THE MONTHLY PROBLEM 11356, APRIL2008.

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Dear Peter.

Proposed by Michael Poghosyan, Yerevan State University, Yerevan, Armenia.
Prove that for any positive integer, n,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$

Proof:
Computing the left side:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!(2k)!(2n-2k)!}{k!(n-k)!(2n)!(2k+1)}$$

$$= \frac{1}{[n+\frac{1}{2}]_{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{n![2k,2]_k[2k-1,2]_k[2n-2k,2]_{n-k}[2n-2k-1,2]_{n-k}[n+\frac{1}{2}]_{n+1}}{k!(n-k)![2n,2]_n[2n-1,2]_n(2k+1)}$$

$$= \frac{2}{[n+\frac{1}{2}]_n} \sum_{k=0}^{n} \binom{n}{k} \frac{2^k[k-\frac{1}{2}]_k 2^k 2^{n-k}[n-k-\frac{1}{2}]_{n-k} 2^{n-k}[n+\frac{1}{2}]_{n-k}(k+\frac{1}{2})[k-\frac{1}{2}]_k}{2^n[n-\frac{1}{2}]_n 2^n(2k+1)}$$

$$= \frac{1}{[n+\frac{1}{2}]_n[n-\frac{1}{2}]_n} \sum_{k=0}^{n} \binom{n}{k} [-\tfrac{1}{2}]_k(-1)^k[-\tfrac{1}{2}]_{n-k}(-1)^{n-k}[n+\tfrac{1}{2}]_{n-k}[-\tfrac{1}{2}]_k(-1)^k$$

$$= \frac{(-1)^n}{[n+\frac{1}{2}]_n[n-\frac{1}{2}]_n} \sum_{k=0}^{n} \binom{n}{k} [-\tfrac{1}{2}]_k^2[-\tfrac{1}{2}]_{n-k}[n+\tfrac{1}{2}]_{n-k}(-1)^k$$

$$= \frac{1}{[n+\frac{1}{2}]_n[n-\frac{1}{2}]_n}[-1]_n^2 = \frac{n!^2}{[n+\frac{1}{2}]_n[n-\frac{1}{2}]_n} = \frac{2^{2n}n!^2}{[2n+1,2]_n[2n-1,2]_n}$$

$$= \frac{2^{4n}n!^4}{[2n+1,2]_n[2n,2]_n^2[2n-1,2]_n} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$

eventually using the Pfaff-Schaalschütz formula, (9.1). to get rid of the sum.
This is obviously the right side suggested.
The references are of course to Summa Summarum.
Best Regards, Mogens.

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