YOUR STRANGE DOUBLE SUM

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October 24, 2007

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Dear Peter.

Computing your strange double sum:

$$\sum_{m=0}^{a} \frac{(-1)^m}{\binom{a}{m}} \left[\sum_{s=0}^{a} \binom{a-i}{s} \binom{i}{m-s} \binom{a-j}{a+r-i-j-s} \binom{j}{i+j+s-r-m} \right]$$

or, equivalently

$$\sum_{s=0}^{a} \binom{a-i}{s} \binom{a-j}{a+r-i-j-s} \left[\sum_{m=0}^{a} \frac{(-1)^m}{\binom{a}{m}} \binom{i}{m-s} \binom{j}{i+j+s-r-m} \right]$$

The second form may be written as

$$\frac{1}{a!} \sum_s \binom{a-j}{s+i-r} \frac{(a-i)!}{s!(a-i-s)!} \sum_m \binom{i}{m-s} \binom{j}{m-s+r-i} m!(a-m)!(-1)^m$$

We may change the variable in the inner sum to $m = k + s$ and get

$$\frac{(a-i)!}{a!} \sum_s \binom{a-j}{s+i-r} \frac{(-1)^s}{s!(a-i-s)!} \sum_k \binom{i}{k} \binom{j}{k+r-i} (s+k)!(a-s-k)!(-1)^k$$

Now, $\frac{(s+k)!}{s!} = [s+k]_k$ and $\frac{(a-s-k)!}{(a-i-s)!} = [a-s-k]_{i-k}$ so we may distribute $\binom{i}{k} = \frac{i!}{k!(i-k)!}$ to write

$$\frac{1}{\binom{a}{i}} \sum_s \binom{a-j}{s+i-r} (-1)^s \sum_k \binom{j}{k+r-i} \binom{s+k}{k} \binom{a-s-k}{i-k} (-1)^k$$

The inner sum is of type II(3,3,1) so it is proportional to its canonical form (5.8)

$$\sum_k \binom{i}{k} [-1-s]_k [i+j-r]_k [i-a+s-1]_{i-k} [r]_{i-k} (-1)^k$$

The factor may be taken as the fractions of the 0–terms by (5.9), i.e.,

$$\frac{\binom{j}{r-i}\binom{a-s}{i}}{[i-a+s-1]_i [r]_i} = \frac{[j]_{r-i}[a-s]_i}{[a-s]_i (-1)^i [r]_i (r-i)! i!} = \frac{[j]_{r-i}(-1)^i}{i! r!}$$

using (2.1). Fortunately, this factor is independent of s.

Hence the sum becomes

$$\frac{[j]_{r-i}(-1)^i}{i!r!\binom{a}{i}}\sum_s\binom{a-j}{s+i-r}(-1)^s\sum_k\binom{i}{k}[-1-s]_k[i+j-r]_k[i-a+s-1]_{i-k}[r]_{i-k}(-1)^k$$

The inner sum is still of type II(3,3,1), so we may apply the transformation (9.7) to write in stead

$$\frac{[j]_{r-i}(-1)^i}{i!r!\binom{a}{i}}\sum_s\binom{a-j}{s+i-r}(-1)^s\sum_k\binom{i}{k}[a+1]_k[r]_k[-1-s]_{i-k}[-1-j]_{i-k}(-1)^k$$

Changing the order of summation we get the inner sum

$$\sum_s\binom{a-j}{s+i-r}[-1-s]_{i-k}(-1)^s$$

this is the well known Chu–Vandermonde (8.9) so it equals

$$(-1)^{r+i}[i+j-a-r-1]_{i+j-a-k}[i-k]_{a-j}$$

As $[i-k]_{a-j}=0$ for $i-k<a-j$ or $i+j-a<k$, we have the zero for $i+j<a$.

If not, let $p=i+j-a\geq 0$. Then we get zero for $k>p$. So, we may write the whole sum as

$$\frac{[j]_{r-i}(-1)^r}{i!r!\binom{a}{i}}\sum_k\binom{i}{k}[a+1]_k[r]_k[-1-j]_{i-k}[i+j-a-r-1]_{i+j-a-k}[i-k]_{a-j}(-1)^k$$

Now, $\binom{i}{k}[i-k]_{a-j}=[i]_{i-p}\binom{p}{k}$ and $[-1-j]_{i-k}=[-1-j]_{i-p}[-1-j-i+p]_{p-k}$ by (2.8) and (2.2). So we may write it as

$$\frac{[j]_{r-i}(-1)^r[i]_{i-p}[-1-j]_{i-p}}{i!r!\binom{a}{i}}\sum_k\binom{p}{k}[a+1]_k[r]_k[p-r-1]_{p-k}[-1-a]_{p-k}(-1)^k$$

This sum is a Pfaff–Saalschütz sum as we have

$$a+1+r+p-r-1-1-a-p+1=0$$

So the sum becomes from (9.1)

$$[0]_p[a+p-r]_p$$

which result is zero for $p>0$. Hence the only nonzero result is the expression for $p=0$ (i.e., $i+j=a$). It becomes using (2.1) to get $[-1-j]_i=[i+j]_i(-1)^i$:

$$\frac{[j]_{r-i}(-1)^r[i]_i[-1-j]_{i-p}}{i!r!\binom{a}{i}}=\frac{(-1)^{r+i}[i+j]_i[j]_{r-i}}{r!\binom{a}{i}}=(-1)^{r+i}\frac{[a]_r}{r!\binom{a}{i}}=(-1)^{r+i}\frac{\binom{a}{r}}{\binom{a}{i}}$$

QED.

The references are of course to Summa Summarum.

I guess the proof is beautiful enough!

Best Regards, Mogens.