonios (2. cent. BC).

Set theory

At the same time the mathematicians have left the physical content and turned to a kind of semantic build on Georg Cantor’s (1845–1918) “set theory,” which one is difficult to deal with. The axioms lead immidiatly to paradoxes or contradictions. The set of subsets of a set is always greater than the set itself, which means that no greatest set can exist, e.g., the set of all sets. This is the so-called *Russell paradox,* after Bertrand Russell (1872–1970).

Proof of Russell's paradox

A finite set of n elements has 2^n subsets, and $n < 2^n$ holds for all n.

The general proof may proceed: Let A be a set and D the set of its subsets. If they have the same size, we may find a function, $f : A \to D$, such that every subset, $d \in D$, is the image of one of the elements, $a \in A$. Let $f : A \to D$ be any function. We shall prove that there must exist a subset of A not the image of any element of A by the function f. Consider the subset

$$b = \{x \in A | x \notin f(x)\}$$

Assume, that $b = f(y)$. If $y \in b$, we conclude from the definition, that $y \notin b = f(y)$. And if $y \notin b = f(y)$, then we conclude from the definition, that $y \in b$. Hence, $b \neq f(y)$ for any $y \in A$, i.e., there are more subsets than elements in every set.

The Banach–Tarski paradox

The counter-intuitivity of set theory is beautifully stressed be the famous Banach–Tarski paradox, due to Felix Hausdorff (1868–1942), but named after Stefan Banach (1892–1945) and Alfred Tarski (1901–83), who wrote about it in 1924, with reference to Hausdorff's book, *Grundzüge der Mengenlehre* from 1914.

In all simplicity it says that we may divide any sphere (without center) in 3 congruent disjoint subsets in such a way, that 2 of them may be joint to fill the whole spere! This phenomenon contradicts our physical intuition.

From a mathematical point of view we have just proved, that we may not define a measure (not identically zero) such that all subsets have one. So, we must introduce the concept of "measurable" among subsets.

We are thrown back to Pythagoras, the numbers do not suffice to describe the relations between the objects under consideration!

A logial paradox

Not only the set theory contradicts our physical intuition. Also pure logic might do.

In a closet we have two drawers. On the first one is a text saying: "The ring is in the other drawer." On the second the text says: "only one of the statements on the drawers is true."

Now, the statements may be true or false. If the second statement is false, we conclude that both statements are false. This means that the ring is en the first drawer. And if the second statement is true, then the first statement is false and the ring is in the first drawer!

But in a physical world nothing prevents a logical ignorant from placing the ring in the second drawer!