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Modules with Cosupport and Injective Functors

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Abstract Several authors have studied the filtered colimit closure $\varinjlim \mathcal{B}$ of a class $\mathcal{B}$ of finitely presented modules. Lenzing called $\varinjlim \mathcal{B}$ the category of modules with support in $\mathcal{B}$, and proved that it is equivalent to the category of flat objects in the functor category $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$. In this paper, we study the category $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ of modules with cosupport in $\mathcal{B}$. We show that $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is equivalent to the category of injective objects in $(\mathcal{B}, \mathsf{Ab})$, and thus recover a classical result by Jensen-Lenzing on pure injective modules. Works of Angeleri-Hügel, Enochs, Krause, Rada, and Saorín make it easy to discuss covering and enveloping properties of $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$, and furthermore we compare the naturally associated notions of $\mathcal{B}$-coherence and $\mathcal{B}$-noetherianness. Finally, we prove a number of stability results for $\varinjlim \mathcal{B}$ and $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. Our applications include a generalization of a result by Gruson-Jensen and Enochs on pure injective envelopes of flat modules.

Keywords Algebraically compact · Coherent · Contravariantly finite · Cosupport · Cotorsion pairs · Covariantly finite · Covers · Direct limits · Envelopes · Equivalence · Filtered colimits · Flat functors · Functor category · Injective functors · Noetherian · Pure injective · Stability · Support

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1 Introduction

Let B be a finitely presented left module over a ring R, and let Λ be its endomorphism ring. Since B is a left-Λ-left-R-bimodule, one can consider the functors

$$R\text{-}\mathsf{Mod} \underset{-\otimes_{\Lambda} B}{\overset{\mathrm{Hom}_R(B,-)}{\rightleftarrows}} \mathsf{Mod}\text{-}\Lambda.$$

An important observation in Auslander's work on representation theory for Artin algebras is that these functors give an equivalence between $\mathsf{add}\,B$ and $\mathsf{proj}\text{-}\Lambda$; see Notation 2.2. Actually, it follows by Lazard [35] that the functors above also induce an equivalence between $\varinjlim(\mathsf{add}\,B)$ and $\mathsf{Flat}\text{-}\Lambda$. In [36] Lenzing generalizes this result even further by proving that for any additive category $\mathcal{B}$ of finitely presented left R-modules, the Yoneda functor,

$$R\text{-}\mathsf{Mod} \longrightarrow (\mathcal{B}^{\mathrm{op}}, \mathsf{Ab}) \quad , \quad M \longmapsto \mathrm{Hom}_R(-, M)|_{\mathcal{B}}$$

restricts to an equivalence between $\varinjlim \mathcal{B}$ and the category $\mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ of flat functors in the sense of Oberst-Röhrl [39] and Stenström [42]. The category $\varinjlim \mathcal{B}$ has several nice properties, and it has been studied in great detail by e.g. the authors of [3–5, 13, 15, 34], and [36].

In this paper, we study the category of modules with *cosupport* in $\mathcal{B}$,

$$(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Prod}\{\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \mid B \in \mathcal{B}\}.$$

The main theorem of Section 3 is a result dual to that of Lenzing [36, prop. 2.4].

Theorem 1.1 *The tensor evaluation functor,*

$$\mathsf{Mod}\text{-}R \longrightarrow (\mathcal{B}, \mathsf{Ab}) \quad , \quad N \longmapsto (N \otimes_R -)|_{\mathcal{B}}$$

restricts to an equivalence between $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ *and* $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$.

Two special cases of Theorem 1.1 are worth mentioning: If $\mathcal{B} = \mathsf{add}\,B$ for some finitely presented module B with endomorphism ring Λ, it follows that the functors

$$\mathsf{Mod}\text{-}R \underset{\mathrm{Hom}_{\Lambda}(B,-)}{\overset{-\otimes_R B}{\rightleftarrows}} \Lambda\text{-}\mathsf{Mod}$$

induce an equivalence between $\mathsf{Prod}\{\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})\}$ and $\Lambda\text{-}\mathsf{Inj}$. For $\mathcal{B} = R\text{-}\mathsf{mod}$ we get an equivalence between the category of pure injective right R-modules and $\mathsf{Inj}(R\text{-}\mathsf{mod}, \mathsf{Ab})$. We refer to Jensen-Lenzing [30, thm. B.16][1] for this classical result.

[1]Unfortunately, the proof of Jensen-Lenzing [30, thm. B.16] does not apply to give a proof of Theorem 1.1, as one key ingredient in their argument is the fact that the tensor evaluation functor

$$\mathsf{Mod}\text{-}R \longrightarrow (R\text{-}\mathsf{mod}, \mathsf{Ab}) \quad , \quad N \longmapsto (N \otimes_R -)|_{R\text{-}\mathsf{mod}}$$

is fully faithful. If $R \notin \mathcal{B}$ the tensor evaluation functor in Theorem 1.1 is, in general, neither full nor faithful as Example 3.3 shows. Our proof of Theorem 1.1 uses techniques—such as tensor products of functors—different from those found in the proof of [30, thm. B.16].

In Section 4 we investigate enveloping and covering properties of $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. One easy consequence of Theorem 1.1 is the following:

Theorem 1.2 *The class* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ *is enveloping in* $\mathsf{Mod}\text{-}R$*. In addition, for a homomorphism* $h\colon N \longrightarrow I$ *with* I *in* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$*, the following conditions are equivalent:*

(i) *h is a* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$*-envelope;*
(ii) *h is an essential $\mathcal{B}$-monomorphism, cf. Definition* 4.4.

Theorem 1.2 is not new, but it does cover several references in the literature: That $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is enveloping also follows from Enochs-Jenda-Xu [19, thm. 2.1] and Krause [32, cor. 3.15]. In the case where R is in $\mathcal{B}$, the class of short $(- \otimes_R \mathcal{B})$-exact sequences constitutes a *proper class* in the sense of Stenström [41, §2], and hence Theorem 1.2 also contains [41, prop. 4.5].

We stress that the hard parts of the proof of Theorem 1.3 below follow from references to works of Angeleri-Hügel, Krause, Rada and Saorín, [2, 32, 33, 40].

Theorem 1.3 *For the full subcategory* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ *of* $\mathsf{Mod}\text{-}R$*, the following conditions are equivalent:*

(i) *It is closed under coproducts;*
(ii) *It is closed under direct limits;*
(iii) *It is precovering;*
(iv) *It is covering;*
(v) *It is closed under pure submodules;*
(vi) *It is closed under pure submodules, pure quotients, and pure extensions;*
(vii) *It equals* $\mathsf{Add}\, E$ *for some right R-module E.*

If the equivalent conditions in Theorem 1.3 are satisfied, R is called *$\mathcal{B}$-noetherian*. In Definition 4.2 we define what it means for R to be *$\mathcal{B}$-coherent*. Using this terminology, we end Section 4 by addressing the question of when a cotorsion pair $(\mathcal{M}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ of finite type exists.

In Section 5 we prove stability results for modules with (co)support in $\mathcal{B}$, e.g.

Theorem 1.4 *A module F is in* $\varinjlim \mathcal{B}$ *if and only if* $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ *is in* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$.

Theorem 1.5 *Assume that R is in $\mathcal{B}$. Then R is $\mathcal{B}$-noetherian if and only if*

(1) *R is $\mathcal{B}$-coherent, and*
(2) *Any right R-module E is in* $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ *if only if* $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ *is in* $\varinjlim \mathcal{B}$.

We point out a couple of applications of the stability theorems above:

Corollary 5.3 gives conditions on a class $\mathcal{E}$ which ensure that $\mathcal{E} \cap \mathsf{PureInj}\text{-}R$ has the form $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. In Example 2.6 we apply 5.3 to describe the modules with cosupport in the category of G-dimension zero modules over a Gorenstein ring.

Corollary 5.7 generalizes a result by Gruson-Jensen [27] and Enochs [16] which asserts that over a coherent ring, the pure injective envelope of a flat module is flat.

The paper ends with Appendix A where we show two results on injective and flat functors. These results are needed to prove the stability theorems in Section 5.

2 Preliminaries

In this preliminary section, we introduce our notation, define modules with cosupport in $\mathcal{B}$, and briefly present some relevant background material.

Setup 2.1 Throughout this paper, R is any unital ring and $\mathcal{B}$ denotes any additive full subcategory of the category of finitely presented left R-modules.

Notation 2.2 We write $R\text{-}\mathsf{Mod}/\mathsf{Mod}\text{-}R$ for the category of left/right R-modules, and Ab for the category of abelian groups. As in Krause-Solberg [34], we define for $\mathcal{C} \subseteq R\text{-}\mathsf{Mod}$ four full subcategories of $R\text{-}\mathsf{Mod}$ by specifying their objects as below.

- $\mathsf{add}\,\mathcal{C}$ – direct summands of finite (co)products of modules from $\mathcal{C}$;
- $\mathsf{Add}\,\mathcal{C}$ – direct summands of arbitrary coproducts of modules from $\mathcal{C}$;
- $\mathsf{Prod}\,\mathcal{C}$ – direct summands of arbitrary products of modules from $\mathcal{C}$;
- $\varinjlim \mathcal{C}$ – filtered colimits, cf. [37, IX.§1], of modules from $\mathcal{C}$.

Some authors [4, 5, 34] use the notation $\varinjlim \mathcal{C}$—others [3], [13, §4] write $\vec{\mathcal{C}}$. The following specific categories of modules play a central role in our examples.

- mod – finitely presented modules;
- proj – finitely generated projective modules;
- $\mathsf{Flat}/\mathsf{Inj}/\mathsf{Proj}$ – flat/injective/projective modules;
- $\mathsf{PureInj}$ – pure injective modules.

Definition 2.3 Modules *with support in* $\mathcal{B}$ were defined by Lenzing [36],

$$(R\text{-}\mathsf{Mod})_{\mathcal{B}} = \varinjlim \mathcal{B}.$$

In this paper we study the category of right R-modules *with cosupport in* $\mathcal{B}$,

$$(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Prod}\{\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \mid B \in \mathcal{B}\}.$$

Example 2.4 The following is well-known.

(a) If $\mathcal{B} = R\text{-}\mathsf{proj}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Inj}\text{-}R$.
(b) If $\mathcal{B} = R\text{-}\mathsf{mod}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{PureInj}\text{-}R$.

Example 2.5 Let R be commutative and noetherian, let C be a semidualizing[2] R-module, and let $\mathcal{B} = \mathsf{add}\,C$. Combining Example 2.4(a) with the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})),$$

it is easily seen that $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ consists exactly of modules of the form $\mathrm{Hom}_R(C, E)$, where E is injective. These modules play a central role in e.g. [20, 21], and [29].

[2] A finitely generated module is semidualizing if the homothety map $R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism. Semidualizing modules have been studied under different names by Foxby [22] (PG-modules of rank one), Golod [26] (suitable modules), and Vasconcelos [43] (spherical modules).

Above it was tacitly used that if M is a direct summand of $\mathrm{Hom}_R(C, E)$, where E is injective, then M also has this form. In fact, as $C \otimes_R \mathrm{Hom}_R(C, E) \cong E$ it follows that $C \otimes_R M$ is injective, and it is not hard to see that $\mathrm{Hom}_R(C, C \otimes_R M) \cong M$.

Example 2.6 Assume that R is Iwanaga-Gorenstein, that is, R is two-sided noetherian and has finite injective dimension from both sides. Consider:

- The class $\mathcal{B}$ of G-dimension zero[3] left R-modules, cf. Auslander-Bridger [7];
- The class $\mathcal{E}$ of Gorenstein injective[4] right R-modules, cf. Enochs-Jenda [17].

Then there is an equality $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathcal{E} \cap \mathsf{PureInj}\text{-}R$.

Proof We apply Corollary 5.3. By [28, thm. 2.6] the class $\mathcal{E}$ is closed under products and direct summands. Condition 5.3(1) holds by [12, prop. 3.8] and [28, thm. 3.6]; and condition 5.3(2) holds by [18, cor. 10.3.9] and [18, thm. 10.3.8]. □

Functor categories 2.7 Let $\mathcal{C}$ be any additive and skeletally small category, for example $\mathcal{C} = \mathcal{B}$ from Setup 2.1. We adopt the notation of [13, 34] and write $(\mathcal{C}, \mathsf{Ab})$ for the category of all additive covariant functors $\mathcal{C} \longrightarrow \mathsf{Ab}$.

It is well-known, cf. [24, II.§1] that $(\mathcal{C}, \mathsf{Ab})$ is an abelian category with small Hom-sets, and that $(\mathcal{C}, \mathsf{Ab})$ admits the same categorical constructions (such as exact direct limits) as Ab does. The representable functors $\mathcal{C}(C, -)$ are projective objects, and they constitute a generating set. Thus $(\mathcal{C}, \mathsf{Ab})$ has injective hulls in the sense of [24, II.§5, §6]. We write $\mathsf{Inj}(\mathcal{C}, \mathsf{Ab})$ for the category of injective objects in $(\mathcal{C}, \mathsf{Ab})$.

A functor F is *finitely generated* if there is an exact sequence $\mathcal{C}(C, -) \to F \to 0$ for some $C \in \mathcal{C}$. Similarly, F is *finitely presented* if there exists an exact sequence $\mathcal{C}(C_1, -) \to \mathcal{C}(C_0, -) \to F \to 0$ with $C_0, C_1 \in \mathcal{C}$.

Flat functors 2.8 Oberst-Röhrl [39, §1] and Stenström [42, §3] construct over any preadditive and skeletally small category $\mathcal{C}$ a right exact tensor product,

$$(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab}) \times (\mathcal{C}, \mathsf{Ab}) \longrightarrow \mathsf{Ab} \quad , \quad (F, G) \longmapsto F \otimes_{\mathcal{C}} G$$

which has the following properties for all F and G as above, and all $A \in \mathsf{Ab}$.

(a) $\mathrm{Hom}_{\mathbb{Z}}(F \otimes_{\mathcal{C}} G, A) \cong (\mathcal{C}, \mathsf{Ab})(G, \mathrm{Hom}_{\mathbb{Z}}(F, A)) \cong (\mathcal{C}^{\mathrm{op}}, \mathsf{Ab})(F, \mathrm{Hom}_{\mathbb{Z}}(G, A))$.
(b) $F \otimes_{\mathcal{C}} \mathcal{C}(C, -) \cong FC$ and $\mathcal{C}(-, C) \otimes_{\mathcal{C}} G \cong GC$.

A functor F in $(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab})$ is *flat* if $F \otimes_{\mathcal{C}} -$ is exact, however, [13, thm. (1.3)], [39, thm. (3.2)], and [42, thm. 3] contain several equivalent characterizations of flatness.

We write $\mathsf{Flat}(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab})$ for the category of flat functors in $(\mathcal{C}^{\mathrm{op}}, \mathsf{Ab})$.

[3] A f.g. R-module B is of G-dimension zero if $\mathrm{Ext}^{\geqslant 1}(B, R) = 0 = \mathrm{Ext}^{\geqslant 1}(\mathrm{Hom}(B, R), R)$ and if the biduality homomorphism $B \longrightarrow \mathrm{Hom}(\mathrm{Hom}(B, R), R)$ is an isomorphism.

[4] M is Gorenstein injective if there is an exact sequence $\boldsymbol{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ of injective modules such that $\mathrm{Hom}(I, \boldsymbol{E})$ is exact for all injective I and $M \cong \mathrm{Ker}(E_0 \to E_{-1})$.

3 An Equivalence Between Two Categories

In this section, we prove that the category $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ of modules with cosupport in $\mathcal{B}$ is equivalent to the category of injective objects in the functor category $(\mathcal{B}, \mathsf{Ab})$.

Definition 3.1 The *tensor evaluation functor* with respect to $\mathcal{B}$ is defined as

$$\mathsf{Mod}\text{-}R \longrightarrow (\mathcal{B}, \mathsf{Ab}) \quad , \quad N \longmapsto (N \otimes_R -)|_{\mathcal{B}}.$$

Remark 3.2 For $\mathcal{B} = R\text{-}\mathsf{mod}$ the tensor evaluation functor is studied in e.g. [6, 25, 30, 32]. In this case, the tensor evaluation is fully faithful, as the inverse of

$$\mathrm{Hom}_{R^{\mathrm{op}}}(M, N) \xrightarrow{\cong} (\mathcal{B}, \mathsf{Ab})\big((M \otimes_R -)|_{\mathcal{B}}, (N \otimes_R -)|_{\mathcal{B}}\big)$$

is given by evaluating a natural transformation on the ground ring R.

Example 3.3 For general $\mathcal{B}$, the tensor evaluation functor is neither full nor faithful. To see this, let $R = \mathbb{Z}$, let $p \neq q$ be prime numbers and set $\mathcal{B} = \mathsf{add}\,\mathbb{Z}/(p)$.

(a) As $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \cong \mathbb{Z}/(p)$ the functors $(\mathbb{Z}/(p) \otimes_{\mathbb{Z}} -)|_{\mathcal{B}}$ and $(\mathbb{Z} \otimes_{\mathbb{Z}} -)|_{\mathcal{B}}$ are equivalent, and since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(p), \mathbb{Z}) \cong 0$, the tensor evaluation is not full.
(b) As $\mathbb{Z}/(q) \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \cong 0$ we get $(\mathbb{Z}/(q) \otimes_{\mathbb{Z}} -)|_{\mathcal{B}} = 0$, so from the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(q), \mathbb{Z}/(q)) \cong \mathbb{Z}/(q)$, the tensor evaluation cannot be faithful.

Part (d) of the next result shows that the tensor evaluation functor does become fully faithful when appropriately restricted.

Proposition 3.4 *The following conclusions hold:*

(a) *The tensor evaluation functor, cf. Definition* 3.1*, is additive and commutes with small filtered colimits and products.*
(b) *For $B \in \mathcal{B}$ there is a natural equivalence of functors $\mathcal{B} \longrightarrow \mathsf{Ab}$,*

$$(\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}} \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\mathcal{B}(-, B), \mathbb{Q}/\mathbb{Z}).$$

(c) *For $F \in (\mathcal{B}, \mathsf{Ab})$ and $B \in \mathcal{B}$ there is a natural isomorphism of abelian groups,*

$$\mathrm{Hom}_{\mathbb{Z}}(FB, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} (\mathcal{B}, \mathsf{Ab})\big(F, (\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}}\big).$$

(d) *Let N and I be right R-modules with I in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. The homomorphism in Ab induced by the tensor evaluation functor is then an isomorphism,*

$$\mathrm{Hom}_{R^{\mathrm{op}}}(N, I) \xrightarrow{\cong} (\mathcal{B}, \mathsf{Ab})\big((N \otimes_R -)|_{\mathcal{B}}, (I \otimes_R -)|_{\mathcal{B}}\big).$$

Proof

"(a)": Clearly, the tensor evaluation functor is additive. It commutes with filtered colimits by [44, cor. 2.6.17], and with products by [18, thm. 3.2.22].
"(b)": As $\mathcal{B}$ consists of finitely presented modules, [9, prop. VI.5.3] gives that

$$\begin{aligned}(\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}} &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(-, B), \mathbb{Q}/\mathbb{Z})|_{\mathcal{B}} \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{B}(-, B), \mathbb{Q}/\mathbb{Z}).\end{aligned}$$

"(c)": By part (b) we get the first isomorphism in:

$$\begin{aligned}(\mathcal{B}, \mathsf{Ab})(F, (\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}}) &\cong (\mathcal{B}, \mathsf{Ab})(F, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{B}(-, B), \mathbb{Q}/\mathbb{Z}))\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{B}(-, B) \otimes_{\mathcal{B}} F, \mathbb{Q}/\mathbb{Z})\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(FB, \mathbb{Q}/\mathbb{Z}).\end{aligned}$$

The second and third isomorphisms are by 2.8(a) and (b), respectively.

"(d)": By Definition 2.3, I is a direct summand of a product of modules of the form $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ where $B \in \mathcal{B}$. Thus, since the tensor evaluation functor and the covariant Hom-functors $\mathrm{Hom}_{R^{\mathrm{op}}}(N, -)$ and $(\mathcal{B}, \mathsf{Ab})((N \otimes_R -)|_{\mathcal{B}}, ?)$ are additive and commute with products, we may assume that $I = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ with $B \in \mathcal{B}$. We then apply part (c) with $F = (N \otimes_R -)|_{\mathcal{B}}$ to get the first isomorphism in:

$$\begin{aligned}(\mathcal{B}, \mathsf{Ab})((N \otimes_R -)|_{\mathcal{B}}, (I \otimes_R -)|_{\mathcal{B}}) &\cong \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R B, \mathbb{Q}/\mathbb{Z})\\ &\cong \mathrm{Hom}_{R^{\mathrm{op}}}(N, I).\end{aligned}$$

The second isomorphism is by adjunction [9, prop. II.5.2] and by definition of I. □

Definition 3.5 A homomorphism of right R-modules $h\colon M \longrightarrow N$ is called a *$\mathcal{B}$-monomorphism* if $h \otimes_R B$ is a monomorphism for all B in $\mathcal{B}$.

Lemma 3.6 *If I has cosupport in $\mathcal{B}$ and $h\colon I \longrightarrow N$ is a $\mathcal{B}$-monomorphism, then h is a split monomorphism.*

Proof By our assumptions and by the isomorphism,

$$\mathrm{Hom}_{R^{\mathrm{op}}}(h, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(h \otimes_R B, \mathbb{Q}/\mathbb{Z}),$$

it follows that $\mathrm{Hom}_{R^{\mathrm{op}}}(h, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))$ is surjective for every B in $\mathcal{B}$. Combining this with Definition 2.3, we see that $\mathrm{Hom}_{R^{\mathrm{op}}}(h, J)$ is surjective for all J with cosupport in $\mathcal{B}$, that is, every homomorphism $I \longrightarrow J$ factors through h. If I has cosupport in $\mathcal{B}$, we apply this to $\mathrm{id}\colon I \longrightarrow I$ to get the desired conclusion. □

Once we have proved Theorem 1.1, the following Lemmas 3.7 and 3.8 will be superfluous. These lemmas are the key ingredients in proving essential surjectivity of the tensor evaluation when viewed as a functor from $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ to $\mathsf{Inj}\,(\mathcal{B}, \mathsf{Ab})$.

Lemma 3.7 *Every functor F in $(\mathcal{B}, \mathsf{Ab})$ can be embedded into a functor of the form $(I \otimes_R -)|_{\mathcal{B}}$ where I has cosupport in $\mathcal{B}$.*

Proof Applying Gabriel's result [24, (proof of) II.§1 prop. 3] to $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ we get a family of index sets $\{U_B\}_{B \in \mathcal{B}}$ and an exact sequence in $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ of the form,

$$\textstyle\coprod_{B \in \mathcal{B}} \mathcal{B}(-, B)^{(U_B)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to this sequence, we get an exact sequence in $(\mathcal{B}, \mathsf{Ab})$,

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\textstyle\coprod_{B \in \mathcal{B}} \mathcal{B}(-, B)^{(U_B)}, \mathbb{Q}/\mathbb{Z}\right).$$

The module I defined by $\prod_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})^{U_B}$ has cosupport in $\mathcal{B}$, and we have a natural equivalence for the latter functor above:

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}\left(\coprod_{B\in\mathcal{B}}\mathcal{B}(-, B)^{(U_B)}, \mathbb{Q}/\mathbb{Z}\right) &\cong \prod_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{B}(-, B), \mathbb{Q}/\mathbb{Z})^{U_B}\\ &\cong \prod_{B\in\mathcal{B}} (\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}}^{U_B}\\ &\cong (I \otimes_R -)|_{\mathcal{B}}.\end{aligned}$$

The second isomorphism follows by Proposition 3.4(b), and the third one since the tensor evaluation functor commutes with products. To finish the proof, note that F embeds into its double Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. □

The author thanks the anonymous referee for making the following elegant proof of Lemma 3.8 available. The author's original proof used Eilenberg's swindle and was more elaborate.

Lemma 3.8 *If a functor F in $(\mathcal{B}, \mathsf{Ab})$ is a direct summand of $(I \otimes_R -)|_{\mathcal{B}}$ where I has cosupport in $\mathcal{B}$, then $F(-) \cong (J \otimes_R -)|_{\mathcal{B}}$ for some J with cosupport in $\mathcal{B}$.*

Proof If F is a direct summand of $(I \otimes_R -)|_{\mathcal{B}}$, then there is a corresponding idempotent e in the endomorphism ring $\mathrm{End}\left((I \otimes_R -)|_{\mathcal{B}}\right)$. By Proposition 3.4(d), e lifts to an idempotent e' in $\mathrm{End}_R(I)$ which, in turn, corresponds to a direct summand J of I. It is straightforward to check that $F(-) \cong (J \otimes_R -)|_{\mathcal{B}}$. □

We are now ready to prove Theorem 1.1 from the Introduction. Note that this result is well-known in the case where $\mathcal{B} = R\text{-}\mathsf{mod}$, see for example [30, thm. B.16].

Proof of Theorem 1.1 First we must argue that the functor $(I \otimes_R -)|_{\mathcal{B}}$ is injective if I has cosupport in $\mathcal{B}$. By Definition 2.3 and Proposition 3.4(a) we may assume that I has the form $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ for some B in $\mathcal{B}$. Now, let

$$\Xi \;=\; 0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

be a short exact sequence in $(\mathcal{B}, \mathsf{Ab})$, in particular,

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F''B, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(FB, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F'B, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \qquad (\dagger)$$

is exact in Ab. By Proposition 3.4(c), the sequence (†) is isomorphic to

$$(\mathcal{B}, \mathsf{Ab})\big(\Xi, (\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \otimes_R -)|_{\mathcal{B}}\big) = (\mathcal{B}, \mathsf{Ab})\big(\Xi, (I \otimes_R -)|_{\mathcal{B}}\big), \qquad (\ddagger)$$

and since Eq. (†) is exact, so is Eq. (‡). Thus, $(I \otimes_R -)|_{\mathcal{B}}$ is injective in $(\mathcal{B}, \mathsf{Ab})$.

To show that the tensor evaluation functor gives the claimed equivalence, we argue that it is fully faithful and essentially surjective when viewed as a functor from $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ to $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$.

By Proposition 3.4(d), the restriction of the tensor evaluation functor to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is fully faithful, and essential surjectivity follows from Lemmas 3.7 and 3.8. □

Remark 3.9 By Theorem 1.1, every F in $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ has the form $F \cong (I \otimes_R -)|_{\mathcal{B}}$ for a unique (up to isomorphism) module I with cosupport in $\mathcal{B}$. However, if the functor $(I \otimes_R -)|_{\mathcal{B}}$ is injective, I need not have cosupport in $\mathcal{B}$.

For example, if as in Example 3.3 we let $R = \mathbb{Z}$ and $\mathcal{B} = \mathsf{add}\,\mathbb{Z}/(p)$, it follows from the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(p), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/(p)$ that $\mathbb{Z}/(q)$ does not have cosupport in $\mathcal{B}$. However, $(\mathbb{Z}/(q) \otimes_{\mathbb{Z}} -)|_{\mathcal{B}}$ is the zero functor and thus it is injective.

Proposition 3.10 *Let N be in $\mathsf{Mod}\text{-}R$. Then $(N \otimes_R -)|_{\mathcal{B}}$ is in $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ if and only if there exists I in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ and a $(- \otimes_R \mathcal{B})$-isomorphism $N \longrightarrow I$.*

Thus, if R is in $\mathcal{B}$ then $(N \otimes_R -)|_{\mathcal{B}} \in \mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ if and only if $N \in (\mathsf{Mod}\text{-}R)^{\mathcal{B}}$.

Proof The first assertion is clear from Proposition 3.4(d) and Theorem 1.1. For the last assertion we note that if φ is a $(- \otimes_R \mathcal{B})$-isomorphism and R is in $\mathcal{B}$, then φ is an isomorphism. □

Proposition 3.11 *Let M be in $R\text{-}\mathsf{Mod}$. Then $\mathrm{Hom}_R(-, M)|_{\mathcal{B}}$ is in $\mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ if and only if there exists F in $\varinjlim \mathcal{B}$ and a $\mathrm{Hom}_R(\mathcal{B}, -)$-isomorphism $F \longrightarrow M$.*

Thus, if $R \in \mathcal{B}$ then $\mathrm{Hom}_R(-, M)|_{\mathcal{B}} \in \mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ if and only if $M \in \varinjlim \mathcal{B}$.

Proof By the proof of [36, Proposition 2.4], the homomorphism of abelian groups,

$$\mathrm{Hom}_R(F, M) \longrightarrow (\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})(\mathrm{Hom}_R(-, F)|_{\mathcal{B}}, \mathrm{Hom}_R(-, M)|_{\mathcal{B}}),$$

induced by the Yoneda functor $R\text{-}\mathsf{Mod} \longrightarrow (\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ is an isomorphism for all F in $\varinjlim \mathcal{B}$. From this fact and from [36, Proposition 2.4] the first assertion follows.

For the last assertion we note that if φ is a $\mathrm{Hom}_R(\mathcal{B}, -)$-isomorphism and R is in $\mathcal{B}$, then φ is an isomorphism. □

4 Covers and Envelopes by Modules with (Co)support

The reader is assumed to be familiar with the notions of precovering (contravariantly finite), preenveloping (covariantly finite), covering, and enveloping subcategories. We refer to e.g. [18, chap. 5.1 and 6.1] for the relevant definitions.

By El Bashir [15, thm. 3.2] the class $\varinjlim \mathcal{B}$ is covering, in particular, it is closed under coproducts in $R\text{-}\mathsf{Mod}$. The next result due to Crawley-Boevey [13, thm.(4.2)] and Krause [32, prop. 3.11] characterizes when $\varinjlim \mathcal{B}$ is closed under products.

Theorem 4.1 *The following conditions are equivalent:*

(i) *$\varinjlim \mathcal{B}$ is closed under products in $R\text{-}\mathsf{Mod}$;*
(ii) *$\varinjlim \mathcal{B}$ is preenveloping in $R\text{-}\mathsf{Mod}$;*
(iii) *$\mathcal{B}$ is preenveloping in $R\text{-}\mathsf{mod}$;*
(iv) *$\varinjlim \mathcal{B}$ is definable.*

Definition 4.2 R is called *$\mathcal{B}$-coherent* if the conditions in Theorem 4.1 are satisfied.

Example 4.3 The following conclusions hold.

(a) If $\mathcal{B} = R\text{-}\mathsf{proj}$ then $\varinjlim \mathcal{B} = R\text{-}\mathsf{Flat}$ by Lazard [35], so by Chase [10, thm. 2.1], R is $\mathcal{B}$-coherent if and only if it is right coherent in the classical sense.
(b) If $\mathcal{B} = R\text{-}\mathsf{mod}$ then $\varinjlim \mathcal{B} = R\text{-}\mathsf{Mod}$ by [30, (7.15)], so all rings are $\mathcal{B}$-coherent.

As an easy application of Theorem 1.1, we now prove Theorem 1.2. In view of Example 2.4, Theorem 1.2 implies the existence of injective hulls and pure injective envelopes. The first of these classical results was proved by Eckmann and Schopf [14], and the second one by Fuchs [23] and Kiełpiński [31].

Definition 4.4 A homomorphism $h\colon N \longrightarrow M$ of right R-modules is called an *essential $\mathcal{B}$-monomorphism* if it is a $\mathcal{B}$-monomorphism in the sense of Definition 3.5 and if any homomorphism $g\colon M \longrightarrow L$ is a $\mathcal{B}$-monomorphism if $g \circ h$ is so.

Proof of Theorem 1.2 Since an envelope is unique up to isomorphism, cf. [45, prop. 1.2.1], it suffices to argue that every N admits an essential $\mathcal{B}$-monomorphism $h\colon N \longrightarrow I$ with I in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$, and that every such map is a $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$-envelope.

To this end, let $u\colon (N \otimes_R -)|_{\mathcal{B}} \longrightarrow U$ be an injective hull in $(\mathcal{B}, \mathsf{Ab})$, see Functor categories 2.7. By Theorem 1.1, the functor U has the form $(I \otimes_R -)|_{\mathcal{B}}$ for an I with cosupport in $\mathcal{B}$, and by Proposition 3.4(d), u is induced by a homomorphism $h\colon N \longrightarrow I$. It is easily seen that h is an essential $\mathcal{B}$-monomorphism. Another application of Theorem 1.1, combined with [24, II.§5, prop. 8], gives that if h is an essential $\mathcal{B}$-monomorphism then it is also a $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$-envelope. □

We are now ready to prove Theorem 1.3, which characterizes when $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is closed under coproducts. The hard parts of the proof of Theorem 1.3 follow from references to work of Angeleri-Hügel, Krause, Rada, and Saorín, [2, 32, 33, 40].

Proof of Theorem 1.3 It suffices to prove the implications:

$$\begin{tikzcd}
(iii) \arrow[r, Rightarrow] & (i) \arrow[d, Rightarrow] & (ii) \arrow[l, Rightarrow] & \\
 & & & (vi) \arrow[ul, Rightarrow] \\
(iv) \arrow[uu, Rightarrow] & (vii) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (v) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

"$(i) \Rightarrow (vii)$": Note that $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Prod}\, J$, where J is $\prod_{\alpha \in A} \mathrm{Hom}_{\mathbb{Z}}(B_\alpha, \mathbb{Q}/\mathbb{Z})$ and $\{B_\alpha\}_{\alpha \in A}$ is a set of representatives for the isomorphism classes in $\mathcal{B}$. By Definition 2.3, all modules in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ are pure injective. Hence (i) implies that J is Σ-pure-injective, and the proof of [2, prop. 6.10] gives the desired conclusion.

"$(vii) \Rightarrow (iv)$": If *(vii)* holds then E is product complete, cf. [33, §3], and it follows by [33, cor. 3.6] that $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is closed under direct limits. By [40, cor. 3.7(a)], the class $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is also precovering, and hence it is covering by [45, thm. 2.2.8].

"$(iv) \Rightarrow (iii) \Rightarrow (i)$": The first implicaton is trivial, and the latter is a consequence of [40, thm. 3.4] since $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is closed under direct summands.

"$(vii) \Rightarrow (v)$": By [32, thm. 6.7], the assumption *(vii)* ensures that $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is definable, in particular, it is closed under pure submodules, cf. [32, thm. 2.1].

"$(v) \Rightarrow (vi)$": Let $\eta = 0 \to N' \to N \to N'' \to 0$ be pure exact. If N' and N'' are in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ then, as N' is pure injective, η splits and

$N \cong N' \oplus N'' \in (\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. If N is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$, the assumption (v) gives that N' is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. As before, the sequence splits, and N'' is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ since it is a direct summand of N.

"$(vi) \Rightarrow (ii)$": Let $\varphi_{\mu\lambda}\colon I_\lambda \longrightarrow I_\mu$ be a direct system of modules from $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. As $\prod E_\lambda$ is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$, as $\coprod E_\lambda \longrightarrow \prod E_\lambda$ is a pure monomorphism, and since $\coprod E_\lambda \longrightarrow \varinjlim E_\lambda$ is a pure epimorphism, we conclude that $\varinjlim E_\lambda$ is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$.

"$(ii) \Rightarrow (i)$": A coproduct is the direct limit of its finite sub-coproducts. □

Definition 4.5 R is *$\mathcal{B}$-noetherian* if the conditions in Theorem 1.3 are satisfied.

Example 4.6 The following conclusions hold.

(a) If $\mathcal{B} = R\text{-}\mathsf{proj}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Inj}\text{-}R$ cf. 2.4(a), so by Bass [8, thm. 1.1], R is $\mathcal{B}$-noetherian if and only if it is right noetherian in the classical sense.
(b) If $\mathcal{B} = R\text{-}\mathsf{mod}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{PureInj}\text{-}R$ cf. 2.4(b), so by [30, thm. B.18], R is $\mathcal{B}$-noetherian if and only if it is right pure semi-simple.

Corollary 4.7 *Assume that $\mathcal{B} \subseteq \mathcal{B}'$ are two additive full subcategories of R-mod. If the ring R is $\mathcal{B}'$-noetherian then it is also $\mathcal{B}$-noetherian.*

Proof Assume that R is $\mathcal{B}'$-noetherian and let $\{E_\lambda\}$ be a family in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. By our assumptions, $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} \subseteq (\mathsf{Mod}\text{-}R)^{\mathcal{B}'}$, and the latter is closed under coproducts. It follows that $\coprod E_\lambda$ belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}'}$ and, in particular, $\coprod E_\lambda$ is pure injective. Thus, the pure monomorphism $\coprod E_\lambda \longrightarrow \prod E_\lambda$ is split, and since $\prod E_\lambda$ belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ then so does $\coprod E_\lambda$. Thus R is $\mathcal{B}$-noetherian by Theorem 1.3. □

Recall that if $\mathcal{M} \subseteq \mathsf{Mod}\text{-}R$ then $(\mathcal{M}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ is a *cotorsion pair* if

$$\operatorname{Ker}\operatorname{Ext}^1_{R^{\mathrm{op}}}(\mathcal{M}, -) = (\mathsf{Mod}\text{-}R)^{\mathcal{B}} \quad \text{and} \quad \mathcal{M} = \operatorname{Ker}\operatorname{Ext}^1_{R^{\mathrm{op}}}\left(-, (\mathsf{Mod}\text{-}R)^{\mathcal{B}}\right).$$

If a cotorsion pair $(\mathcal{M}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ exists, then it is said to be of *finite type* provided that there is a set $\mathcal{S} \subseteq \mathsf{mod}\text{-}R$ such that $\operatorname{Ker}\operatorname{Ext}^1_{R^{\mathrm{op}}}(\mathcal{S}, -) = (\mathsf{Mod}\text{-}R)^{\mathcal{B}}$.

Note that if R is right coherent and $(\mathcal{M}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ is a cotorsion pair of finite type, then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is closed under coproducts, that is, R is $\mathcal{B}$-noetherian. In Example 4.8 and Corollary 4.10 below we give examples of situations where $\mathcal{B}$-noetherianness of R is enough to ensure the existence of a cotorsion pair $(\mathcal{M}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ of finite type.

Example 4.8 The following is well-known, however, it also follows from Proposition 4.9.

(a) If $\mathcal{B} = R\text{-}\mathsf{proj}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{Inj}\text{-}R$, cf. 2.4(a). Clearly $(\mathsf{Mod}\text{-}R, \mathsf{Inj}\text{-}R)$ is a cotorsion pair. If R is right noetherian—equivalently, $\mathcal{B}$-noetherian by 4.6(a)—then this cotorsion pair is of finite type by Baer's Criterion.
(b) If $\mathcal{B} = R\text{-}\mathsf{mod}$ then $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathsf{PureInj}\text{-}R$, cf. 2.4(b). In general, there does not exist a cotorsion pair of the form $(\mathcal{M}, \mathsf{PureInj}\text{-}R)$. If R is right pure semi-simple—equivalently, $\mathcal{B}$-noetherian by 4.6(b)—then $\mathsf{PureInj}\text{-}R$ equals $\mathsf{Mod}\text{-}R$, and hence $(\mathsf{Proj}\text{-}R, \mathsf{PureInj}\text{-}R)$ is a cotorsion pair of finite type.

Proposition 4.9 *Assume that $\mathcal{B}$ satisfies the following two conditions:*

(1) *R is $\mathcal{B}$-noetherian, and*
(2) *If $\mathrm{Tor}_1^R(M,F)=0$ for all M in $\mathrm{Ker}\,\mathrm{Tor}_1^R(-,\mathcal{B})\cap\mathsf{mod}\text{-}R$ then F is in $\varinjlim\mathcal{B}$.*

Then $(\mathcal{M},(\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ is a cotorsion pair of finite type, where $\mathcal{M}=\mathrm{Ker}\,\mathrm{Tor}_1^R(-,\mathcal{B})$.

Proof By the isomorphism [9, VI.§5],

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Tor}_1^R(-,B),\mathbb{Q}/\mathbb{Z}\right)\cong\mathrm{Ext}^1_{R^{\mathrm{op}}}(-,\mathrm{Hom}_{\mathbb{Z}}(B,\mathbb{Q}/\mathbb{Z})),$$

it follows that $\mathcal{M}=\mathrm{Ker}\,\mathrm{Tor}_1^R(-,\mathcal{B})=\mathrm{Ker}\,\mathrm{Ext}^1_{R^{\mathrm{op}}}(-,(\mathsf{Mod}\text{-}R)^{\mathcal{B}})$. To establish that $(\mathcal{M},(\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ is a cotorsion pair of finite type, we will show that E is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ provided that $\mathrm{Ext}^1_{R^{\mathrm{op}}}(\mathcal{M}\cap\mathsf{mod}\text{-}R,E)=0$.

First we argue that R is right noetherian: By (2) it follows that R-proj is contained in $\varinjlim\mathcal{B}$, and hence also in $\mathcal{B}$, as $(\varinjlim\mathcal{B})\cap R\text{-}\mathsf{mod}=\mathcal{B}$. By (1) and Corollary 4.7 we then conclude that R is $(R\text{-}\mathsf{proj})$-noetherian, that is, R is right noetherian by Example 4.6(a). Now, since R is right noetherian, [9, prop. VI.5.3] gives that

$$\mathrm{Tor}_1^R(M,\mathrm{Hom}_{\mathbb{Z}}(E,\mathbb{Q}/\mathbb{Z}))\cong\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Ext}^1_{R^{\mathrm{op}}}(M,E),\mathbb{Q}/\mathbb{Z}\right)\qquad(\dagger)$$

for all M in $\mathsf{mod}\text{-}R$. It follows by Eq. ($\dagger$) and (2) that if $\mathrm{Ext}^1_{R^{\mathrm{op}}}(\mathcal{M}\cap\mathsf{mod}\text{-}R,E)=0$ then $\mathrm{Hom}_{\mathbb{Z}}(E,\mathbb{Q}/\mathbb{Z})$ is in $\varinjlim\mathcal{B}$. Thus, by (1) and Theorem 1.5 (the latter is proved in the next section), we conclude that the module E belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. $\square$

Corollary 4.10 *Assume that R is commutative, noetherian, local, and Gorenstein. As in Example 2.6, let $\mathcal{B}$ be the class of R-modules with G-dimension zero. If R is $\mathcal{B}$-noetherian then there exists a cotorsion pair $(\mathcal{M},(\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ of finite type.*

Proof It suffices to prove that $\mathcal{B}$ satisfies 4.9(2). By [18, thm. 10.3.8], $\varinjlim\mathcal{B}$ is the class of Gorenstein flat R-modules, and thus it follows by [18, prop. 11.5.9] that $\mathcal{M}=\mathrm{Ker}\,\mathrm{Tor}_1^R(-,\mathcal{B})$ is the class of R-modules with finite projective dimension. Hence in order to prove 4.9(2), we need to argue that an R-module F is Gorenstein flat if $\mathrm{Tor}_1^R(M,F)=0$ for all finitely generated R-modules M with finite projective dimension. By considering an exact sequence $0\to M'\to P\to M\to 0$ where P is finitely generated free, the latter condition is easily seen to imply $\mathrm{Tor}^R_{\geqslant 1}(M,F)=0$ for all finitely generated R-modules M with finite projective dimension. Now it follows by [11, cor. (3.3)] and [28, thm. 3.19] that F is Gorenstein flat. $\square$

5 Stability Results

In this section we prove a number of stability results for modules with (co)support in $\mathcal{B}$, and we also present some applications. The terminology in Definitions 4.2 and 4.5 plays a central role in this section.

Injective structures 5.1 Maranda [38] defines an *injective structure* as a pair $(\mathcal{H}, \mathcal{Q})$ where $\mathcal{H}$ is a class of homomorphisms and $\mathcal{Q}$ is a class of modules satisfying:

(1) $Q \in \mathcal{Q}$ if and only if $\mathrm{Hom}_R(h, Q)$ is surjective for all $h \in \mathcal{H}$;
(2) $h \in \mathcal{H}$ if and only if $\mathrm{Hom}_R(h, Q)$ is surjective for all $Q \in \mathcal{Q}$;
(3) For every R-module M there exists $h\colon M \longrightarrow Q$ where $h \in \mathcal{H}$ and $Q \in \mathcal{Q}$.

Given (2), condition (3) means exactly that $\mathcal{Q}$ is preenveloping in $\mathsf{Mod}\text{-}R$.

Enochs-Jenda-Xu [19, thm. 2.1] prove that if $\mathcal{H}$ is the class of $\mathcal{B}$-monomorphisms, cf. Definition 3.5, then $(\mathcal{H}, (\mathsf{Mod}\text{-}R)^{\mathcal{B}})$ is an injective structure, and $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ is enveloping (not just preenveloping). The last fact also follows from Theorem 1.2.

Lemma 5.2 *Let ξ be a complex of left R-modules, and let η be a complex of right R-modules. Then the following conclusions hold:*

(a) *ξ is $\mathrm{Hom}_R(\mathcal{B}, -)$-exact if and only if $\mathrm{Hom}_{\mathbb{Z}}(\xi, \mathbb{Q}/\mathbb{Z})$ is $(- \otimes_R \mathcal{B})$-exact.*
(b) *η is $(- \otimes_R \mathcal{B})$-exact if and only if $\mathrm{Hom}_{\mathbb{Z}}(\eta, \mathbb{Q}/\mathbb{Z})$ is $\mathrm{Hom}_R(\mathcal{B}, -)$-exact.*

Proof For a finitely presented left R-module B, there are natural isomorphisms,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(B, \xi), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\xi, \mathbb{Q}/\mathbb{Z}) \otimes_R B,$$

$$\mathrm{Hom}_{\mathbb{Z}}(\eta \otimes_R B, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(\eta, \mathbb{Q}/\mathbb{Z})),$$

see [9, prop. VI.5.3] and [9, prop. II.5.2]. From these the lemma easily follows. □

Proof of Theorem 1.4 “Only if”: If $F \in \varinjlim \mathcal{B}$ and h is a $\mathcal{B}$-monomorphism, then $h \otimes_R F$ is injective since $\otimes$ commutes with $\varinjlim$ by [44, cor. 2.6.17]. Thus

$$\mathrm{Hom}_{R^{\mathrm{op}}}(h, \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(h \otimes_R F, \mathbb{Q}/\mathbb{Z})$$

is surjective, and it follows from Injective structures 5.1 that $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$.

“If”: By [40, cor. 3.7(a)] the class of modules consisting of coproducts of modules from $\mathcal{B}$ is precovering. Hence there is a left-exact and $\mathrm{Hom}_R(\mathcal{B}, -)$-exact sequence,

$$\xi \;=\; 0 \longrightarrow K \longrightarrow P \xrightarrow{\pi} F \longrightarrow 0,$$

where P is a set-indexed coproduct of modules from $\mathcal{B}$. A priori we do not know if ξ is exact at F, but we will argue that ξ is, in fact, pure exact. Having showed this, it will follow from [36, prop. 2.1] that F belongs to $\varinjlim \mathcal{B}$, as desired.

Exactness and pure exactness of ξ can be proved simultaneously by showing that $\mathrm{Hom}_{\mathbb{Z}}(\pi, \mathbb{Q}/\mathbb{Z})$ is a split monomorphism, cf. [30, thm. 6.4]. As $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$, it suffices by Lemma 3.6 to see that $\mathrm{Hom}_{\mathbb{Z}}(\pi, \mathbb{Q}/\mathbb{Z})$ is a $\mathcal{B}$-monomorphism, but this follows from Lemma 5.2(a) and $\mathrm{Hom}_R(\mathcal{B}, -)$-surjectivity of π. □

Corollary 5.3 *Assume that $\mathcal{E}$ is a class of right R-modules that is closed under direct summands and products in $\mathsf{Mod}\text{-}R$ and satisfies the two conditions:*

(1) *$\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ belongs to $\mathcal{E}$ for every $B \in \mathcal{B}$;*
(2) *$\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ belongs to $\varinjlim \mathcal{B}$ for every $E \in \mathcal{E}$.*

Then there is an equality $(\mathsf{Mod}\text{-}R)^{\mathcal{B}} = \mathcal{E} \cap \mathsf{PureInj}\text{-}R$.

Proof The inclusion "$\subseteq$" is clear from (1). To prove "$\supseteq$" we assume that $E \in \mathcal{E}$ is pure injective. As E is in $\mathcal{E}$, it follows by (2) and Theorem 1.4 that the module $D(E)$ defined by $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. As the canonical homomorphism $E \longrightarrow D(E)$ is a pure monomorphism, and since E is pure injective, E is a direct summand of $D(E)$. Consequently, E belongs to $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$. □

Applying 3.6, 5.1, 5.2, and Theorem 1.4, it is easy to prove the following properties for modules with support in $\mathcal{B}$. Corollary 5.4 is not new, in fact, it is not hard to see that part (b) also follows from Lenzing [36, prop. 2.2].

Corollary 5.4 *The following conclusions hold:*

(a) *A left R-module F belongs to $\varinjlim \mathcal{B}$ if and only if $h \otimes_R F$ is a monomorphism for every $\mathcal{B}$-monomorphism h.*
(b) *If $0 \to F' \to F \to F'' \to 0$ is an exact and $\mathrm{Hom}_R(\mathcal{B}, -)$-exact sequence with F'' in $\varinjlim \mathcal{B}$, then F' is in $\varinjlim \mathcal{B}$ if and only if F is in $\varinjlim \mathcal{B}$.* □

Observation 5.5 Note that if $R \in \mathcal{B}$, Proposition 3.11/3.10 implies that the ring R is $\mathcal{B}$-coherent/-noetherian in the sense of Definition 4.2/4.5 if and only if the category $\mathcal{B}$ is left coherent/noetherian in the sense of Definition 5.10.

Theorem 5.6 *Assume that R is in $\mathcal{B}$ and that R is $\mathcal{B}$-coherent. Then a right R-module E is in $(\mathsf{Mod}\text{-}R)^{\mathcal{B}}$ only if $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ is in $\varinjlim \mathcal{B}$.*

Proof We have the following implications,

$$\begin{aligned} E \text{ is in } (\mathsf{Mod}\text{-}R)^{\mathcal{B}} &\iff (E \otimes_R -)|_{\mathcal{B}} \text{ is in } \mathsf{Inj}(\mathcal{B}, \mathsf{Ab}) \\ &\implies \mathrm{Hom}_{\mathbb{Z}}((E \otimes_R -)|_{\mathcal{B}}, \mathbb{Q}/\mathbb{Z}) \text{ is in } \mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab}) \\ &\iff \mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}))|_{\mathcal{B}} \text{ is in } \mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab}) \\ &\iff \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \text{ is in } \varinjlim \mathcal{B}. \end{aligned}$$

The first and last equivalence follows from Propositions 3.10 and 3.11, and the penultimate equivalence is by adjunction. The implication in the second line is immediate by Observation 5.5 and Proposition 5.12(a).

A result by Gruson-Jensen [27] and Enochs [16, lem. 1.1] asserts that over a right coherent ring, the pure injective envelope of a flat left R-module is again flat. In view of Example 4.3(b), we have the following generalization.

Corollary 5.7 *Assume that R is in $\mathcal{B}$, and that R is $\mathcal{B}$-coherent. If F is in $\varinjlim \mathcal{B}$ then its pure injective envelope $PE(F)$ and the quotient $PE(F)/F$ are in $\varinjlim \mathcal{B}$.*

Proof We have a pure monomorphism $F \longrightarrow D(F) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, and since $D(F)$ is pure injective it contains $PE(F)$ as a direct summand. Theorems 1.4 and 5.6 imply that $D(F)$ is in $\varinjlim \mathcal{B}$, and we conclude that $PE(F)$ is in $\varinjlim \mathcal{B}$. That $PE(F)/F$ is in $\varinjlim \mathcal{B}$ now follows from [36, prop. 2.2].

Proof of Theorem 1.5 In view of the proof of Theorem 5.6, Theorem 1.5 is an immediate consequence of Observation 5.5 and Proposition 5.13.

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Appendix A: Two Results on Flat and Injective Functors

Propositions 5.12 and 5.13 below play a central role in the proofs of Theorems 5.6 and 1.5. Since we have not been able to find proofs of 5.12 or 5.13 in the literature, they are included in this Appendix.

Lemma 5.8 *For $F, G \in (\mathcal{B}, \mathsf{Ab})$ and $A \in \mathsf{Ab}$ there is a canonical homomorphism,*

$$\mathrm{Hom}_{\mathbb{Z}}(G, A) \otimes_{\mathcal{B}} F \xrightarrow{\omega_{GAF}} \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(F, G), A).$$

If A is injective (divisible) and F is finitely presented then ω_{GAF} is an isomorphism.

Proof For each B in $\mathcal{B}$ there is a canonical homomorphism of abelian groups,

$$\mathrm{Hom}_{\mathbb{Z}}(GB, A) \xrightarrow{\varpi_B} \mathrm{Hom}_{\mathbb{Z}}(FB, \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(F, G), A)),$$

It is given by $\varpi_B(f)(x)(\theta) = (f \circ \theta_B)(x)$ where $f : GB \longrightarrow A$ is a homomorphism, $x \in FB$ is an element, and $\theta : F \longrightarrow G$ is a natural transformation. It is easily seen that ϖ is a natural transformation of functors $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$. By applying 2.8(a),

$$\begin{aligned} \varpi &\in (\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})\big(\mathrm{Hom}_{\mathbb{Z}}(G, A), \mathrm{Hom}_{\mathbb{Z}}(F, \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(F, G), A))\big) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_{\mathbb{Z}}(G, A) \otimes_{\mathcal{B}} F, \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(F, G), A)\big), \end{aligned}$$

it follows that ϖ corresponds to the homomorphism which is denoted by ω in the lemma. It is straightforward to verify that ω is natural in F, G, and A.

To see that ω_{GAF} is an isomorphism when A is injective and F is finitely presented, note that $\omega_{GA?}$ is a natural transformation between right exact and additive functors $(\mathcal{B}, \mathsf{Ab}) \longrightarrow \mathsf{Ab}$. As every finitely presented F fits into an exact sequence,

$$\mathcal{B}(B_1, -) \longrightarrow \mathcal{B}(B_0, -) \longrightarrow F(-) \longrightarrow 0,$$

it suffices, by the five-lemma, to check that $\omega_{G,A,\mathcal{B}(B,-)}$ is an isomorphism. However, this homomorphism is the composite of the following two isomorphisms:

$$\mathrm{Hom}_{\mathbb{Z}}(G, A) \otimes_{\mathcal{B}} \mathcal{B}(B, -) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(GB, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(\mathcal{B}(B, -), G), A),$$

where the left-hand isomorphism is by 2.8(b), and the right-hand isomorphism is by Yoneda's Lemma, cf. [37, III.§2]. This finishes the proof. □

Remark 5.9 By Oberst-Röhrl [39, (proof of) thm. (3.2)] a functor T in $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ is flat if for every finitely generated additive subfunctor F of a representable functor $\mathcal{B}(B, -)$, one has exactness of the sequence:

$$0 \longrightarrow T \otimes_{\mathcal{B}} F \longrightarrow T \otimes_{\mathcal{B}} \mathcal{B}(B, -).$$

Although the author was not able to find a reference, it is well-known that Baer's Criterion holds in functor categories[5], that is, E in $(\mathcal{B}, \mathsf{Ab})$ is injective if for every additive subfunctor G of $\mathcal{B}(B, -)$, one has exactness of the sequence:

$$(\mathcal{B}, \mathsf{Ab})(\mathcal{B}(B, -), E) \longrightarrow (\mathcal{B}, \mathsf{Ab})(G, E) \longrightarrow 0.$$

Definition 5.10 The category $\mathcal{B}$ is *left coherent* if $\mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ is closed under products in $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$; and $\mathcal{B}$ is *right coherent* if $\mathcal{B}^{\mathrm{op}}$ is left coherent.

The category $\mathcal{B}$ is *left noetherian* if $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ is closed under coproducts in $(\mathcal{B}, \mathsf{Ab})$; and $\mathcal{B}$ is *right noetherian* if $\mathcal{B}^{\mathrm{op}}$ is left noetherian.

Remark 5.11 The references Jensen-Lenzing [30, thm. B.17] and Oberst-Röhrl [39, thm. (4.1)] contain several equivalent characterizations of the notions above. For example, $\mathcal{B}$ if left coherent if and only if every *finitely generated* additive subfunctor of $\mathcal{B}(B, -)$ is finitely presented; and $\mathcal{B}$ if left noetherian if and only if *every* additive subfunctor of $\mathcal{B}(B, -)$ is finitely generated (and thus, finitely presented).

Proposition 5.12 *For every E in $(\mathcal{B}, \mathsf{Ab})$ the following hold:*

(a) *If $E \in \mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ and $\mathcal{B}$ is left coherent then* $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \in \mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$.
(b) *If* $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \in \mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ *and $\mathcal{B}$ is left noetherian then $E \in \mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$.*

Proof Let F be a finitely presented additive subfunctor of a representable functor $\mathcal{B}(B, -)$. By Lemma 5.8 there is a commutative diagram in Ab, where the vertical homomorphisms are isomorphisms:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathcal{B}} F \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathcal{B}} \mathcal{B}(B, -) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(F, E), \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}((\mathcal{B}, \mathsf{Ab})(\mathcal{B}(B, -), E), \mathbb{Q}/\mathbb{Z}). \end{tikzcd}$$

The desired conclusions now follow from Remarks 5.9 and 5.11. □

Proposition 5.13 *The category $\mathcal{B}$ is left noetherian if and only if it satisfies:*

(1) *$\mathcal{B}$ is left coherent, and*
(2) *Any functor E is in $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$ if only if* $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ *is in* $\mathsf{Flat}(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$.

[5] One way to prove this is by combining the proof of Anderson-Fuller [1, prop. 16.13] with the first sentence in the proof of Lemma 3.7.

Proof "If": Let $\{E_\lambda\}$ be a family in $\mathsf{Inj}(\mathcal{B}, \mathsf{Ab})$. Combining the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}\left(\coprod E_\lambda, \mathbb{Q}/\mathbb{Z}\right) \cong \prod \mathrm{Hom}_{\mathbb{Z}}(E_\lambda, \mathbb{Q}/\mathbb{Z})$$

with (1) and (2), it follows by Definition 5.10 that $\mathcal{B}$ is left noetherian.

"Only if": If $\mathcal{B}$ is left noetherian then (1) holds by Remark 5.11, and hence part (2) follows from Proposition 5.12.

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