

THE MONTHLY PROBLEM 11164

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Solution to the problem 11164 from José Luis Díaz-Barrero in Amer. Math. Monthly 112,6 p. 568.

Proof of the formula

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \frac{1}{n^2}$$

Introducing harmonic numbers as

$$H_n = \sum_{k=1}^n \frac{1}{k}$$

I shall prefer to write the formula as

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{j=1}^k \frac{1}{j} H_j = \frac{1}{n^2}$$

Now, summation by parts yields this sum to equal

$$1 - \sum_{k=1}^n (-1)^{k+1} \binom{n-1}{k} \frac{1}{k+1} H_{k+1} = 1 - \frac{1}{n} \sum_{k=2}^{n+1} (-1)^k \binom{n}{k} H_k$$

As we may write the terms as a difference of the function

$$g(n, k) = (-1)^{k-1} \left(\binom{n-1}{k-1} H_k - \frac{1}{n} \binom{n-1}{k} \right)$$

that is

$$(-1)^k \binom{n}{k} H_k = g(n, k+1) - g(n, k)$$

the sum becomes telescoping to yield only the last term for $k = 2$:

$$1 + \frac{1}{n} g(n, 2) = 1 - \frac{1}{n} \left(\binom{n-1}{1} H_2 - \frac{1}{n} \binom{n-1}{2} \right) = \frac{1}{n^2}$$