Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Pure and Applied Algebra 205 (2006) 423–445
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Semi-dualizing modules and related Gorenstein homological dimensions

Henrik Holm[a,*], Peter Jørgensen[b]

[a]Department of Mathematical Sciences, University of Aarhus, Ny MunKegade, Building 530, DK-8000 Aarhus C, Denmark
[b]Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, UK

Received 28 June 2004; received in revised form 28 April 2005
Available online 1 September 2005
Communicated by I. Reiten

Abstract

A semi-dualizing module over a commutative noetherian ring A is a finitely generated module C with $\mathrm{RHom}_A(C, C) \simeq A$ in the derived category $\mathsf{D}(A)$.

We show how each such module gives rise to three new homological dimensions which we call C-Gorenstein projective, C-Gorenstein injective, and C-Gorenstein flat dimension, and investigate the properties of these dimensions.

MSC: 13D05; 13D07; 13D25; 18G10; 18G25

0. Introduction

It is by now a well-established fact that over any associative ring A, there exists a Gorenstein injective, Gorenstein projective and Gorenstein flat dimension defined for complexes of A-modules. These are usually denoted $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$, respectively. Some references are [2,4,10,14].

In this paper, we need to consider *semi-dualizing* A-modules C (see Definition 1.1), and in order to make things less technical, we only consider commutative and noetherian rings.

* Corresponding author.
E-mail addresses: holm@imf.au.dk (H. Holm), popjoerg@maths.leeds.ac.uk (P. Jørgensen)
URL: http://www.maths.leeds.ac.uk/~popjoerg (P. Jørgensen).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.07.010

For any semi-dualizing module (in fact, complex) C over A, and any complex Z with bounded and finitely generated homology, Christensen [3] introduced the dimension $G\text{-}\dim_C Z$, and developed a satisfactory theory for this new invariant.

If C is a semi-dualizing A-module and M is any A-complex, then we suggested in [12] the viewpoint that one should change rings from A to $A \ltimes C$ (the *trivial extension* of A by C; see Definition 1.2), and then consider the three "changed ring" Gorenstein dimensions:

$$\operatorname{Gid}_{A\ltimes C} M, \quad \operatorname{Gpd}_{A\ltimes C} M \quad \text{and} \quad \operatorname{Gfd}_{A\ltimes C} M.$$

The usefulness of this viewpoint was demonstrated as it enabled us to introduce three new *Cohen–Macaulay dimensions*, which characterize Cohen–Macaulay rings in a way one could hope for.

In this paper, we define for every semi-dualizing A-module C, three new Gorenstein dimensions:

$$C\text{-}\operatorname{Gid}_A(-), \quad C\text{-}\operatorname{Gpd}_A(-) \quad \text{and} \quad C\text{-}\operatorname{Gfd}_A(-),$$

which are called the *C-Gorenstein injective*, *C-Gorenstein projective* and *C-Gorenstein flat dimension*, respectively (see Definition 2.9).

It is worth pointing out that the, say, C-Gorenstein injective dimension is defined in terms of resolutions consisting of so-called *C-Gorenstein injective A-modules* (see Definition 2.7); and it does not involve a change of rings. The C-Gorenstein dimensions have at least five nice properties:

(1) For complexes with bounded and finitely generated homology, our $C\text{-}\operatorname{Gpd}_A(-)$ agrees with Christensen's $G\text{-}\dim_C(-)$ (Proposition 3.1).
(2) The three C-Gorenstein dimensions always agree with the "changed ring" dimensions $\operatorname{Gid}_{A\ltimes C}(-)$, $\operatorname{Gpd}_{A\ltimes C}(-)$ and $\operatorname{Gfd}_{A\ltimes C}(-)$, which were so important in [12] (Theorem 2.16).
(3) If $C = A$, the C-Gorenstein dimensions agree with the classical Gorenstein dimensions $\operatorname{Gid}_A(-)$, $\operatorname{Gpd}_A(-)$ and $\operatorname{Gfd}_A(-)$.

If A admits a dualizing complex D; cf. [4, Definition (1.1)], then finiteness of the C-Gorenstein dimensions can be interpreted in terms of *Auslander and Bass categories* (see Remark 4.1):

(4) If we define $C^\dagger = \operatorname{RHom}_A(C, D)$, then for all (appropriately homologically bounded) A-complexes M and N, we have the following implications (Theorem 4.6):

$$M \in \mathsf{A}_{C^\dagger}(A) \Leftrightarrow C\text{-}\operatorname{Gpd}_A M < \infty \Leftrightarrow C\text{-}\operatorname{Gfd}_A M < \infty,$$
$$N \in \mathsf{B}_{C^\dagger}(A) \Leftrightarrow C\text{-}\operatorname{Gid}_A N < \infty.$$

This generalizes the main results in [4, Theorems (4.3) and (4.5)].

Finally, each of the three C-Gorenstein dimensions has a related *proper* variant, giving us three additional dimensions (Definitions 5.2 and 5.3):

$$C\text{-}\mathsf{Gid}_A(-), \quad C\text{-}\mathsf{Gpd}_A(-) \quad \text{and} \quad C\text{-}\mathsf{Gfd}_A(-).$$

It turns out that the best one could hope for really happens, as we in Theorems 5.6, 5.8 and 5.11 prove:

(5) The proper C-Gorenstein dimensions (whenever these are defined) agree with the ordinary C-Gorenstein dimensions.

The paper is organized as follows:

In Section 1 we have collected some fundamental facts about the trivial extension $A \ltimes C$, which will be important later on. Section 2 defines the three new C-Gorenstein dimensions and proves how they are related to the "changed ring" Gorenstein dimensions over $A \ltimes C$. Section 3 compares our $C\text{-Gpd}_A(-)$ with Christensen's $G\text{-dim}_C(-)$. In Section 4 we interpret the C-Gorenstein dimensions in terms of Auslander and Bass categories. Finally, Section 5 investigates the proper C-Gorenstein dimensions.

Setup and notation. Throughout this paper, A is a fixed commutative and noetherian ring with unit, and C is a fixed semi-dualizing A-module; cf. Definition 1.1.

We work within the derived category $\mathsf{D}(A)$ of the category of A-modules; cf. e.g. [9, Chapter I; 15, Chapter 10]; and complexes $M \in \mathsf{D}(A)$ have differentials going to the right:

$$M = \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots .$$

We consistently use the hyper-homological notation from [2, Appendix], in particular we use $\mathrm{RHom}_A(-,-)$ for the right derived Hom functor, and $-\otimes^{\mathsf{L}}_A-$for the left derived tensor product functor.

1. A few results about the trivial extension

In this section we collect some fundamental results about the trivial extension, which will be important later on.

Definition 1.1. A finitely generated A-module C with $\mathrm{RHom}_A(C, C) \simeq A$ in $\mathsf{D}(A)$ is called *semi-dualizing* ($C = A$ is such an example).

Definition 1.2. If C is any A-module, then the direct sum $A \oplus C$ can be equipped with the product:

$$(a, c) \cdot (a', c') = (aa', ac' + a'c).$$

This turns $A \oplus C$ into a ring which is called the *trivial extension* of A by C and denoted $A \ltimes C$.

There are canonical ring homomorphisms, $A \rightleftarrows A \ltimes C$, which enable us to view A-modules as $(A \ltimes C)$-modules, and vice versa. This will be done frequently.

We import from [12, Lemma 3.2] the following facts about the interplay between the rings A and $A \ltimes C$:

Lemma 1.3. *Let A be a ring with a semi-dualizing module C.*

(1) *There is an isomorphism in* $\mathsf{D}(A \ltimes C)$:

$$\mathrm{RHom}_A(A \ltimes C, C) \cong A \ltimes C.$$

(2) *There is a natural equivalence of functors on* $\mathsf{D}(A)$:

$$\mathrm{RHom}_{A \ltimes C}(-, A \ltimes C) \simeq \mathrm{RHom}_A(-, C).$$

(3) *If M is in* $\mathsf{D}(A)$ *then the two biduality morphisms*:

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, C), C) \quad \textit{and}$$
$$M \longrightarrow \mathrm{RHom}_{A \ltimes C}(\mathrm{RHom}_{A \ltimes C}(M, A \ltimes C), A \ltimes C)$$

are equal.

(4) *There is an isomorphism in* $\mathsf{D}(A \ltimes C)$:

$$\mathrm{RHom}_{A \ltimes C}(A, A \ltimes C) \cong C.$$

Furthermore, we have the next result [12, Lemma 3.1] about injective modules over A and $A \ltimes C$:

Lemma 1.4. *The following two conclusions hold*:

(1) *If I is a (faithfully) injective A-module then* $\mathrm{Hom}_A(A \ltimes C, I)$ *is a (faithfully) injective* $(A \ltimes C)$*-module.*
(2) *Each injective* $(A \ltimes C)$*-module is a direct summand in a module* $\mathrm{Hom}_A(A \ltimes C, I)$ *where I is some injective A-module.*

Using the same methods, we obtain:

Lemma 1.5. *The following two conclusions hold*:

(1) *If P is a projective A-module then* $(A \ltimes C) \otimes_A P$ *is a projective* $(A \ltimes C)$*-module.*
(2) *Each projective* $(A \ltimes C)$*-module is a direct summand in a module* $(A \ltimes C) \otimes_A P$ *where P is some projective A-module.*

2. C-Gorenstein homological dimensions

Let M be an (appropriately homologically bounded) A-complex. In [12] we demonstrated the usefulness of changing rings from A to $A \ltimes C$, and then considering the "changed ring" Gorenstein dimensions:

$$\mathrm{Gid}_{A \ltimes C} M, \quad \mathrm{Gpd}_{A \ltimes C} M \quad \text{and} \quad \mathrm{Gfd}_{A \ltimes C} M.$$

This point of view enabled us to introduce three *Cohen–Macaulay dimensions* which characterize Cohen–Macaulay local rings in a way one could hope for. The next result is taken from [12, Lemma 4.6].

Proposition 2.1. *If E is a faithfully injective A-module, and M is any homologically right-bounded A-complex, then*:

$$\mathrm{Gid}_{A\ltimes C}\mathrm{Hom}_A(M, E) = \mathrm{Gfd}_{A\ltimes C}M.$$

Lemma 2.2. *Let J be an injective A-module and Q a projective A-module. Then we have a natural equivalence of functors on $\mathsf{D}(A\ltimes C)$*:

(1) $\mathrm{RHom}_{A\ltimes C}(\mathrm{Hom}_A(A\ltimes C, J), -) \simeq \mathrm{RHom}_A(\mathrm{Hom}_A(C, J), -)$.
(2) $\mathrm{RHom}_{A\ltimes C}(-, (A\ltimes C)\otimes_A Q) \simeq \mathrm{RHom}_A(-, C\otimes_A Q)$.

Proof. (1) is [12, Lemma 3.4], and (2) is proved similarly. □

Corollary 2.3. *For any A-module M, and integer n we have*:

(1) $\mathrm{Ext}^n_A(\mathrm{Hom}_A(C, J), M) = 0$ *for all injective A-modules J if and only if* $\mathrm{Ext}^n_{A\ltimes C}(U, M) = 0$ *for all injective $(A\ltimes C)$-modules U.*
(2) $\mathrm{Ext}^n_A(M, C\otimes_A P) = 0$ *for all projective A-modules P if and only if* $\mathrm{Ext}^n_{A\ltimes C}(M, S) = 0$ *for all projective $(A\ltimes C)$-modules S.*

Proof. (1) follows from Lemmas 2.2(1) and 1.4, while (2) is a consequence of Lemmas 2.2(2) and 1.5. □

We need to recall the next result from [12, Lemma 4.1]. Its proof uses, in fact, Lemmas 2.2(1) and 1.4.

Lemma 2.4. *Let M be an A-module which is Gorenstein injective over $A\ltimes C$. Then there exists a short exact sequence of A-modules*,

$$0 \to M' \longrightarrow \mathrm{Hom}_A(C, I) \longrightarrow M \to 0,$$

where I is injective over A and M' is Gorenstein injective over $A\ltimes C$. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ *for any injective A-module J.*

"Dualizing" the proof of Lemma 2.4; this time using Lemmas 2.2(2) and 1.5, we establish the next:

Lemma 2.5. *Let M be an A-module which is Gorenstein projective over $A\ltimes C$. Then there exists a short exact sequence of A-modules*,

$$0 \to M \longrightarrow C\otimes_A P \longrightarrow M' \to 0,$$

where P is projective over A and M' is Gorenstein projective over $A\ltimes C$. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_A(-, C\otimes_A Q)$ *for any projective A-module Q.*

The last result we will need to get started is [12, Lemma 3.3]:

Lemma 2.6. *The A-modules A and C are Gorenstein projective over $A \ltimes C$. If I is an injective A-module, then* $\mathrm{Hom}_A(A, I) \cong I$ *and* $\mathrm{Hom}_A(C, I)$ *are Gorenstein injective over* $A \ltimes C$.

Next, we introduce three new classes of modules:

Definition 2.7. An A-module M is called *C-Gorenstein injective* if:

(I1) $\mathrm{Ext}_A^{\geqslant 1}(\mathrm{Hom}_A(C, I), M) = 0$ for all injective A-modules I.
(I2) There exist injective A-modules $I_0, I_1, \ldots$ together with an exact sequence:

$$\cdots \to \mathrm{Hom}_A(C, I_1) \to \mathrm{Hom}_A(C, I_0) \to M \to 0,$$

and also, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ for any injective A-module J.

M is called *C-Gorenstein projective* if:

(P1) $\mathrm{Ext}_A^{\geqslant 1}(M, C \otimes_A P) = 0$ for all projective A-modules P.
(P2) There exist projective A-modules $P^0, P^1, \ldots$ together with an exact sequence:

$$0 \to M \to C \otimes_A P^0 \to C \otimes_A P^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(-, C \otimes_A Q)$ for any projective A-module Q.

Finally, M is called *C-Gorenstein flat* if:

(F1) $\mathrm{Tor}^A_{\geqslant 1}(\mathrm{Hom}_A(C, I), M) = 0$ for all injective A-modules I.
(F2) There exist flat A-modules $F^0, F^1, \ldots$ together with an exact sequence:

$$0 \to M \to C \otimes_A F^0 \to C \otimes_A F^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(C, I) \otimes_A -$ for any injective A-module I.

Example 2.8. (a) If I is an injective A-module, then $\mathrm{Hom}_A(C, I)$ and I are C-Gorenstein injective because:

It is easy to see that $\mathrm{Hom}_A(C, I)$ is C-Gorenstein injective. Concerning I itself it is clear that condition (I1) of Definition 2.7 is satisfied. From Lemma 2.6 it follows that I is Gorenstein injective over $A \ltimes C$, so iterating Lemma 2.4 we also get condition (I2).

(b) Similarly, if P is a projective A-module, then $C \otimes_A P$ and P are C-Gorenstein projective. The last claim uses Lemmas 2.6 and 2.5.

(c) If F is a flat A-module, then $C \otimes_A F$ and F are C-Gorenstein flat. The last claim uses (a) together with Propositions 2.1, 2.13(1), 2.15 (the last two can be found below).

Definition 2.9. By Example 2.8(a), there exists for every homologically left-bounded complex N a left-bounded complex Y of C-Gorenstein injective modules with $Y \simeq N$ in $\mathsf{D}(A)$

(as one could take Y to be an injective resolution of N). Every such Y is called a *C-Gorenstein injective resolution* of N.

C-Gorenstein projective and *C-Gorenstein flat resolutions* of homologically right-bounded complexes are defined in a similar way, and they always exist by Examples 2.8(b) and (c). Thus, we may define:

For any homologically left-bounded A-complex N we introduce:

$$C\text{-}\mathrm{Gid}_A\, N = \inf_Y \left(\sup\{n \in \mathbb{Z} \mid Y_{-n} \neq 0\}\right),$$

where the infimum is taken over all C-Gorenstein injective resolutions Y of N. For a homologically right-bounded A-complex M we define:

$$C\text{-}\mathrm{Gpd}_A\, M = \inf_X (\sup\{n \in \mathbb{Z} \mid X_n \neq 0\}),$$

where the infimum is taken over all C-Gorenstein projective resolutions X of M. Finally, we define $C\text{-}\mathrm{Gfd}_A\, M$ analogously to $C\text{-}\mathrm{Gpd}_A\, M$.

Observation 2.10. Note that when $C = A$ in Definition 2.7, we recover the categories of ordinary Gorenstein injective, Gorenstein projective, and Gorenstein flat A-modules.

Thus, $A\text{-}\mathrm{Gid}_A(-)$, $A\text{-}\mathrm{Gpd}_A(-)$, and $A\text{-}\mathrm{Gfd}_A(-)$ are the usual Gorenstein injective, Gorenstein projective and Gorenstein flat dimensions over A, which one usually denotes $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$, respectively.

Lemma 2.11. *Let M be an A-module which is C-Gorenstein injective. Then there exists a short exact sequence of $(A \ltimes C)$-modules,*

$$0 \to M' \longrightarrow U \longrightarrow M \to 0,$$

where U is injective over $A \ltimes C$ and M' is C-Gorenstein injective over A. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_{A \ltimes C}(V, -)$ for any injective $(A \ltimes C)$-module V.

Proof. Since M is C-Gorenstein injective, we in particular get a short exact sequence of A-modules:

$$0 \to N \longrightarrow \mathrm{Hom}_A(C, I) \longrightarrow M \to 0,$$

where I is injective and N is C-Gorenstein injective, which stays exact under $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ when J is injective. Applying the functor $\mathrm{Hom}_A(-, I)$ to the exact sequence:

$$0 \to C \longrightarrow A \ltimes C \longrightarrow A \to 0 \tag{$*$}$$

gives an exact sequence of $(A \ltimes C)$-modules:

$$0 \to I \longrightarrow \mathrm{Hom}_A(A \ltimes C, I) \longrightarrow \mathrm{Hom}_A(C, I) \to 0. \tag{$**$}$$

If viewed as a sequence of A-modules then this is split, because the same holds for $(*)$. Combining these data gives a commutative diagram of $(A \ltimes C)$-modules with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & \mathrm{Hom}_A(A \ltimes C, I) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & \mathrm{Hom}_A(C, I) \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

We will prove that the upper row here has the properties claimed in the lemma:

First, $\mathrm{Hom}_A(A \ltimes C, I)$ is an injective $(A \ltimes C)$-module by Lemma 1.4(1). Secondly, using the Snake Lemma on the diagram embeds the vertical arrows into exact sequences. The leftmost of these is:

$$0 \to I \longrightarrow M' \longrightarrow N \to 0,$$

proving that as A-modules, $M' \cong I \oplus N$. Here N is C-Gorenstein injective by construction, and I is by Example 2.8(a). So M' is clearly also C-Gorenstein injective.

Finally, by construction, the lower row in the diagram stays exact under $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ when J is injective. If viewed as a sequence of A-modules then the sequence $(**)$ is split, so the surjection $\mathrm{Hom}_A(A \ltimes C, I) \longrightarrow \mathrm{Hom}_A(C, I)$ is split, and therefore the upper row in the diagram also stays exact under $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$.

By applying $\mathrm{H}_0(-)$ to Lemma 2.2(1), we see that the upper row in the diagram stays exact under $\mathrm{Hom}_{A \ltimes C}(\mathrm{Hom}_A(A \ltimes C, J), -)$ when J is an injective A-module. Thus, it also stays exact under $\mathrm{Hom}_{A \ltimes C}(V, -)$ for any injective $(A \ltimes C)$-module V, because of Lemma 1.4(2). □

By a similar argument we get:

Lemma 2.12. *Let M be an A-module which is C-Gorenstein projective. Then there exists a short exact sequence of $(A \ltimes C)$-modules,*

$$0 \to M \longrightarrow R \longrightarrow M' \to 0,$$

where R is projective over $A \ltimes C$ and M' is C-Gorenstein projective over A. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_{A \ltimes C}(-, S)$ for any projective $(A \ltimes C)$-module S.

Proposition 2.13. *For any A-module M the two conclusions hold*:

(1) *M is C-Gorenstein injective if and only if M is Gorenstein injective over $A \ltimes C$.*
(2) *M is C-Gorenstein projective if and only if M is Gorenstein projective over $A \ltimes C$.*

Proof. (1) If M is C-Gorenstein injective, then Lemma 2.11 gives the "left half" of a complete injective resolution of M over $A \ltimes C$.

Conversely, if M is Gorenstein injective over $A \ltimes C$, then Lemma 2.4 gives the existence of a sequence like the one in Definition 2.7(I2). Now, to finish the proof we only need to refer to Corollary 2.3(1).

(2) Similar, but using Lemmas 2.12, 2.5 and Corollary 2.3(2). □

Before turning to C-Gorenstein flat modules, we need to recall the notion of *Kaplansky classes* from [8, Definition 2.1], which is reformulated in Definition 5.4, Section 5. The following lemma will be central:

Lemma 2.14. *The class* $\mathsf{F} = \{C \otimes_A F \mid F \text{ flat } A\text{-module}\}$ *is Kaplansky, and furthermore it is closed under direct limits.*

Proof. Every homomorphism $\varphi : C \otimes_A F_1 \to C \otimes_A F_2$, where F_i is flat, has the form $\varphi = C \otimes_A \psi$ for some homomorphism $\psi : F_1 \to F_2$; namely $\psi = \mathrm{Hom}_A(C, \varphi)$, because $\mathrm{Hom}_A(C, C \otimes_A F_i) \cong F_i$.

With this observation in mind it is clear that F is closed under direct limits, since the class of flat modules has this property.

To see that F is Kaplansky, we first note that a finitely generated A-module has cardinality at most $\kappa = \max\{|A|, \aleph_0\}$.

Now, assume that x is an element of $G = C \otimes_A F$, where F is a flat A-module. Write $x = \sum_{i=1}^{n} c_i \otimes x_i$ for some $c_1, \ldots, c_n \in C$ and $x_1, \ldots, x_n \in F$. Let S be the A-submodule of F generated by $x_1, \ldots, x_n$, and then use [16, Lemma 2.5.2] (or [6, Lemma 5.3.12]) to enlarge S to a pure submodule F' in F with cardinality:

$$|F'| \leqslant \max\{|S| \cdot |A|, \aleph_0\} \leqslant \kappa.$$

Since F is flat and $F' \subseteq F$ is a pure submodule, then F' and F/F' are flat as well. Furthermore, exactness of:

$$0 \to C \otimes_A F' \to C \otimes_A F \to C \otimes_A (F/F') \to 0$$

shows that $G' = C \otimes_A F'$ is a submodule of $G = C \otimes_A F$ which contains x. Clearly, G' and $G/G' \cong C \otimes_A (F/F')$ belong to F, and:

$$|G'| = |C \otimes_A F'| \leqslant |\mathbb{Z}^{(C \times F')}| \leqslant |(2^{\mathbb{Z}})^{(C \times F')}| = |2^{(\mathbb{Z} \times C \times F')}| \leqslant 2^{\kappa}.$$

The last inequality comes from the fact that all three cardinal numbers $|\mathbb{Z}|$, $|C|$ and $|F'|$ are less than κ. Note that the cardinal number 2^{κ} only depends on the ring A. $\square$

The next proof is modelled on that of [2, Theorem (6.4.2)].

Proposition 2.15. *Let M be an A-module. Then M is C-Gorenstein flat if and only if M is Gorenstein flat over $A \ltimes C$. In the affirmative case, M has the next property, which implies Definition* 2.7(F2):

(F2′) *There exist flat A-modules $F^0, F^1, \ldots$ together with an exact sequence*:

$$0 \to M \to C \otimes_A F^0 \to C \otimes_A F^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(-, C \otimes_A G)$ *for any flat A-module G.*

Proof. For the first statement, it suffices by Propositions 2.1 and 2.13(1) to show that if E is a faithfully injective A-module, then:

$$M \text{ is } C\text{-Gorenstein flat} \Leftrightarrow \mathrm{Hom}_A(M, E) \text{ is } C\text{-Gorenstein injective.}$$

For any injective A-module I we have (adjointness) isomorphisms:

$$\begin{aligned}&\mathrm{Ext}^i_A(\mathrm{Hom}_A(C, I), \mathrm{Hom}_A(M, E))\\ &\quad\cong \mathrm{Hom}_A(\mathrm{Tor}^A_i(\mathrm{Hom}_A(C, I), M), E).\end{aligned}$$

Thus, Definition 2.7(F1) for M is equivalent to (I1) for $\mathrm{Hom}_A(M, E)$. The rest of the proof will concern the conditions (F2) for M and (I2) for $\mathrm{Hom}_A(M, E)$ in Definition 2.7.

If $\mathbb{S} = 0 \to M \to C\otimes_A F^0 \to C\otimes_A F^1 \to \cdots$ is a sequence for M like the one in Definition 2.7(F2), then, using adjointness, it is easy to see that $\mathrm{Hom}_A(\mathbb{S}, E)$ is a sequence for $\mathrm{Hom}_A(M, E)$ like the one in (I2). Therefore, we have proved the implication "$\Rightarrow$".

To show "$\Leftarrow$", we assume that $\mathrm{Hom}_A(M, E)$ is C-Gorenstein injective. As already noted, we only have to show Definition 2.7(F2) for M. First note that (F2$'$) really implies Definition 2.7(F2), since:

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_A(C, I)\otimes_A -, E) &\simeq \mathrm{Hom}_A(-, \mathrm{Hom}_A(\mathrm{Hom}_A(C, I), E))\\ &\simeq \mathrm{Hom}_A(-, C\otimes_A \mathrm{Hom}_A(I, E)),\end{aligned}$$

and when I is injective, then $G = \mathrm{Hom}_A(I, E)$ is flat. In order prove (F2$'$), it suffices to show the existence of a short exact sequence:

$$0 \to M \to C\otimes_A F \to M' \to 0, \tag{$\dagger$}$$

satisfying the following three conditions:

(1) F is flat,
(2) $\mathrm{Hom}_A(M', E)$ is C-Gorenstein injective,
(3) $\mathrm{Hom}_A\ ((\dagger), C\otimes_A G)$ is exact for any flat A-module G.

Because then one obtains the sequence in (F2$'$) by iterating ($\dagger$). By Lemma 2.14, the class of A-modules:

$$\mathsf{F} = \{C\otimes_A F \mid F \text{ flat } A\text{-module}\}.$$

is Kaplansky. Furthermore, it is closed under arbitrary direct products; since C is finitely generated and A is noetherian, and hence [8, Theorem 2.5] implies that every A-module has an F-preenvelope.

Note that since $\mathrm{Hom}_A(M, E)$ is C-Gorenstein injective, there in particular exists an epimorphism $\mathrm{Hom}_A(C, I) \twoheadrightarrow \mathrm{Hom}_A(M, E)$, where I is injective. Applying $\mathrm{Hom}_A(-, E)$, we get a monomorphism:

$$\begin{aligned}M &\hookrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M, E), E)\\ &\hookrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(C, I), E) \cong C\otimes_A \mathrm{Hom}_A(I, E) \in \mathsf{F}.\end{aligned}$$

Thus, M can be embedded into a module from F. Therefore, taking an F-preenvelope $\varphi : M \to C \otimes_A F$ of M, it is automatically injective; and defining $M' = \operatorname{Coker} \varphi$, we certainly get an exact sequence (†) satisfying (1) and (3).

Finally, we argue that (2) is true. Keeping Proposition 2.13(1) in mind we must prove that $\operatorname{Hom}_A(M', E)$ is Gorenstein injective over $A \ltimes C$. Applying $\operatorname{Hom}_A(-, E)$ to (†) we get:

$$0 \to \operatorname{Hom}_A(M', E) \to \operatorname{Hom}_A(C, J) \to \operatorname{Hom}_A(M, E) \to 0, \tag{‡}$$

where $J \cong \operatorname{Hom}_A(F, E)$ is injective. $\operatorname{Hom}_A(C, J)$ and $\operatorname{Hom}_A(M, E)$ are both Gorenstein injective over $A \ltimes C$—the last module by assumption. Hence, if we can prove that $\operatorname{Ext}^1_{A \ltimes C}(U, \operatorname{Hom}_A(M', E)) = 0$ for every injective $(A \ltimes C)$-module U, then [5, Theorem 2.13] gives the desired conclusion. Using Corollary 2.3(1), we must prove that:

$$\operatorname{Ext}^1_A(\operatorname{Hom}_A(C, I), \operatorname{Hom}_A(M', E)) = 0 \tag{♮}$$

for all injective A-modules I. Consider the commutative diagram with exact columns:

$$\begin{tikzcd}
0 & \\
\operatorname{Ext}^1_A(\operatorname{Hom}_A(C, I), \operatorname{Hom}_A(M', E)) \arrow[u] & 0 \\
\operatorname{Hom}_A(\operatorname{Hom}_A(C, I), \operatorname{Hom}_A(M, E)) \arrow[u] & \operatorname{Hom}_A(\operatorname{Hom}_A(C, I) \otimes_A M, E) \arrow[l, "\cong"'] \arrow[u] \\
\operatorname{Hom}_A(\operatorname{Hom}_A(C, I), \operatorname{Hom}_A(C, J)) \arrow[u] & \operatorname{Hom}_A(\operatorname{Hom}_A(C, I) \otimes_A (C \otimes_A F), E) \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd}$$

The first column is the induced long exact sequence which comes from applying $\operatorname{Hom}_A(\operatorname{Hom}_A(C, I), -)$ to (‡). We get another monomorphism when we apply $\operatorname{Hom}_A(C, I) \otimes_A -$ to the one $0 \to M \to C \otimes_A F$ from (†); this follows from property (3) which (†) satisfies together with the calculation preceding (†). Turning this into an epimorphism with $\operatorname{Hom}_A(-, E)$ we get the second column. The vertical isomorphisms are by adjointness. The diagram implies that the module in (♮) is zero. □

Theorem 2.16. *Let N be a homologically left-bounded complex, and M a homologically right-bounded complex of A-modules. Then we have the following equalities*:

$$C\text{-}\operatorname{Gid}_A N = \operatorname{Gid}_{A \ltimes C} N,$$
$$C\text{-}\operatorname{Gpd}_A M = \operatorname{Gpd}_{A \ltimes C} M,$$
$$C\text{-}\operatorname{Gfd}_A M = \operatorname{Gfd}_{A \ltimes C} M.$$

Proof. The proof uses Propositions 2.13(1), (2) and 2.15 in combination with [4, Theorems (2.5), (2.2) and (2.8)]. We only prove that $C\text{-}\operatorname{Gid}_A N = \operatorname{Gid}_{A \ltimes C} N$, since the proofs of the other two equalities are similar:

From Proposition 2.13(1) we get that every C-Gorenstein injective A-module is also Gorenstein injective over $A \ltimes C$, and this give us the inequality "$\geqslant$".

For the opposite inequality "$\leqslant$", we may assume that $n = \mathrm{Gid}_{A \ltimes C} N$ is finite. Pick a left-bounded complex I of injective A-modules such that $I \simeq N$ in $\mathsf{D}(A)$. By Lemma 2.6 the modules I_i are Gorenstein injective over $A \ltimes C$, and therefore [4, Theorem (2.5)] implies that the A-module Z^{I}_{-n} is Gorenstein injective over $A \ltimes C$.

Now, Proposition 2.13(1) shows that Z^{I}_{-n} is C-Gorenstein injective. By Example 2.8(a), the complex $I_{-n}\supset = \cdots \to I_{-n+1} \to \mathrm{Z}^{I}_{-n} \to 0$ consists of C-Gorenstein injective A-modules, and since $I_{-n}\supset \simeq I \simeq N$ we see that $C\text{-}\mathrm{Gid}_A N \leqslant n$. □

Corollary 2.17. *For any A-complexes N and M, homologically bounded as in Theorem* 2.16, *we have the following equalities*:

$$
\begin{aligned}
&\mathrm{Gid}_{A \ltimes A} N = \mathrm{Gid}_{A[x]/(x^2)} N = \mathrm{Gid}_A N,\\
&\mathrm{Gpd}_{A \ltimes A} M = \mathrm{Gpd}_{A[x]/(x^2)} M = \mathrm{Gpd}_A M,\\
&\mathrm{Gfd}_{A \ltimes A} M = \mathrm{Gfd}_{A[x]/(x^2)} M = \mathrm{Gfd}_A M.
\end{aligned}
$$

Proof. This follows immediately from Theorem 2.16; we only have to note that $A \ltimes A \cong A[x]/(x^2)$ (sometimes referred to as the *dual numbers* over A). □

Having realized that, on the level of A-complexes, the three (classical) Gorenstein dimensions cannot distinguish between A and $A \ltimes A$, we can reap a nice result from the work of [12]:

Theorem 2.18. *If $(A, \mathfrak{m}, k)$ is local, then the following conditions are equivalent*:

(1) *A is Gorenstein.*
(2) *There exists a homologically bounded A-complex M such that all three numbers* $\mathrm{fd}_A M$, $\mathrm{Gid}_A M$ *and* $\mathrm{width}_A M$ *are finite.*
(3) *There exists a homologically bounded A-complex N such that all three numbers* $\mathrm{id}_A N$, $\mathrm{Gpd}_A N$ *and* $\mathrm{depth}_A N$ *are finite.*
(4) *There exists a homologically bounded A-complex N such that all three numbers* $\mathrm{id}_A N$, $\mathrm{Gfd}_A N$ *and* $\mathrm{depth}_A N$ *are finite.*

Proof. It is well-known that over a Gorenstein ring, every homologically bounded complex has finite Gorenstein injective, Gorenstein projective and Gorenstein flat dimension, and thus (1) $\Rightarrow$ (2), (3), (4).

Of course, (3) $\Rightarrow$ (4); and using Corollary 2.17, the remaining implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) follow immediately from [12, Propositions 4.5 and 4.7]. □

Remark 2.19. There already exist special cases of this result in the literature: If A admits a dualizing complex, then [2, (3.3.5)] compared with [4, Theorems (4.3) and (4.5)] gives Theorem 2.18. If one drops the assumption that a dualizing complex should exist, then Theorem 2.18 is proved in [11, Corollary (3.3)], but only for modules.

3. Comparison with Christensen's G-$\dim_C(-)$

In [3, Definition (3.11)], Christensen introduced the number $G\text{-}\dim_C Z$ for any semi-dualizing complex C, and any complex Z with bounded and finitely generated homology. When $C = A$ (and Z is a module), we recover Auslander–Bridger's G-dimension by [2, Theorem (2.2.3)].

Proposition 3.1. *If C is a semi-dualizing A-module, and M an A-complex with bounded and finitely generated homology, then*:

$$C\text{-}\mathrm{Gpd}_A\, M = G\text{-}\dim_C M.$$

Proof. By Theorem 2.16, the proposition amounts to:

$$\mathrm{Gpd}_{A\ltimes C}\, M = G\text{-}\dim_C M. \tag{$*$}$$

The homology of M is bounded and finitely generated over A, and hence it is also bounded and finitely generated over $A\ltimes C$. So by e.g. [4, Theorem (2.12)(b); 2, Theorem (4.2.6)], the left hand side in $(*)$ equals $G\text{-}\dim_{A\ltimes C} M$ (Auslander–Bridger's G-dimension over the ring $A\ltimes C$). We must therefore prove that:

$$G\text{-}\dim_{A\ltimes C} M = G\text{-}\dim_C M. \tag{$**$}$$

The left hand side is finite precisely if the biduality morphism:

$$M \longrightarrow \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_{A\ltimes C}(M, A\ltimes C), A\ltimes C)$$

is an isomorphism, and the right hand side is finite precisely when

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, C), C)$$

is an isomorphism. But these two morphisms are equal by Lemma 1.3(3), so the left hand side and right hand side of $(**)$ are simultaneously finite. When the left hand side of $(**)$ is finite, it equals:

$$-\inf\ \mathrm{RHom}_{A\ltimes C}(M, A\ltimes C),$$

and when the right hand side is finite, it is equal to:

$$-\inf\ \mathrm{RHom}_A(M, C).$$

But these two numbers are equal by Lemma 1.3(2). □

Observation 3.2. Christensen's $G\text{-}\dim_C(-)$ only works when the argument has bounded and finitely generated homology, but it has the advantage that C is allowed to be a semi-dualizing *complex*.

By Theorem 2.16, we get that for A-complexes M, the C-Gorenstein projective dimension $C\text{-}\mathrm{Gpd}_A\, M$ agrees with the "changed ring" Gorenstein projective dimension $\mathrm{Gpd}_{A\ltimes C}\, M$.

It is not immediately clear how one should make either of these dimensions work when C is a semi-dualizing *complex*. Because in this case, $A\ltimes C$ becomes a differential graded algebra, and the C-Gorenstein projective objects in Definition 2.7 (from which we build our resolutions) become complexes.

In [1, P. 28] we find an interesting comment, which makes it even more clear why we run into trouble when C is a complex:

"*On the other hand, let C be a semi-dualizing complex with* amp $C = s > 0$. *We are free to assume that* inf $C = 0$, *and it is then immediate from the definition that* G-$\dim_C C = 0$; *but a resolution of C must have length at least s, so the G-dimension with respect to C cannot be interpreted in terms of resolutions*."

It is notable that the number $\operatorname{Gpd}_A \operatorname{RHom}_A(C, N)$, $N \in \mathsf{B}_C(A)$, occuring in Theorem 4.3 makes perfect sense even if C is a complex.

4. Interpretations via Auslander and Bass categories

In this section, we interpret the C-Gorenstein homological dimensions from Section 2 in terms of the Auslander and Bass categories.

Remark 4.1. Let C be a semi-dualizing A-complex. In [3, Section 4] is considered the adjoint pair of functors:

$$\mathsf{D}(A) \underset{\operatorname{RHom}_A(C,-)}{\overset{C\otimes^{\mathsf{L}}_A -}{\rightleftarrows}} \mathsf{D}(A)$$

and the full subcategories (where $\mathsf{D}_{\mathrm{b}}(A)$ is the full subcategory of $\mathsf{D}(A)$ consisting of homologically bounded complexes):

$$\mathsf{A}_C(A) = \left\{ M \in \mathsf{D}(A) \,\middle|\, \begin{array}{l} M \text{ and } C\otimes^{\mathsf{L}}_A M \text{ are in } \mathsf{D}_{\mathrm{b}}(A) \text{ and} \\ M \to \operatorname{RHom}_A(C, C\otimes^{\mathsf{L}}_A M) \text{ is an isomorphism} \end{array} \right\}$$

and

$$\mathsf{B}_C(A) = \left\{ N \in \mathsf{D}(A) \,\middle|\, \begin{array}{l} N \text{ and } \operatorname{RHom}_A(C, N) \text{ are in } \mathsf{D}_{\mathrm{b}}(A) \text{ and} \\ C\otimes^{\mathsf{L}}_A \operatorname{RHom}_A(C, N) \to N \text{ is an isomorphism} \end{array} \right\}.$$

It is an exercise in adjoint functors that the adjoint pair above restricts to a pair of quasi-inverse equivalences of categories:

$$\mathsf{A}_C(A) \underset{\operatorname{RHom}_A(C,-)}{\overset{C\otimes^{\mathsf{L}}_A -}{\rightleftarrows}} \mathsf{B}_C(A).$$

Theorem 4.2. *For any complex* $M \in \mathsf{A}_C(A)$ *we have an equality*:

$$C\text{-}\operatorname{Gid}_A M = \operatorname{Gid}_A(C\otimes^{\mathsf{L}}_A M).$$

Proof. Throughout the proof we make use of the nice descriptions of the *modules* in $\mathsf{A}_C(A)$ and $\mathsf{B}_C(A)$ from [3, Observation (4.10)].

Step 1: In order to prove the equality $C\text{-Gid}_A M = \text{Gid}_A(C \otimes^{\mathrm{L}}_A M)$, we first justify the (necessary) bi-implication:

$$M \text{ is } C\text{-Gorenstein injective} \iff$$
$$C \otimes_A M \text{ is Gorenstein injective} \tag{$\natural$}$$

for any *module* $M \in \mathsf{A}_C(A)$.

"⇒": By Definition 2.7(I2) there is an exact sequence:

$$\cdots \to \text{Hom}_A(C, I_1) \to \text{Hom}_A(C, I_0) \to M \to 0, \tag{$*$}$$

where $I_0, I_1, \ldots$ are injective A-modules. Furthermore, we have exactness of $\text{Hom}_A(\text{Hom}_A(C, J), (*))$ for all injective A-modules J.

M belongs to $\mathsf{A}_C(A)$, and so does $\text{Hom}_A(C, I)$ for any injective A-module I, since $I \in \mathsf{B}_C(A)$ by [3, Proposition (4.4)]. In particular, C is Tor-independent with both of the modules M and $\text{Hom}_A(C, I)$ (two A-modules U and V are Tor-independent if $\text{Tor}^A_{\geqslant 1}(U, V) = 0$). Hence the sequence $(*)$ stays exact if we apply to it the functor $C \otimes_A -$, and doing so we obtain:

$$\cdots \to I_1 \to I_0 \to C \otimes_A M \to 0. \tag{$**$}$$

By similar arguments we see that if we apply $\text{Hom}_A(C, -)$ to the sequence $(**)$, then we get $(*)$ back. If J is any injective A-module, then we have exactness of $\text{Hom}_A(J, (**))$ because:

$$\begin{aligned} \text{Hom}_A(J, (**)) &\cong \text{Hom}_A(C \otimes_A \text{Hom}_A(C, J), (**)) \\ &\cong \text{Hom}_A(\text{Hom}_A(C, J), \text{Hom}_A(C, (**))) \\ &\cong \text{Hom}_A(\text{Hom}_A(C, J), (*)). \end{aligned}$$

Thus, $(**)$ is a "left half" of a complete injective resolution of the A-module $C \otimes_A M$. We also claim that $\text{Ext}^i_A(J, C \otimes_A M) = 0$ for all $i > 0$ and all injective A-modules J. First note that:

$$\begin{aligned} \text{Ext}^i_A(J, C \otimes_A M) &\overset{\text{(a)}}{\cong} \mathrm{H}^i\, \mathrm{RHom}_A(C \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C, J), C \otimes^{\mathrm{L}}_A M) \\ &\overset{\text{(b)}}{\cong} \mathrm{H}^i\, \mathrm{RHom}_A\, (\mathrm{RHom}_A(C, J)\, , \mathrm{RHom}_A(C, C \otimes^{\mathrm{L}}_A M)) \\ &\overset{\text{(c)}}{\cong} \mathrm{H}^i\, \mathrm{RHom}_A(\mathrm{RHom}_A(C, J), M) \\ &\cong \text{Ext}^i_A(\text{Hom}_A(C, J), M). \end{aligned} \tag{$\diamond$}$$

Here (a) follows as $J \in \mathsf{B}_C(A)$ by [3, Proposition (4.4)]; (b) is by adjointness; and (c) is because $M \in \mathsf{A}_C(A)$. This last module is zero because M is C-Gorenstein injective. These considerations prove that $C \otimes_A M$ is Gorenstein injective over A.

"⇐": If $C \otimes_A M$ is Gorenstein injective over A, we have by definition an exact sequence:

$$\cdots \to I_1 \to I_0 \to C \otimes_A M \to 0, \tag{$\dagger$}$$

where $I_0, I_1, \ldots$ are injective A-modules. Furthermore, we have exactness of $\text{Hom}_A(J, (\dagger))$ for all injective A-modules J.

Since $I_0, I_1, \ldots$ and $C \otimes_A M$ are modules from $\mathsf{B}_C(A)$, then so are all the kernels in (†), as $\mathsf{B}_C(A)$ is a triangulated subcategory of $\mathsf{D}(A)$. If $N \in \mathsf{B}_C(A)$, then $\mathrm{Ext}_A^{\geqslant 1}(C, N) = 0$, and consequently, the sequence (†) stays exact if we apply to it the functor $\mathrm{Hom}_A(C, -)$. Doing so we obtain:

$$\cdots \to \mathrm{Hom}_A(C, I_1) \to \mathrm{Hom}_A(C, I_0) \to M \to 0. \tag{‡}$$

If J is any injective A-module, then we have exactness of the complex $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), (‡))$ because:

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), (‡)) &\cong \mathrm{Hom}_A(\mathrm{Hom}_A(C, J), \mathrm{Hom}_A(C, (†)))\\ &\cong \mathrm{Hom}_A(C \otimes_A \mathrm{Hom}_A(C, J), (†))\\ &\cong \mathrm{Hom}_A(J, (†)).\end{aligned}$$

Furthermore, $(\diamond)$ gives that:

$$\mathrm{Ext}_A^{\geqslant 1}(\mathrm{Hom}_A(C, J), M) \cong \mathrm{Ext}_A^{\geqslant 1}(J, C \otimes_A M) = 0,$$

for all injective A-modules J. The last zero is because $C \otimes_A M$ is Gorenstein injective over A. Hence M is C-Gorenstein injective.

Step 2: To prove the inequality $C\text{-}\mathrm{Gid}_A\, M \geqslant \mathrm{Gid}_A(C \otimes_A^{\mathrm{L}} M)$ for any complex $M \in \mathsf{A}_C(A)$, we may assume that $m = C\text{-}\mathrm{Gid}_A\, M = \mathrm{Gid}_{A \ltimes C}\, M$ (cf. Theorem 2.16) is finite. Since $C \otimes_A^{\mathrm{L}} M$ is homologically bounded, there exists a left-bounded injective resolution I of $C \otimes_A^{\mathrm{L}} M$, that is, $I \simeq C \otimes_A^{\mathrm{L}} M$ in $\mathsf{D}(A)$.

We wish to prove that the A-module Z_{-m}^{I} is Gorenstein injective. Since M belongs to $\mathsf{A}_C(A)$, we get isomorphisms:

$$M \simeq \mathrm{RHom}_A(C, C \otimes_A^{\mathrm{L}} M) \simeq \mathrm{RHom}_A(C, I) \simeq \mathrm{Hom}_A(C, I).$$

Now, $\mathrm{Hom}_A(C, I)$ is a complex of Gorenstein injective $A \ltimes C$-modules, and thus the A-module $L := \mathrm{Z}_{-m}^{\mathrm{Hom}_A(C,I)}$ is Gorenstein injective over $A \ltimes C$ by [4, Theorem (2.5)]. By Proposition 2.13(1), L is also C-Gorenstein injective. Note that:

$$-m = -\mathrm{Gid}_{A \ltimes C}\, M \leqslant \inf\, M \overset{(a)}{=} \inf(C \otimes_A^{\mathrm{L}} M) = \inf\, I,$$

where the equality (a) comes from [3, Lemma(4.11)(b)]. Therefore, $0 \to \mathrm{Z}_{-m}^{I} \to I_{-m} \to I_{-m-1}$ is exact, and applying the left exact functor $\mathrm{Hom}_A(C, -)$ to this sequence we get an isomorphism of A-modules:

$$L = \mathrm{Z}_{-m}^{\mathrm{Hom}_A(C,I)} \cong \mathrm{Hom}_A(C, \mathrm{Z}_{-m}^{I}). \tag{$\flat$}$$

We have a degreewise split exact sequence of complexes:

$$0 \to \Sigma^{-m}\mathrm{Z}_{-m}^{I} \longrightarrow I_{-m} \supset \longrightarrow I_{-m+1} \sqsupset \longrightarrow 0,$$

where we have used the notation from [2, Appendix (A.1.14)] to denote soft and hard truncations. Since $I_{-m+1} \sqsupset$ has finite injective dimension it belongs to $\mathsf{B}_C(A)$ by [3, Proposition (4.4)], and furthermore,

$$I_{-m} \supset \simeq I \simeq C \otimes_A^{\mathrm{L}} M \in \mathsf{B}_C(A).$$

Thus, the module Z^I_{-m} is also in $\mathsf{B}_C(A)$, as $\mathsf{B}_C(A)$ is a triangulated subcategory of $\mathsf{D}(A)$. Consequently, the module L from ($\flat$) belongs to $\mathsf{A}_C(A)$ and has the property that $C \otimes_A L \cong \mathsf{Z}^I_{-m}$. Therefore, the implication "$\Rightarrow$" in ($\natural$) gives that Z^I_{-m} is Gorenstein injective over A, as desired.

Step 3: To prove the opposite inequality $C\text{-Gid}_A\, M \leqslant \text{Gid}_A(C \otimes^{\mathrm{L}}_A M)$ for any complex $M \in \mathsf{A}_C(A)$, we assume that $n = \text{Gid}_A(C \otimes^{\mathrm{L}}_A M)$ is finite. Pick any left-bounded injective resolution I of $C \otimes^{\mathrm{L}}_A M$. Then the A-module Z^I_{-n} is Gorenstein injective by [4, Theorem (2.5)].

As in Step 2 we get $M \simeq \text{Hom}_A(C, I)$, and thus it suffices to show that the module:

$$N := \mathsf{Z}^{\text{Hom}_A(C,I)}_{-n} \cong \text{Hom}_A(C, \mathsf{Z}^I_{-n})$$

is C-Gorenstein injective, because then $M \simeq \text{Hom}_A(C, I)_{-n}\supset$ shows that $C\text{-Gid}_A\, M \leqslant n$. As before we get that N is a module in $\mathsf{A}_C(A)$ with $C \otimes_A N \cong \mathsf{Z}^I_{-n}$, which this time is Gorenstein injective over A. Therefore, the implication "$\Leftarrow$" in ($\natural$) gives that N is C-Gorenstein injective. $\square$

Using Proposition 2.13(2), a similar argument gives:

Theorem 4.3. *For any complex $N \in \mathsf{B}_C(A)$ we have an equality*:

$$C\text{-Gpd}_A\, N = \text{Gpd}_A\, \text{RHom}_A(C, N).$$

From Theorems 4.2 and 2.16, and Proposition 2.1 we can easily derive:

Theorem 4.4. *For any complex $N \in \mathsf{B}_C(A)$ we have an equality*:

$$C\text{-Gfd}_A\, N = \text{Gfd}_A\, \text{RHom}_A(C, N).$$

Proof. Let E be a faithfully injective A-module. Since $N \in \mathsf{B}_C(A)$ it is easy to see that $\text{RHom}_A(N, E) \simeq \text{Hom}_A(N, E)$ is in $\mathsf{A}_C(A)$. Hence

$$\begin{aligned} C\text{-Gfd}_A N &\overset{(a)}{=} C\text{-Gid}_A\, \text{RHom}_A(N, E) \\ &\overset{(b)}{=} \text{Gid}_A(C \otimes^{\mathrm{L}}_A \text{RHom}_A(N, E)) \\ &\overset{(c)}{=} \text{Gid}_A\, \text{RHom}_A(\text{RHom}_A(C, N), E) \\ &\overset{(d)}{=} \text{Gfd}_A\, \text{RHom}_A(C, N). \end{aligned}$$

Here (a) is by Proposition 2.1 and Theorem 2.16; (b) is by Theorem 4.2; (c) is by the isomorphism [2, (A.4.24)]; and finally, (d) is by Proposition 2.1 and Corollary 2.17. $\square$

In the rest of this section, we assume that A admits a *dualizing complex* D^A; cf. [4, Definition (1.1)]. The canonical homomorphism of rings, $A \to A \ltimes C$, turns $A \ltimes C$ into a finitely generated A-module, and thus

$$D^{A \ltimes C} = \text{RHom}_A(A \ltimes C, D^A)$$

is a dualizing complex for $A \ltimes C$.

Lemma 4.5. *There is an isomorphism over A,*

$$D^{A\ltimes C} \otimes^{\mathrm{L}}_{A\ltimes C} A \cong \mathrm{RHom}_A(C, D^A).$$

Proof. This is a computation:

$$\begin{aligned} D^{A\ltimes C} \otimes^{\mathrm{L}}_{A\ltimes C} A &= \mathrm{RHom}_A(A\ltimes C, D^A) \otimes^{\mathrm{L}}_{A\ltimes C} A \\ &\overset{(a)}{\cong} \mathrm{RHom}_A(\mathrm{RHom}_{A\ltimes C}(A, A\ltimes C), D^A) \\ &\overset{(b)}{\cong} \mathrm{RHom}_A(C, D^A), \end{aligned}$$

where (a) holds because D^A has finite injective dimension over A and (b) is by Lemma 1.3(4). □

By [3, Corollary (2.12)], the complex $C^\dagger = \mathrm{RHom}_A(C, D^A)$ is semi-dualizing for A. We now have the following generalization of the main results in [4, Theorems (4.3) and (4.5)]:

Theorem 4.6. *Let M and N be A-complexes such that the homology of M is right-bounded and the homology of N is left-bounded. Then*:

(1) $M \in \mathsf{A}_{C^\dagger}(A) \iff C\text{-}\mathrm{Gpd}_A\, M < \infty \iff C\text{-}\mathrm{Gfd}_A\, M < \infty$.
(2) $N \in \mathsf{B}_{C^\dagger}(A) \iff C\text{-}\mathrm{Gid}_A\, N < \infty$.

Proof. Recall that $D^{A\ltimes C} = \mathrm{RHom}_A(A\ltimes C, D^A)$ is a dualizing complex for $A\ltimes C$. If M is a complex of A-modules then

$$\begin{aligned} C^\dagger \otimes^{\mathrm{L}}_A M &= \mathrm{RHom}_A(C, D^A) \otimes^{\mathrm{L}}_A M \\ &\overset{(a)}{\cong} \left(D^{A\ltimes C} \otimes^{\mathrm{L}}_{A\ltimes C} A\right) \otimes^{\mathrm{L}}_A M \\ &\cong D^{A\ltimes C} \otimes^{\mathrm{L}}_{A\ltimes C} M \end{aligned}$$

and

$$\begin{aligned} \mathrm{RHom}_A(C^\dagger, M) &= \mathrm{RHom}_A(\mathrm{RHom}_A(C, D^A), M) \\ &\overset{(b)}{\cong} \mathrm{RHom}_A(D^{A\ltimes C} \otimes^{\mathrm{L}}_{A\ltimes C} A, M) \\ &\overset{(c)}{\cong} \mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C}, \mathrm{RHom}_A(A, M)) \\ &\cong \mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C}, M), \end{aligned}$$

where (a) and (b) are by Lemma 4.5 and (c) is by adjunction. So using the adjoint pair:

$$\mathsf{D}(A) \underset{\mathrm{RHom}_A(C^\dagger,-)}{\overset{C^\dagger \otimes^{\mathrm{L}}_A -}{\rightleftarrows}} \mathsf{D}(A)$$

on complexes of A-modules is the same as viewing these complexes as complexes of $(A\ltimes C)$-modules and using the adjoint pair:

$$\mathsf{D}(A \ltimes C) \underset{\mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C},-)}{\overset{D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}-}{\rightleftarrows}} \mathsf{D}(A \ltimes C)$$

Hence a complex M of A-modules is in $\mathsf{A}_{C^\dagger}(A)$ if and only if it is in $\mathsf{A}_{D^{A\ltimes C}}(A\ltimes C)$ when viewed as a complex of $(A\ltimes C)$-modules. If M has right-bounded homology, this is equivalent both to $\mathrm{Gpd}_{A\ltimes C} M < \infty$ and $\mathrm{Gfd}_{A\ltimes C} M < \infty$ by [4, Theorem (4.3)], and by Theorem 2.16 this is the same as C-$\mathrm{Gpd}_A M < \infty$ and C-$\mathrm{Gfd}_A M < \infty$.

So part (1) of the theorem follows, and a similar method using [4, Theorem (4.5)] deals with part (2). $\square$

5. Proper dimensions

In this section, we define and study the *proper* variants of the dimensions from Theorem 2.16. The results to follow depend strongly on the work in [8].

In Definition 2.9 we introduced the dimensions C-$\mathrm{Gid}_A(-)$, C-$\mathrm{Gpd}_A(-)$ and C-$\mathrm{Gfd}_A(-)$ for A-complexes. When M is an A-module it is not hard to see that these dimensions specialize to:

$$C\text{-}\mathrm{Gid}_A M = \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to I^0 \to \cdots \to I^n \to 0 \text{ is exact} \\ \text{and } I^0, \ldots, I^n \text{ are } C\text{-Gorenstein injective} \end{array} \right\},$$

and similarly for C-$\mathrm{Gpd}_A M$ and C-$\mathrm{Gfd}_A M$.

Definition 5.1. Let Q be a class of A-modules (which contains the zero-module), and let M be any A-module. A *proper left* Q*-resolution* of M is a complex (not necessarily exact):

$$\cdots \to Q_1 \to Q_0 \to M \to 0, \tag{‡}$$

where $Q_0, Q_1, \ldots \in \mathsf{Q}$ and such that (†) becomes exact when we apply to it the functor $\mathrm{Hom}_A(Q, -)$ for every $Q \in \mathsf{Q}$. A *proper right* Q*-resolution* of M is a complex (not necessarily exact):

$$0 \to M \to Q^0 \to Q^1 \to \cdots, \tag{‡}$$

where $Q^0, Q^1, \ldots \in \mathsf{Q}$ and such that (‡) becomes exact when we apply the functor $\mathrm{Hom}_A(-, Q)$ to it for every $Q \in \mathsf{Q}$.

Definition 5.2. Let Q be a class of A-modules, and let M be any A-module. If M has a proper left Q-resolution, then we define the *proper left* Q*-dimension* of M by

$$\mathscr{L}\text{-}\dim_{\mathsf{Q}} M = \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to Q_n \to \cdots \to Q_0 \to M \to 0 \text{ is} \\ \text{a proper left } \mathsf{Q}\text{-resolution of } M \end{array} \right\}.$$

Similarly, if M has a proper right Q-resolution, then we define the *proper right* Q*-dimension* of M by

$$\mathscr{R}\text{-}\dim_{\mathsf{Q}} M = \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to Q^0 \to \cdots \to Q^n \to 0 \text{ is} \\ \text{a proper right } \mathsf{Q}\text{-resolution of } M \end{array} \right\}.$$

Definition 5.3. We use $\mathsf{GI}_C(A)$, $\mathsf{GP}_C(A)$ and $\mathsf{GF}_C(A)$ to denote the classes of C-Gorenstein injective, C-Gorenstein projective and C-Gorenstein flat A-modules, respectively.

A proper right $\mathsf{GI}_C(A)$-resolution is called a *proper C-Gorenstein injective resolution*, and a proper left $\mathsf{GP}_C(A)/\mathsf{GF}_C(A)$-resolution is called a *proper C-Gorenstein projective/flat resolution*.

Finally, we introduce the (more natural) notation:

- $C\text{-}\mathsf{Gid}_A(-)$ for the proper right $\mathsf{GI}_C(A)$-dimension,
- $C\text{-}\mathsf{Gpd}_A(-)$ for the proper left $\mathsf{GP}_C(A)$-dimension,
- $C\text{-}\mathsf{Gfd}_A(-)$ for the proper left $\mathsf{GF}_C(A)$-dimension,

whenever these dimensions are defined.

The next definition is taken directly from [8, Definition 2.1]:

Definition 5.4. Let F be a class of A-modules. Then F is called *Kaplansky* if there exists a cardinal number κ such that for every module $M \in \mathsf{F}$ and every element $x \in M$ there is a submodule $N \subseteq M$ satisfying $x \in N$ and $N, M/N \in \mathsf{F}$ with $|N| \leqslant \kappa$.

Lemma 5.5. *The class of C-Gorenstein injective A-modules is Kaplansky.*

Proof. The class of Gorenstein injective $(A \ltimes C)$-modules is Kaplansky by [8, Proposition 2.6]. Let κ be a cardinal number which implements the Kaplansky property for this class.

Now assume that M is a C-Gorenstein injective A-module, and that $x \in M$ is an element. By Proposition 2.13(1), M is Gorenstein injective over $A \ltimes C$, and thus there exists a Gorenstein injective $(A \ltimes C)$-submodule $N \subseteq M$ with $x \in N$ and $|N| \leqslant \kappa$, and such that the $(A \ltimes C)$-module M/N is Gorenstein injective.

Since M is an A-module, when we consider it as a module over $A \ltimes C$, it is annihilated by the ideal $C \subseteq A \ltimes C$. Consequently, the two $(A \ltimes C)$-modules N and M/N are also annihilated by C. This means that N and M/N really are A-modules which are viewed as $(A \ltimes C)$-modules. Hence Proposition 2.13(1) implies that N and M/N are C-Gorenstein injective A-modules; and we are done. $\square$

Theorem 5.6. *Every A-module M has a proper C-Gorenstein injective resolution, and we have an equality*:

$$C\text{-}\mathsf{Gid}_A\, M = C\text{-}\mathrm{Gid}_A\, M.$$

Proof. By Lemma 5.5, the class of C-Gorenstein injective A-modules is Kaplansky, and it is obviously also closed under arbitrary direct products. Therefore, [8, Theorem 2.5 and Remark 3] implies that every A-module admits a proper C-Gorenstein injective resolution.

Every injective A-module is also Gorenstein injective by Example 2.8(a), and hence a proper C-Gorenstein injective resolution is exact. Consequently, we immediately get the inequality:

$$C\text{-}\mathsf{Gid}_A\, M \geqslant C\text{-}\mathrm{Gid}_A\, M.$$

To show the opposite inequality, we may assume that $n = C\text{-Gid}_A M$ is finite. Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a proper C-Gorenstein injective resolution of M. Defining $D^n = \text{Coker}(E^{n-2} \to E^{n-1})$ we get an exact sequence:

$$0 \to M \to E^0 \to \cdots \to E^{n-1} \to D^n \to 0,$$

which also stays exact when we apply the left exact functor $\text{Hom}_A(-, E)$ to it for every C-Gorenstein injective A-module E. Since $C\text{-Gid}_A M = \text{Gid}_{A \ltimes C} M = n$, we get by [10, Theorem 2.22] and Proposition 2.13(1) that D^n is C-Gorenstein injective, so $C\text{-}\mathsf{Gid}_A M \leqslant n$. □

Sometimes, nice proper C-Gorenstein injective resolutions exist:

Proposition 5.7. *If M is module in $\mathsf{A}_C(A)$ such that $n = C\text{-Gid}_A M$ is finite, then there exists a proper C-Gorenstein injective resolution of the form*:

$$0 \to M \to H^0 \to \text{Hom}_A(C, I^1) \to \cdots \to \text{Hom}_A(C, I^n) \to 0, \tag{$*$}$$

where H^0 is C-Gorenstein injective and $I^1, \ldots, I^n$ are injective.

Proof. As in the proof of Theorem 4.2, the assumption $M \in \mathsf{A}_C(A)$ gives the existence of an exact sequence of A-modules:

$$0 \to M \to \text{Hom}_A(C, J^0) \to \cdots \to \text{Hom}_A(C, J^{n-1}) \to D^n \to 0,$$

where $J^0, \ldots, J^{n-1}$ are injective, and D^n is Gorenstein injective over $A \ltimes C$. Applying Lemma 2.4 to D^n we get a commutative diagram of A-modules with exact rows:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & \text{Hom}_A(C, J^0) & \longrightarrow & \cdots & \longrightarrow & \text{Hom}_A(C, J^{n-1}) & \longrightarrow & D^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & D^0 & \longrightarrow & \text{Hom}_A(C, U^0) & \longrightarrow & \cdots & \longrightarrow & \text{Hom}_A(C, U^{n-1}) & \longrightarrow & D^n & \longrightarrow & 0
\end{array}$$

where $U^0, \ldots, U^{n-1}$ are injective and D^0 is C-Gorenstein injective. The mapping cone of this chain map is of course exact, and furthermore, it has $0 \to D^n \xrightarrow{=} D^n \to 0$ as a subcomplex.

Consequently, we get the exact sequence $(*)$, where $I^i = U^{i-1} \oplus J^i$ for $i = 1, \ldots, n-1$ together with $I^n = U^{n-1}$ are injective; and $H^0 = D^0 \oplus \text{Hom}_A(C, J^0)$ is C-Gorenstein injective.

We claim that the sequence $(*)$ remains exact when we apply the functor $\text{Hom}_A(-, N)$ to it for any C-Gorenstein injective A-module N (and this will finish the proof):

Splitting $(*)$ into short exact sequences, we get sequences of the form $0 \to X \to Y \to Z \to 0$, where Z has the property that it fits into an exact sequence:

$$0 \to Z \to \text{Hom}_A(C, E^0) \to \text{Hom}_A(C, E^m) \to 0,$$

where $E^0, \ldots, E^m$ are injective. Therefore, it suffices to prove that every such module Z satisfies $\text{Ext}^1_A(Z, N) = 0$ for all C-Gorenstein injective modules N. But as $\text{Ext}^{\geqslant 1}_A(\text{Hom}_A(C, E^i), N) = 0$ for $i = 0, \ldots, m$, this follows easily. □

We do not know if every module has a proper C-Gorenstein projective resolution. However, in the case where A admits a dualizing complex and where $C = A$, then the answer is positive by [13, Theorem 3.2].

"Dualizing" the proof of Theorem 5.6 (except the first part about existence of proper resolutions) and Proposition 5.7, we get:

Theorem 5.8. *Assume that M is an A-module which has a proper C-Gorenstein projective resolution. Then we have an equality*:

$$C\text{-}\mathsf{Gpd}_A\, M = C\text{-}\mathrm{Gpd}_A\, M.$$

Proposition 5.9. *If M is module in* $\mathsf{B}_C(A)$ *such that* $n = C\text{-}\mathrm{Gpd}_A\, M$ *is finite, then there exists a proper C-Gorenstein projective resolution of the form*:

$$0 \to C \otimes_A P_n \to \cdots \to C \otimes_A P_1 \to G_0 \to M \to 0,$$

where G_0 *is C-Gorenstein projective and* $P_1, \ldots, P_n$ *are projective. Furthermore, if M is finitely generated, then* $G_0, P_1, \ldots, P_n$ *may be taken to be finitely generated as well.*

The C-Gorenstein flat case is more subtle. We begin with the next:

Lemma 5.10. *The class of C-Gorenstein flat A-modules is Kaplansky, and closed under direct limits.*

Proof. As in the proof of Lemma 5.5; this time using [8, Proposition 2.10], we see that the class of C-Gorenstein flat A-modules is Kaplansky.

By Proposition 2.15, a module M is C-Gorenstein flat if and only if M satisfies conditions (F1) in Definition 2.7 and (F2′) in Proposition 2.15. Clearly, the condition (F1) is closed under direct limits.

Concerning condition (F2′), we recall from Lemma 2.14 that the class of A-modules $\mathsf{F} = \{C \otimes_A F \mid F \text{ flat } A\text{-module}\}$ is closed under direct limits. Condition (F2′) states that M admits an infinite proper right F-resolution, or in the language of [7,8], that $\mu_{\mathsf{F}}(M) = \infty$. Hence [8, Theorem 2.4] implies that also (F2′) is closed under direct limits. □

Theorem 5.11. *Every A-module M has a proper C-Gorenstein flat resolution, and there is an equality*:

$$C\text{-}\mathsf{Gfd}_A\, M = C\text{-}\mathrm{Gfd}_A\, M.$$

Proof. The class $\mathsf{GF}_C(A)$ of C-Gorenstein flat modules contains the projective (in fact, flat) modules by Example 2.8(c), and furthermore, it is closed under extensions by [10, Theorem 3.7] and Proposition 2.15.

Thus, by Lemma 5.10 and [8, Theorem 2.9] we conclude that the pair $(\mathsf{GF}_C(A), \mathsf{GF}_C(A)^{\perp})$ is a *perfect cotorsion theory* according to [8, Definition 2.2]. In particular, every module admits a C-Gorenstein flat (pre)cover, and hence proper C-Gorenstein flat resolutions always exist.

The equality $C\text{-}\mathsf{Gfd}_A M = C\text{-}\mathsf{Gfd}_A M$ follows as in Theorem 5.6; this time using [10, Theorem 3.14] instead of [10, Theorem 2.22]. $\square$

References

[1] L.W. Christensen, Functorial dimensions, Ph.D. Thesis, 1999, available from http://www.math.ku.dk/~winther/papers.html.
[2] L.W. Christensen, Gorenstein dimensions, Lecture Notes in Mathematics, vol. 1747, Springer, Berlin, 2000.
[3] L.W. Christensen, Semi-dualizing complexes and their Auslander categories, Trans. Amer. Math. Soc. 353 (2001) 1839–1883.
[4] L.W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions—a functorial description with applications, preprint, 2003, available from http://arXiv.org/abs/math.AC/0403156.
[5] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995) 611–633.
[6] E.E. Enochs, O.M.G. Jenda, Relative homological algebra, de Gruyter Exp. Math., vol. 30, de Gruyter, Berlin, 2000.
[7] E.E. Enochs, O.M.G. Jenda, L. Oyonarte, λ and μ-dimensions of modules, Rend. Sem. Mat. Univ. Padova 105 (2001) 111–123.
[8] E.E. Enochs, J.A. López-Ramos, Kaplansky classes, Rend. Sem. Mat. Univ. Padova 107 (2002) 67–79.
[9] R. Hartshorne, Residues and duality, Lecture Notes in Mathematics, vol. 20, Springer, Berlin, 1966 (Notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne).
[10] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (2004) 167–193.
[11] H. Holm, Rings with finite Gorenstein injective dimension, Proc. Amer. Math. Soc. 132 (2004) 1279–1283.
[12] H. Holm, P. Jørgensen, Cohen–Macaulay injective, projective, and flat dimension, preprint, 2004, http://www.arxiv.org/abs/math.AC/0405523.
[13] P. Jørgensen, The Gorenstein projective modules are precovering, preprint, 2003, available from http://arXiv.org/abs/math.RA/0312263.
[14] O. Veliche, Gorenstein projective dimension for complexes, Trans. Amer. Math. Soc., to appear, available from http://www.math.purdue.edu/~oveliche.
[15] C.A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
[16] J. Xu, Flat covers of modules, Lecture Notes in Mathematics, vol. 1634, Springer, Berlin, 1996.