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Auslander and Buchweitz have proved that every finitely generated module over a Cohen–Macaulay (CM) ring with a dualizing module admits a so-called maximal CM approximation. In terms of relative homological algebra, this means that every finitely generated module has a special maximal CM precover. In this paper, we prove the existence of special maximal CM preenvelopes and, in the case where the ground ring is henselian, of maximal CM envelopes. We also characterize the rings over which every finitely generated module has a maximal CM envelope with the unique lifting property. Finally, we show that cosyzygies with respect to the class of maximal CM modules must eventually be maximal CM, and we compute some examples.

1. Introduction

Let R be a commutative noetherian local Cohen–Macaulay (CM) ring with a dualizing module Ω and denote by MCM the class of maximal CM R-modules. Auslander and Buchweitz [1989, Theorem A] construct a *maximal CM approximation* for every finitely generated R-module M, that is, a short exact sequence

$$0 \longrightarrow I \longrightarrow X \stackrel{\pi}{\longrightarrow} M \longrightarrow 0,$$

where X belongs to MCM and I has finite injective dimension. By a result form [Ischebeck 1969] one has $\operatorname{Ext}^1_R(Y, I) = 0$ for all Y in MCM, so in terms of relative homological algebra, this means that the homomorphism $\pi : X \twoheadrightarrow M$ is a *special* MCM-*precover* of M. Corollary 2.5 of [Takahashi 2005] shows that if R is henselian (for example, if R is complete), then every MCM-precover can be refined to an MCM-*cover*. The corollary follows from Takahashi's Proposition 2.4, which the author attributes to Yoshino [1993, Lemma 2.2]. We summarize these results in the following theorem.

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Theorem [Auslander and Buchweitz 1989; Takahashi 2005; Yoshino 1993].

(a) *Every finitely generated R-module has a special* MCM*-precover* (*also called a special right* MCM*-approximation*).

(b) *If R is henselian*, *then every finitely generated R-module has an* MCM*-cover* (*also called a minimal right* MCM*-approximation*).

This paper is concerned with the existence and the construction of special MCM-preenvelopes and MCM-envelopes of finitely generated modules. Our first main result, which is proved in Section 3, is the following "dual" of the theorem above.

Theorem A. (a) *Every finitely generated R-module M has a special* MCM*-preenvelope* (*also called a special left* MCM*-approximation*).

(b) *If R is henselian*, *then every finitely generated R-module has an* MCM*-envelope* (*also called a minimal left* MCM*-approximation*).

(c) *Every special* MCM*-preenvelope* (*and hence every* MCM*-envelope*) $\mu : M \to X$ *of a finitely generated R-module M has the property that* $\operatorname{Hom}_R(\operatorname{Coker}\mu, \Omega)$ *has finite injective dimension.*

Theorem C of [Holm 2014] showed the existence of (nonspecial!) MCM-preenvelopes, but its proof is not constructive: it is a consequence of an abstract result — Theorem (4.2) of [Crawley-Boevey 1994] — combined with the fact, also proved in [Holm 2014], that the direct limit closure of MCM is closed under products. Theorem A above is not only stronger than [Holm 2014, Theorem C]; our proof, modeled on that of [Holm and Jørgensen 2011, Theorem 1.6], also shows how (special) MCM-(pre)envelopes can be constructed from (special) MCM-(pre)covers.

In Section 4 we compute the MCM-envelope of some specific modules. In Section 5 we turn our attention to MCM-envelopes with the *unique lifting property*, and we characterize the rings over which every finitely generated module admits such an envelope:

Theorem B. *The following conditions are equivalent.*

(i) *For every finitely generated R-module M*, *the module* $\operatorname{Hom}_R(M, \Omega)$ *is maximal CM.*

(ii) *The Krull dimension of R is* $\leqslant 2$.

(iii) *The inclusion functor* $\mathsf{MCM} \hookrightarrow \mathsf{mod}$ *has a left adjoint.*

(iv) *Every finitely generated R-module has an* MCM*-envelope with the unique lifting property.*

From a homological point of view, maximal CM modules are interesting because every module can be finitely resolved by such modules. More precisely, if d denotes the Krull dimension of the CM ring R, and if M is any finitely generated R-module with a resolution

$$\cdots \longrightarrow X_d \longrightarrow X_{d-1} \longrightarrow X_{d-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

by finitely generated free R-modules $X_0, X_1, \ldots$, then the n-th syzygy of M, i.e., the module $\mathrm{Syz}_n(M) = \mathrm{Ker}(X_{n-1} \to X_{n-2})$, is maximal CM for every $n \geqslant d$. Actually, the same conclusion holds if $X_0, X_1, \ldots$ are just assumed to be maximal CM (but not necessarily free); this well-known fact follows from the behavior of depth in short exact sequences; see [Bruns and Herzog 1993, Proposition 1.2.9] or Lemma 2.5. Given a finitely generated R-module M, one can *not* always construct an *exact* sequence

$$(*) \qquad 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

where $X^0, X^1, \ldots$ are maximal CM; however, there is a canonical way to construct a *complex* of the form $(*)$. Theorem A guarantees the existence of MCM-preenvelopes, which makes the following construction possible: take an MCM-preenvelope $\mu^0 : M \to X^0$ of M and set $C^1 = \mathrm{Coker}\,\mu^0$; take an MCM-preenvelope $\mu^1 : C^1 \to X^1$ of C^1 and set $C^2 = \mathrm{Coker}\,\mu^1$; etc. The hereby constructed complex $(*)$ — which is called a *proper* MCM-*coresolution* or an MCM-*resolvent* of M — is not necessarily exact, but it becomes exact if one applies the functor $\mathrm{Hom}_R(-, Y)$ to it for any Y in MCM. The module $C^n = \mathrm{Coker}(X^{n-2} \to X^{n-1})$ is called the *n-th cosyzygy of M with respect to* MCM, and it is denoted by $\mathrm{Cosyz}^n_{\mathsf{MCM}}(M)$. In Section 6 we prove that such cosyzygies must eventually be maximal CM:

Theorem C. *Let M be a finitely generated R-module. For every $n \geqslant d$, any n-th cosyzygy $\mathrm{Cosyz}^n_{\mathsf{MCM}}(M)$ of M with respect to* MCM *is maximal CM.*

2. Preliminaries

Setup 2.1. Throughout, $(R, \mathfrak{m}, k)$ is a commutative noetherian local CM ring of Krull dimension d. It is assumed that R has a dualizing (or canonical) module Ω.

Let M be a finitely generated R-module. The *depth* of M is the number

$$\mathrm{depth}_R M = \inf\{i \mid \mathrm{Ext}^i_R(k, M) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\};$$

see [Bruns and Herzog 1993, Definitions 1.2.6 and 1.2.7]. If $M \neq 0$, then $\mathrm{depth}_R M$ is the common length of a maximal M-regular sequence (in $\mathfrak{m}$). The depth can also be computed from the dualizing module:

$$\mathrm{depth}_R M = d - \sup\{i \mid \mathrm{Ext}^i_R(M, \Omega) \neq 0\};$$

see [Bruns and Herzog 1993, Corollary 3.5.11]. One calls M *maximal CM* if $\mathrm{depth}_R M \geqslant d$, that is, if $\mathrm{Ext}^i_R(M, \Omega) = 0$ for all $i > 0$. The category of all such

R-modules is denoted by MCM. Note that the zero module is maximal CM and has depth ∞. The category of all finitely generated R-modules is denoted by mod.

We recall a few notions from relative homological algebra.

Definition 2.2. Let $\mathcal{A}$ be a full subcategory of an abelian category $\mathcal{M}$ (e.g., $\mathcal{M} = \mathsf{mod}$ and $\mathcal{A} = \mathsf{MCM}$), and let M be an object in $\mathcal{M}$. Following [Enochs and Jenda 2000, Definition 6.1.1], a morphism $\varepsilon : M \to A$ with $A \in \mathcal{A}$ is called an *$\mathcal{A}$-preenvelope* (or a *left $\mathcal{A}$-approximation*) of M if every other morphism $\varepsilon' : M \to A'$ with $A' \in \mathcal{A}$ factors through ε, as illustrated below.

$$\begin{array}{ccc} M & \xrightarrow{\ \varepsilon\ } & A \\ {\scriptstyle \varepsilon'}\big\downarrow & \swarrow & \\ A' & & \end{array}$$

A *special $\mathcal{A}$-preenvelope* (or a *special left $\mathcal{A}$-approximation*) is an $\mathcal{A}$-preenvelope $\varepsilon : M \to A$ such that $\operatorname{Ext}^1_{\mathcal{M}}(\operatorname{Coker}\varepsilon, A') = 0$ for every $A' \in \mathcal{A}$. An *$\mathcal{A}$-envelope* (or a *minimal left $\mathcal{A}$-approximation*) is an $\mathcal{A}$-preenvelope ε with the property that every endomorphism φ of A that satisfies $\varphi\varepsilon = \varepsilon$ is an automorphism.

Remark 2.3. The notions of *$\mathcal{A}$-precover* (or *right $\mathcal{A}$-approximation*), *special $\mathcal{A}$-precover* (or *special right $\mathcal{A}$-approximation*), and *$\mathcal{A}$-cover* (or *minimal right $\mathcal{A}$-approximation*) are categorically dual to the notions defined above.

By definition, a special $\mathcal{A}$-precover/preenvelope is also an (ordinary) $\mathcal{A}$-precover/preenvelope. If $\mathcal{A}$ is closed under extensions in $\mathcal{M}$, then every $\mathcal{A}$-cover/envelope is a special $\mathcal{A}$-precover/preenvelope; this is the content of Wakamatsu's lemma.[1]

Remark 2.4. It is well-known that the dualizing module Ω gives rise to a duality on the category of maximal CM modules; more precisely, there is an equivalence of categories:

$$\mathsf{MCM} \underset{\operatorname{Hom}_R(-,\Omega)}{\overset{\operatorname{Hom}_R(-,\Omega)}{\rightleftarrows}} \mathsf{MCM}^{\mathrm{op}}.$$

We use the shorthand notation $(-)^\dagger$ for the functor $\operatorname{Hom}_R(-, \Omega)$. For any finitely generated R-module M there is a canonical homomorphism $\delta_M : M \to M^{\dagger\dagger}$, called the *biduality homomorphism*, which is natural in M. An alternative way of phrasing the equivalence above is to say δ_M is an isomorphism if M belongs to MCM; see [Bruns and Herzog 1993, Theorem 3.3.10].

We will need the following result about depth; it is folklore and easily proved.

[1]This result is implicit in [Wakamatsu 1988]. It is explicitly stated in [Auslander and Reiten 1991, Lemma 1.3], but without a proof. It is stated and proved in [Xu 1996, Lemmas 2.1.1 and 2.1.2].

Lemma 2.5. *Let $m \geqslant 0$ be an integer and let $0 \to K_m \to X_{m-1} \to \cdots \to X_0 \to M \to 0$ be an exact sequence of finitely generated R-modules. If $X_0, \dots, X_{m-1}$ are maximal CM, then one has $\operatorname{depth}_R K_m \geqslant \min\{d, \operatorname{depth}_R M + m\}$. In particular, if $m \geqslant d$ then the R-module K_m is maximal CM.* □

3. Special MCM-preenvelopes and MCM-envelopes

In this section, we prove Theorem A from the introduction. Our proof follows that of [Holm and Jørgensen 2011, Theorem 1.6] with some adjustments.

Lemma 3.1. *For every R-module M, the composition $M^\dagger \xrightarrow{\delta_{M^\dagger}} M^{\dagger\dagger\dagger} \xrightarrow{\delta_M^\dagger} M^\dagger$ is the identity map on $M^\dagger$.*

Proof. Straightforward; see [Jans 1961, Theorem 1.4]. □

Lemma 3.2. *For every finitely generated R-module M, the next conditions are equivalent.*

(i) $\operatorname{Ext}^1_R(M, \Omega) = 0$ *and* $\operatorname{Ext}^1_R(X, M^\dagger) = 0$ *for every* $X \in \mathsf{MCM}$.

(ii) $\operatorname{Ext}^1_R(M, Y) = 0$ *for every* $Y \in \mathsf{MCM}$.

Proof. (i)⟹(ii): Given any $Y \in \mathsf{MCM}$ we must argue that $\operatorname{Ext}^1_R(M, Y) = 0$, i.e., that every short exact sequence $0 \to Y \xrightarrow{\alpha} E \to M \to 0$ splits. As $\operatorname{Ext}^1_R(M, \Omega) = 0$, the functor $(-)^\dagger$ leaves this sequence exact; in fact, the induced short exact sequence

$$0 \longrightarrow M^\dagger \longrightarrow E^\dagger \xrightarrow{\alpha^\dagger} Y^\dagger \longrightarrow 0$$

splits as $Y^\dagger$ belongs to MCM and hence $\operatorname{Ext}^1_R(Y^\dagger, M^\dagger) = 0$ by assumption. Let $\beta : Y^\dagger \to E^\dagger$ be a right inverse of $\alpha^\dagger$. Then $\delta_Y^{-1}\beta^\dagger\delta_E : E \to Y$ is a left inverse of α since one has

$$\delta_Y^{-1}\beta^\dagger\delta_E\alpha = \delta_Y^{-1}\beta^\dagger\alpha^{\dagger\dagger}\delta_Y = \delta_Y^{-1}(\alpha^\dagger\beta)^\dagger\delta_Y = \delta_Y^{-1}1_{Y^{\dagger\dagger}}\delta_Y = 1_Y.$$

(ii)⟹(i): Assumption (ii) implies that $\operatorname{Ext}^1_R(M, \Omega) = 0$ since $\Omega \in \mathsf{MCM}$. Given $X \in \mathsf{MCM}$ we must show that $\operatorname{Ext}^1_R(X, M^\dagger) = 0$, i.e., that every short exact sequence $0 \to M^\dagger \xrightarrow{\alpha} E \to X \to 0$ splits. Since X is in MCM we in particular have $\operatorname{Ext}^1_R(X, \Omega) = 0$, so an application of the functor $(-)^\dagger$ yields another short exact sequence:

$$(*) \qquad 0 \longrightarrow X^\dagger \longrightarrow E^\dagger \xrightarrow{\alpha^\dagger} M^{\dagger\dagger} \longrightarrow 0.$$

As $X^\dagger$ belongs to MCM we have $\operatorname{Ext}^1_R(M, X^\dagger) = 0$, so the functor $\operatorname{Hom}_R(M, -)$ leaves the sequence $(*)$ exact. Surjectivity of $\operatorname{Hom}_R(M, \alpha^\dagger)$ yields a homomorphism $\beta : M \to E^\dagger$ with $\alpha^\dagger\beta = \delta_M$. It follows that $\beta^\dagger\delta_E : E \to M^\dagger$ is a left inverse of α since one has $\beta^\dagger\delta_E\alpha = \beta^\dagger\alpha^{\dagger\dagger}\delta_{M^\dagger} = (\alpha^\dagger\beta)^\dagger\delta_{M^\dagger} = \delta_M^\dagger\delta_{M^\dagger} = 1_{M^\dagger}$, where the last equality follows from Lemma 3.1. □

Proof of Theorem A. We begin by proving the last assertion in the theorem. Let $\mu : M \to X$ be any special MCM-preenvelope of M. By assumption, we have $\operatorname{Ext}^1_R(\operatorname{Coker}\mu, Y) = 0$ for every $Y \in \mathsf{MCM}$. Hence Lemma 3.2 implies that $\operatorname{Ext}^1_R(Z, (\operatorname{Coker}\mu)^\dagger) = 0$ for every $Z \in \mathsf{MCM}$. By [Auslander and Buchweitz 1989, Theorem A], we can take a *hull of finite injective dimension* for the finitely generated module $(\operatorname{Coker}\mu)^\dagger$, that is, a short exact sequence

$$0 \longrightarrow (\operatorname{Coker}\mu)^\dagger \longrightarrow I \longrightarrow Z \longrightarrow 0,$$

where I has finite injective dimension and Z is maximal CM. This sequence splits since $\operatorname{Ext}^1_R(Z, (\operatorname{Coker}\mu)^\dagger) = 0$, and $(\operatorname{Coker}\mu)^\dagger$ is therefore a direct summand in I. Since I has finite injective dimension, so has $(\operatorname{Coker}\mu)^\dagger$.

To prove parts (a) and (b), let M be a finitely generated R-module and let $\pi : Z \to M^\dagger$ be a homomorphism with $Z \in \mathsf{MCM}$. We will show that if π is a special MCM-precover, respectively, an MCM-cover of $M^\dagger$ (recall that by the theorem by Auslander, Buchweitz, Takahashi and Yoshino from the introduction, special MCM-precovers always exist, and MCM-covers exist if R is henselian), then the homomorphism

$$\mu := \pi^\dagger \delta_M : M \longrightarrow Z^\dagger$$

is a special MCM-preenvelope, respectively, an MCM-envelope, of M.

First assume that π is a special MCM-precover. We begin by proving that μ is an MCM-preenvelope. Note that $Z^\dagger$ is in MCM by Remark 2.4. We must show that $\operatorname{Hom}_R(\mu, Y)$ is surjective for every $Y \in \mathsf{MCM}$. By Remark 2.4 every such Y has the form $Y \cong X^\dagger$ for some $X \in \mathsf{MCM}$ (namely for $X = Y^\dagger$), so it suffices to show that $\operatorname{Hom}_R(\mu, X^\dagger)$ is surjective for every $X \in \mathsf{MCM}$. By definition of μ, the homomorphism $\operatorname{Hom}_R(\mu, X^\dagger)$ is the composition of the maps

$$(*) \qquad \operatorname{Hom}_R(Z^\dagger, X^\dagger) \xrightarrow{\operatorname{Hom}_R(\pi^\dagger, X^\dagger)} \operatorname{Hom}_R(M^{\dagger\dagger}, X^\dagger) \xrightarrow{\operatorname{Hom}_R(\delta_M, X^\dagger)} \operatorname{Hom}_R(M, X^\dagger)\ .$$

Via the "swap" isomorphism, see [Christensen 2000, (A.2.9)], the homomorphisms in $(*)$ are identified with the ones in the top row of the following diagram:

$$(**) \qquad \begin{array}{ccccc} \operatorname{Hom}_R(X, Z^{\dagger\dagger}) & \xrightarrow{\operatorname{Hom}_R(X,\pi^{\dagger\dagger})} & \operatorname{Hom}_R(X, M^{\dagger\dagger\dagger}) & \xrightarrow{\operatorname{Hom}_R(X,\delta_M^\dagger)} & \operatorname{Hom}_R(X, M^\dagger) \\ {\scriptstyle \operatorname{Hom}_R(X,\delta_Z)}\uparrow\cong & & \uparrow{\scriptstyle \operatorname{Hom}_R(X,\delta_{M^\dagger})} & \diagup\!\!\diagup & \\ \operatorname{Hom}_R(X, Z) & \xrightarrow{\operatorname{Hom}_R(X,\pi)} & \operatorname{Hom}_R(X, M^\dagger) & & \end{array}$$

The left square in $(**)$ is commutative since the biduality homomorphism δ is natural, and the right triangle in $(**)$ is commutative by Lemma 3.1. The map δ_Z is an isomorphism since Z is in MCM; and $\operatorname{Hom}_R(X, \pi)$ is surjective as π is an MCM-precover and $X \in \mathsf{MCM}$. It follows that the composition of the maps in the top row of (∗∗), and therefore also the map $\mathrm{Hom}_R(\mu, X^\dagger)$, is surjective. Thus, μ is an MCM-preenvelope.

To see that μ is a special MCM-preenvelope, we must prove that $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, Y)$ vanishes for every $Y \in$ MCM. As the functor $(-)^\dagger$ is left exact, $(\mathrm{Coker}\,\mu)^\dagger$ is isomorphic to $\mathrm{Ker}(\mu^\dagger)$. By definition we have $\mu^\dagger = \delta_M^\dagger \pi^{\dagger\dagger}$, and hence $\mu^\dagger$ fits into the commutative diagram:

$$(***) \qquad \begin{array}{ccc} Z^{\dagger\dagger} & \xrightarrow{\mu^\dagger} & M^\dagger \\ \| & & \uparrow{\scriptstyle \delta_M^\dagger} \\ Z^{\dagger\dagger} & \xrightarrow{\pi^{\dagger\dagger}} & M^{\dagger\dagger\dagger} \\ {\scriptstyle \delta_Z}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \delta_{M^\dagger}} \\ Z & \xrightarrow{\pi} & M^\dagger \end{array} \qquad 1_{M^\dagger} \quad \text{(by Lemma 3.1)}$$

It follows that $\mu^\dagger$ and π are isomorphic maps, and hence they also have isomorphic kernels, that is, $\mathrm{Ker}(\mu^\dagger) \cong \mathrm{Ker}\,\pi$. It follows that $(\mathrm{Coker}\,\mu)^\dagger \cong \mathrm{Ker}\,\pi$. Since π is a special MCM-precover, we now have

$$\mathrm{Ext}^1_R(X, (\mathrm{Coker}\,\mu)^\dagger) \cong \mathrm{Ext}^1_R(X, \mathrm{Ker}\,\pi) = 0$$

for every $X \in$ MCM. Thus, to see that $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, Y) = 0$ for every $Y \in$ MCM, it suffices by Lemma 3.2 to prove that $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, \Omega) = 0$. To this end, set $X = Z^\dagger \in$ MCM and consider the factorization of $\mu : M \to Z^\dagger = X$ given by

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X \\ & {\scriptstyle \mu_0}\searrow \quad \nearrow{\scriptstyle \iota} & \\ & \mathrm{Im}\,\mu & \end{array}$$

where μ_0 is the corestriction of μ to its image and ι is the inclusion map. As μ_0 is surjective and $(-)^\dagger$ is left exact, the map $\mu_0^\dagger$ is injective. As $\Omega \in$ MCM and μ is an MCM-preenvelope, the map $\mu^\dagger = \mathrm{Hom}_R(\mu, \Omega)$ is surjective; and hence so is $\mu_0^\dagger$ since $\mu^\dagger = \mu_0^\dagger \iota^\dagger$. Thus, $\mu_0^\dagger$ is an isomorphism. Hence $\iota^\dagger$ and $\mu^\dagger$ are isomorphic maps, and since $\mu^\dagger$ is surjective, so is $\iota^\dagger$. Thus, application of $(-)^\dagger$ to $0 \to \mathrm{Im}\,\mu \xrightarrow{\iota} X \to \mathrm{Coker}\,\mu \to 0$ yields an exact sequence

$$X^\dagger \xrightarrow{\iota^\dagger} (\mathrm{Im}\,\mu)^\dagger \xrightarrow{0} \mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, \Omega) \longrightarrow \mathrm{Ext}^1_R(X, \Omega) = 0,$$

which forces $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, \Omega) = 0$, as desired.

Finally, assume that π is an MCM-cover. We show that $\mu = \pi^\dagger \delta_M$ is an MCM-envelope. We have already seen that μ is an MCM-preenvelope. To show that it is an envelope, let φ be an endomorphism of $Z^\dagger$ with $\varphi\mu = \mu$. It follows that $\mu^\dagger \varphi^\dagger = \mu^\dagger$. The diagram (∗∗∗) shows that $\mu^\dagger \delta_Z = \pi$, and thus $\pi(\delta_Z^{-1} \varphi^\dagger \delta_Z) =$

$\mu^\dagger\varphi^\dagger\delta_Z = \mu^\dagger\delta_Z = \pi$. As π is an MCM-cover, we conclude that $\delta_Z^{-1}\varphi^\dagger\delta_Z$, and therefore also $\varphi^\dagger$, is an automorphism. It follows that $\varphi^{\dagger\dagger}$ is an automorphism of $Z^{\dagger\dagger\dagger}$, and finally that $\varphi = \delta_{Z^\dagger}^{-1}\varphi^{\dagger\dagger}\delta_{Z^\dagger}$ is an automorphism of $Z^\dagger$. □

The proof of Theorem A (above) shows that one can construct MCM-envelopes from MCM-covers. We do not know if the converse is true, that is, we do not know if existence of MCM-envelopes is logically equivalent to existence of MCM-covers. The next result provides a partial answer to this question; it shows that existence of MCM-envelopes for *all* finitely generated modules implies existence of MCM-covers for *some* finitely generated modules (namely for modules N of the form $N \cong M^\dagger$ for some M).

Proposition 3.3. *Let M be a finitely generated R-module. If $\mu : M \to X$ is an* MCM*-preenvelope, a special* MCM*-preenvelope, or an* MCM*-envelope of M, then $\mu^\dagger : X^\dagger \to M^\dagger$ is an* MCM*-precover, a special* MCM*-precover, or an* MCM*-cover of $M^\dagger$, respectively.*

Proof. This is left as an exercise to the reader. □

4. Examples

We compute the MCM-envelope of some specific modules. We begin with a characterization of modules with trivial MCM-envelope.

Proposition 4.1. *For a finitely generated R-module M, one has $\dim_R M < d$ if and only if the zero map $M \to 0$ is an* MCM*-envelope of M.*

Proof. If $\dim_R M < d$ then [Bruns and Herzog 1993, Corollary 3.5.11(a)] shows that $\mathrm{Hom}_R(M, \Omega) = 0$. It follows that every homomorphism $\varphi : M \to X$ with $X \in$ MCM is zero. Indeed, since Ω cogenerates the category MCM, there exists a monomorphism $\iota : X \to \Omega^n$ for some natural number n. As $\mathrm{Hom}_R(M, \Omega) = 0$, the homomorphism $\iota\varphi : M \to \Omega^n$ must be zero, and thus $\varphi = 0$ since ι is injective. Since every homomorphism from M to a maximal CM module is zero, the zero map $M \to 0$ is an MCM-envelope of M.

Conversely, if $M \to 0$ is an MCM-(pre)envelope then, since Ω is in MCM, every homomorphism $\varphi : M \to \Omega$ factors through 0, and hence $\varphi = 0$. Thus $\mathrm{Hom}_R(M, \Omega) = 0$, and it follows from [Bruns and Herzog 1993, Corollary 3.5.11(b)] that one can not have $\dim_R M = d$; so $\dim_R M < d$. □

In general, MCM-(pre)envelopes need not be injective. In fact:

Corollary 4.2. *The ring R is artinian if and only if every finitely generated R-module admits an injective (that is, monic)* MCM*-(pre)envelope.*

Proof. If R is artinian, then every finitely generated R-module M is maximal CM, and therefore $1_M : M \to M$ is an injective MCM-envelope of M. Conversely, if R is not artinian, then the residue field k, which has dimension $\dim_R k = 0$, does not have an injective MCM-preenvelope by Proposition 4.1. □

Next we give a somewhat "general" example.

Example 4.3. Let M be a finitely generated R-module. If $M^\dagger$ is maximal CM, then the identity homomorphism $\pi = 1_{M^\dagger} : M^\dagger \to M^\dagger$ is an MCM-cover of $M^\dagger$. The proof of Theorem A shows that the homomorphism $\mu = \pi^\dagger \delta_M = \delta_M$, i.e., the biduality homomorphism $\delta_M : M \to M^{\dagger\dagger}$, is an MCM-envelope M.

Here is a concrete application of the example above.

Example 4.4. Let M be a submodule of a maximal CM R-module X with the property that $\dim_R(X/M) < d-1$. For example, $M = \mathfrak{a}$ could be an ideal in $X = R$ with $\mathrm{height}_R(\mathfrak{a}) > 1$; see [Bruns and Herzog 1993, Corollary 2.1.4]. Or M could be the submodule $M = (f_1, f_2, \ldots)X$, where $f_1, f_2, \ldots$ is an X-regular sequence of length at least two. We claim that, in this case, the inclusion map $\iota : M \hookrightarrow X$ is an MCM-envelope of M.

To see why, note that the short exact sequence $0 \to M \xrightarrow{\iota} X \to X/M \to 0$ is mapped by the functor $(-)^\dagger$ to the exact sequence

$$0 \longrightarrow (X/M)^\dagger \longrightarrow X^\dagger \xrightarrow{\iota^\dagger} M^\dagger \longrightarrow \mathrm{Ext}^1_R(X/M, \Omega).$$

Since $d - \dim_R(X/M) > 1$, it follows from Corollary 3.5.11(a) of [Bruns and Herzog 1993] that $\mathrm{Hom}_R(X/M, \Omega) = 0$ and $\mathrm{Ext}^1_R(X/M, \Omega) = 0$. Hence the sequence displayed above shows that $\iota^\dagger$ is an isomorphism and, in particular, $M^\dagger \cong X^\dagger$ is maximal CM. Thus Example 4.3 shows that the biduality homomorphism $\delta_M : M \to M^{\dagger\dagger}$ is an MCM-envelope of M. It remains to argue that δ_M can be identified with $\iota : M \hookrightarrow X$; however, this follows from the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & X \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \cong\ \delta_X} \\ M^{\dagger\dagger} & \underset{\cong}{\xrightarrow{\iota^{\dagger\dagger}}} & X^{\dagger\dagger} \end{array}$$

Indeed, δ_X is an isomorphism as $X \in \mathrm{MCM}$, and $\iota^{\dagger\dagger} = (\iota^\dagger)^\dagger$ is an isomorphism because $\iota^\dagger$ is.

Remark 4.5. For a special MCM-precover $\pi : X \to M$ of a finitely generated module M, the kernel $\mathrm{Ker}\,\pi$ has finite injective dimension, and hence one has $\mathrm{Ext}^i_R(X, \mathrm{Ker}\,\pi) = 0$ for every $X \in \mathrm{MCM}$ and every $i > 0$—not just for $i = 1$. A similar phenomenon does not occur for special MCM-preenvelopes. Indeed, if in Example 4.4 one has $\dim_R(X/M) = d-2$, say, then $\operatorname{Coker}\iota = X/M$ satisfies $\operatorname{Ext}^2_R(X/M,\Omega) \neq 0$ by [Bruns and Herzog 1993, Corollary 3.5.11(b)].

5. MCM-envelopes with the unique lifting property

If $\mu : M \to X$ is an MCM-preenvelope of a finitely generated R-module M, then the induced homomorphism $\operatorname{Hom}_R(\mu, Y) : \operatorname{Hom}_R(X, Y) \to \operatorname{Hom}_R(M, Y)$ is surjective for every $Y \in \mathsf{MCM}$; see Definition 2.2. If $\operatorname{Hom}_R(\mu, Y)$ is an isomorphism for every $Y \in \mathsf{MCM}$, then we say that the MCM-preenvelope μ has the *unique lifting property*. Indeed, in this case, there exists for every homomorphism $\nu : M \to Y$ with $Y \in \mathsf{MCM}$ a unique homomorphism $\varphi : X \to Y$ that makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle\nu}\downarrow & \swarrow{\scriptstyle\varphi} & \\ Y & & \end{array}$$

Note that an MCM-preenvelope $\mu : M \to X$ with the unique lifting property must necessarily be an MCM-envelope. Indeed, the only endomorphism φ of X with $\varphi\mu = \mu$ is $\varphi = 1_X$. Evidently, every surjective MCM-preenvelope has the unique lifting property.

Lemma 5.1. *For any finitely generated R-module M, one has* $\operatorname{depth}_R(M^\dagger) \geqslant \min\{d, 2\}$.

Proof. Take an exact sequence $L_1 \to L_0 \to M \to 0$ where L_0 and L_1 are finitely generated and free. Since the functor $(-)^\dagger = \operatorname{Hom}_R(-,\Omega)$ is left exact, we get an exact sequence, $0 \to M^\dagger \to L_0^\dagger \to L_1^\dagger \to C \to 0$, where C is the cokernel of the homomorphism $L_0^\dagger \to L_1^\dagger$. Since the modules $L_0^\dagger$ and $L_1^\dagger$ are maximal CM, Lemma 2.5 yields

$$\operatorname{depth}_R(M^\dagger) \geqslant \min\{d, \operatorname{depth}_R C + 2\} \geqslant \min\{d, 2\}. \qquad \square$$

Proof of Theorem B. (i)$\Longrightarrow$(ii): Consider an exact sequence of finitely generated modules

$$0 \longrightarrow K \longrightarrow L_1 \xrightarrow{\ \alpha\ } L_0 \longrightarrow N \longrightarrow 0,$$

where L_0 and L_1 are free and $K = \operatorname{Ker}\alpha$. From [Bruns and Herzog 1993, Proposition 1.2.9] (last inequality) one gets

$$(*) \qquad \operatorname{depth}_R N \geqslant \operatorname{depth}_R K - 2.$$

Set $C = \operatorname{Coker}(\alpha^\dagger)$ and consider the exact sequence $L_0^\dagger \xrightarrow{\alpha^\dagger} L_1^\dagger \longrightarrow C \longrightarrow 0$. As the functor $(-)^\dagger$ is left exact, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & L_1 & \xrightarrow{\alpha} & L_0 \\ & & & & \cong \big\downarrow \delta_{L_1} & & \cong \big\downarrow \delta_{L_0} \\ 0 & \longrightarrow & C^\dagger & \longrightarrow & L_1^{\dagger\dagger} & \xrightarrow{\alpha^{\dagger\dagger}} & L_0^{\dagger\dagger} \end{array}$$

which shows that $K \cong C^\dagger$, since δ_{L_0} and δ_{L_1} are isomorphisms. By assumption (i), the module K is therefore maximal CM, and hence inequality $(*)$ yields $\operatorname{depth}_R N \geqslant d-2$. As this holds for every finitely generated R-module N, it holds in particular for the residue field $N = k$. We get $0 = \operatorname{depth}_R k \geqslant d-2$, and thus $d \leqslant 2$.

(ii)$\Longrightarrow$(iii): In the case where R is reduced, a proof of this implication can be found in [Burban and Drozd 2008, Proposition 3.2]. We give a slightly different argument.

If $d \leqslant 2$, then Lemma 5.1 shows that for every finitely generated R-module M, the module $M^\dagger$ is maximal CM, and hence so is $M^{\dagger\dagger}$. Thus $\mathrm{F} = (-)^{\dagger\dagger}$ is a functor from mod to MCM, which we claim is a left adjoint of the inclusion $\mathrm{G} : \mathsf{MCM} \to \mathsf{mod}$. For each finitely generated R-module M and each maximal CM R-module X, the homomorphism $\varphi_{M,X} = \operatorname{Hom}_R(\delta_M, X)$ given by

$$\operatorname{Hom}_R(\mathrm{F}M, X) = \operatorname{Hom}_R(M^{\dagger\dagger}, X) \xrightarrow{\varphi_{M,X}} \operatorname{Hom}_R(M, X) = \operatorname{Hom}_R(M, \mathrm{G}X)$$

is evidently natural in M and X; and it is surjective since the biduality map $\delta_M : M \to M^{\dagger\dagger}$ is an MCM-preenvelope of M by Example 4.3. It remains to see that $\operatorname{Hom}_R(\delta_M, X)$ is injective. To this end, let $\mu : M^{\dagger\dagger} \to X$ be a homomorphism with $\mu\delta_M = \operatorname{Hom}_R(\delta_M, X)(\mu) = 0$. It follows that $\delta_M^\dagger \mu^\dagger = (\mu\delta_M)^\dagger = 0$. As $M^\dagger$ is maximal CM, the biduality map $\delta_{M^\dagger}$ is an isomorphism, and hence so is $\delta_M^\dagger$ by Lemma 3.1. Since $\delta_M^\dagger \mu^\dagger = 0$ we conclude that $\mu^\dagger = 0$. Thus $\mu^{\dagger\dagger} = (\mu^\dagger)^\dagger = 0$ and consequently $\mu = \delta_X^{-1} \mu^{\dagger\dagger} \delta_{M^{\dagger\dagger}} = 0$, as desired.

(iii)$\Longrightarrow$(iv): Let $\mathrm{F} : \mathsf{mod} \to \mathsf{MCM}$ be a left adjoint of the inclusion $\mathrm{G} : \mathsf{MCM} \to \mathsf{mod}$. For every finitely generated R-module M, the unit of adjunction $\eta_M : M \to \mathrm{GF}M$ induces, for every maximal CM R-module Y, an isomorphism:

$$\varphi_{M,Y} : \operatorname{Hom}_R(\mathrm{F}M, Y) \xrightarrow{\sim} \operatorname{Hom}_R(M, \mathrm{G}Y) \quad \text{given by} \quad \alpha \mapsto \mathrm{G}(\alpha)\eta_M;$$

see [MacLane 1971, IV.1 Theorem 1]. If we suppress the inclusion functor G and set $X = \mathrm{GF}M = \mathrm{F}M$, which is maximal CM by the assumption on F, we see that unit of adjunction $\eta_M : M \to X$ has the property that the map

$$\operatorname{Hom}_R(X, Y) \xrightarrow{\sim} \operatorname{Hom}_R(M, Y) \quad \text{given by} \quad \alpha \mapsto \alpha\eta_M = \operatorname{Hom}_R(\eta_M, Y)(\alpha)$$

is an isomorphism. Thus, η_M is an MCM-envelope of M with the unique lifting property.

(iv)$\Longrightarrow$(i): Let M be a finitely generated R-module. By assumption, M has an MCM-envelope $\mu : M \to X$ with the unique lifting property. Since Ω is maximal CM, the homomorphism $\mu^\dagger : X^\dagger \to M^\dagger$ is an isomorphism, and as $X^\dagger$ is maximal CM, so is $M^\dagger$. □

6. Cosyzygies with respect to MCM

Let $\mathcal{A}$ be a full subcategory of an abelian category $\mathcal{M}$ (for example, $\mathcal{M} = \mathsf{mod}$ and $\mathcal{A} = \mathsf{MCM}$).

Assume that every object in $\mathcal{M}$ has an $\mathcal{A}$-precover. In this case, every $M \in \mathcal{M}$ admits a *proper $\mathcal{A}$-resolution*, meaning a, not necessarily exact, complex $\mathbb{A} = \cdots \to A_1 \to A_0 \to M \to 0$ with $A_i \in \mathcal{A}$ such that the sequence $\mathrm{Hom}_{\mathcal{M}}(A, \mathbb{A})$ is exact for every $A \in \mathcal{A}$. Such a resolution is constructed recursively as follows: take an $\mathcal{A}$-precover $\pi_0 : A_0 \to M$ of M and set $K_1 = \mathrm{Ker}\,\pi_0$; take an $\mathcal{A}$-precover $\pi_1 : A_1 \to K_1$ of K_1 and set $K_2 = \mathrm{Ker}\,\pi_1$; etc. The object K_n is denoted by $\mathrm{Syz}_n^{\mathcal{A}}(M)$ and it is called the *n-th syzygy of M with respect to $\mathcal{A}$*. A given object $M \in \mathcal{M}$ has, typically, many different $\mathcal{A}$-precovers and proper $\mathcal{A}$-resolutions, so $\mathrm{Syz}_n^{\mathcal{A}}(M)$ is not uniquely determined by M; but it almost is: the version of Schanuel's lemma found in [Enochs et al. 2001, Lemma 2.2] shows that if K_n and $\overline{K}_n$ are both n-th syzygies of M with respect to $\mathcal{A}$, then there exist $A, \overline{A} \in \mathcal{A}$ such that $K_n \oplus \overline{A} \cong \overline{K}_n \oplus A$. In particular, if $\mathcal{A}$ is closed under direct summands (as is the case if $\mathcal{A} = \mathsf{MCM}$), then K_n belongs to $\mathcal{A}$ if and only if $\overline{K}_n$ belongs to $\mathcal{A}$; and thus it makes sense to ask if $\mathrm{Syz}_n^{\mathcal{A}}(M)$ belongs to $\mathcal{A}$.

If every object in $\mathcal{M}$ admits an $\mathcal{A}$-cover, then $\pi_0, \pi_1, \ldots$ in the construction above can be chosen to be $\mathcal{A}$-covers, and the resulting proper $\mathcal{A}$-resolution is then called a *minimal proper $\mathcal{A}$-resolution* of M. In this case, K_n is called the *minimal n-th syzygy of M with respect to $\mathcal{A}$*, and it is denoted by $\text{min-Syz}_n^{\mathcal{A}}(M)$. Since an $\mathcal{A}$-cover (of a given object in $\mathcal{M}$) is unique up to isomorphism, see [Xu 1996, Theorem 1.2.6], the object $\text{min-Syz}_n^{\mathcal{A}}(M)$ is uniquely determined, up to isomorphism, by M.

Dually, if every $M \in \mathcal{M}$ has an $\mathcal{A}$-preenvelope (resp. $\mathcal{A}$-envelope), then a *proper $\mathcal{A}$-coresolution* (resp. *minimal proper $\mathcal{A}$-coresolution*) $0 \to M \to A^0 \to A^1 \to \cdots$ can always be constructed as follows: take an $\mathcal{A}$-preenvelope (resp. $\mathcal{A}$-envelope) $\mu^0 : M \to A^0$ of M and set $C^1 = \mathrm{Coker}\,\mu^0$; take an $\mathcal{A}$-preenvelope (resp. $\mathcal{A}$-envelope) $\mu^1 : C^1 \to A^1$ of C^1 and set $C^2 = \mathrm{Coker}\,\mu^1$; etc. The object C^n is called the *n-th cosyzygy of M with respect to $\mathcal{A}$* (resp. the *minimal n-th cosyzygy of M with respect to $\mathcal{A}$*) and it is denoted by $\mathrm{Cosyz}^n_{\mathcal{A}}(M)$ (resp. $\text{min-Cosyz}^n_{\mathcal{A}}(M)$). The object $\text{min-Cosyz}^n_{\mathcal{A}}(M)$ is uniquely determined, up to isomorphism, by M. The object $\mathrm{Cosyz}^n_{\mathcal{A}}(M)$ is almost uniquely determined by M in the sense that if C^n and $\overline{C}^n$ are both n-th cosyzygies of M with respect to $\mathcal{A}$, then there exist $A, \overline{A} \in \mathcal{A}$ such that $C^n \oplus \overline{A} \cong \overline{C}^n \oplus A$. Thus, if $\mathcal{A}$ is closed under direct summands, then it makes sense to ask if $\mathrm{Cosyz}^n_{\mathcal{A}}(M)$ belongs to $\mathcal{A}$.

We supplement the definitions above by setting $\mathrm{Syz}_0^A(M) = \text{min-}\mathrm{Syz}_0^A(M) = M$, and similarly $\mathrm{Cosyz}^0_{\mathcal{A}}(M) = \text{min-}\mathrm{Cosyz}^0_{\mathcal{A}}(M) = M$.

Example 6.1. Let $(A, \mathfrak{n}, \ell)$ be any local ring and let $\mathcal{F}$ be the class of finitely generated free A-modules. Every finitely generated A-module M has an $\mathcal{F}$-cover; to construct it one takes a minimal set $x_1, \dots, x_b$ of generators of M (here $b = \beta_0^A(M)$ is the zeroth Betti number of M) and then defines $A^b \twoheadrightarrow M$ by $e_i \mapsto x_i$; see [Enochs and Jenda 2000, Theorem 5.3.3]. A minimal proper $\mathcal{F}$-resolution $\cdots \to F_1 \to F_0 \to M \to 0$ of a finitely generated A-module M is nothing but a *minimal free resolution* of M in the classical sense, that is, where each homomorphism $F_n \to F_{n-1}$ becomes zero when tensored with the residue field ℓ of A.

In this section, we are interested in cosyzygies with respect to the class MCM of maximal CM R-modules. We begin with a characterization of modules for which the first such cosyzygy is maximal CM.

Proposition 6.2. *For a finitely generated R-module M the next conditions are equivalent*:

(i) *M has an* MCM*-preenvelope whose cokernel is maximal CM*, *meaning that* $\mathrm{Cosyz}^1_{\mathsf{MCM}}(M)$ *is a maximal CM module.*

(ii) *M has a surjective* MCM*-envelope*, *that is*, $\text{min-}\mathrm{Cosyz}^1_{\mathsf{MCM}}(M) = 0$.

Proof. Evidently, (ii) implies (i). Conversely, let $\mu : M \to X$ be an MCM-preenvelope such that $C = \operatorname{Coker}\mu$ is maximal CM. Since X and $C = X/\operatorname{Im}\mu$ are maximal CM, so is $\operatorname{Im}\mu$. It follows that the corestriction $\mu : M \twoheadrightarrow \operatorname{Im}\mu$ is a surjective MCM-envelope of M. ☐

Next we give a sufficient condition for the second cosyzygy to be maximal CM.

Proposition 6.3. *Let M be a finitely generated R-module such that $M^\dagger$ is maximal CM. Then any second cosyzygy*, $\mathrm{Cosyz}^2_{\mathsf{MCM}}(M)$, *of M with respect to* MCM *is maximal CM.*

Proof. By Example 4.3, the homomorphism $\delta_M : M \to M^{\dagger\dagger}$ is an MCM-envelope of M. Set $C^1 = \text{min-}\mathrm{Cosyz}^1_{\mathsf{MCM}}(M) = \operatorname{Coker}\delta_M$. By application of the left exact functor $(-)^\dagger$, the exact sequence $M \xrightarrow{\delta_M} M^{\dagger\dagger} \longrightarrow C^1 \longrightarrow 0$ induces an exact sequence

$$0 \longrightarrow (C^1)^\dagger \longrightarrow M^{\dagger\dagger\dagger} \xrightarrow{\delta_M^\dagger} M^\dagger.$$

As $M^\dagger$ is maximal CM, the biduality homomorphism $\delta_{M^\dagger}$ is an isomorphism, and hence so is $\delta_M^\dagger$ by Lemma 3.1. It follows that $\operatorname{Hom}_R(C^1, \Omega) = (C^1)^\dagger = 0$, so [Bruns and Herzog 1993, Corollary 3.5.11(b)] implies that $\dim_R(C^1) < d$. Thus

Proposition 4.1 shows that $C^1 \to 0$ is an MCM-envelope of C^1, and therefore the minimal second cosyzygy of M with respect to MCM is zero:

$$\text{min-Cosyz}^2_{\mathsf{MCM}}(M) = \text{min-Cosyz}^1_{\mathsf{MCM}}(C^1) = \text{Coker}(C^1 \to 0) = 0.$$

Hence any second cosyzygy of M with respect to MCM must be maximal CM. □

Proof of Theorem C. First note, that if X is a maximal CM R-module, then $\text{Cosyz}^i_{\mathsf{MCM}}(X)$ is clearly maximal CM for every $i \geqslant 0$. If $n \geqslant d$, then the n-th cosyzygy of M is an $(n-d)^{\text{th}}$ cosyzygy of $\text{Cosyz}^d_{\mathsf{MCM}}(M)$, that is,

$$\text{Cosyz}^n_{\mathsf{MCM}}(M) = \text{Cosyz}^{n-d}_{\mathsf{MCM}}(\text{Cosyz}^d_{\mathsf{MCM}}(M));$$

so it suffices to argue that $\text{Cosyz}^d_{\mathsf{MCM}}(M)$ is maximal CM.

If $d = 0$, then certainly $\text{Cosyz}^0_{\mathsf{MCM}}(M) = M$ is maximal CM, since every finitely generated R-module is maximal CM over an artinian ring.

Assume that $d = 1$. By Theorem A we can take a special MCM-preenvelope $\mu : M \to X$ of M. We must show that $C^1 = \text{Cosyz}^1_{\mathsf{MCM}}(M) = \text{Coker}\,\mu$ is maximal CM. By definition, we have $\text{Ext}^1_R(C^1, Y) = 0$ for all $Y \in \mathsf{MCM}$, in particular, $\text{Ext}^1_R(C^1, \Omega) = 0$. Since Ω has injective dimension $d = 1$, we also have $\text{Ext}^i_R(-, \Omega) = 0$ for all $i > 1$, and consequently, $\text{Ext}^i_R(C^1, \Omega) = 0$ for all $i > 0$. Thus C^1 is maximal CM.

Finally, assume that $d \geqslant 2$. Let $0 \to M \to X^0 \to \cdots \to X^{d-3} \to C^{d-2} \to 0$ be part of a proper MCM-coresolution of M, where $C^{d-2} = \text{Cosyz}^{d-2}_{\mathsf{MCM}}(M)$. In the case $d = 2$, this just means that we consider the module $C^0 = \text{Cosyz}^0_{\mathsf{MCM}}(M) = M$. Since the module Ω is maximal CM, the sequence

$$0 \longrightarrow (C^{d-2})^\dagger \longrightarrow (X^{d-3})^\dagger \longrightarrow \cdots \longrightarrow (X^0)^\dagger \longrightarrow M^\dagger \longrightarrow 0$$

is exact. From Lemma 2.5 and Lemma 5.1 we derive that $\text{depth}_R (C^{d-2})^\dagger \geqslant \min\{d, \text{depth}_R M^\dagger + d - 2\} = d$, so $(C^{d-2})^\dagger = (\text{Cosyz}^{d-2}_{\mathsf{MCM}}(M))^\dagger$ is maximal CM. Proposition 6.3 now yields that

$$\text{Cosyz}^d_{\mathsf{MCM}}(M) = \text{Cosyz}^2_{\mathsf{MCM}}(\text{Cosyz}^{d-2}_{\mathsf{MCM}}(M))$$

is maximal CM, as desired. □

Dutta [1989] shows that if R is not regular, then no syzygy in the minimal free resolution of the residue field k (see Example 6.1) can contain a nonzero free direct summand. The following result has the same flavor, but its proof is easy. Actually, the proof of [Takahashi 2006, Proposition 2.6] applies to prove Proposition 6.4 as well, but since it is so short, we repeat it here.

Proposition 6.4. *Assume that every finitely generated R-module has an* MCM-*envelope* (*by Theorem A, this is the case if R is henselian*). *Let M be a finitely generated R-module and let $n \geqslant 1$ be an integer. The minimal n-th cosyzygy,* min-$\mathrm{Cosyz}^n_{\mathsf{MCM}}(M)$, *of* M *with respect to* MCM *contains no nonzero free direct summand.*

Proof. It suffices to consider the case $n=1$. Let $\mu : M \to X$ be an MCM-envelope of M, set $C = \text{min-}\mathrm{Cosyz}^1_{\mathsf{MCM}}(M) = \operatorname{Coker}\mu$, and write $\pi : X \twoheadrightarrow C$ for the canonical homomorphism. Let F be a free direct summand in C and denote by $\rho : C \twoheadrightarrow F$ the projection onto this direct summand. We have a commutative diagram

$$\begin{array}{ccccccccc} & & M & \xrightarrow{\mu} & X & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_0} & & \| & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & K & \xrightarrow{\iota} & X & \xrightarrow{\rho\pi} & F & \longrightarrow & 0, \end{array}$$

where $\iota : K \to X$ is the kernel of $\rho\pi$, and μ_0 is the corestriction of μ to K. Since F is free, the lower short exact sequence splits, so ι has a left inverse $\sigma : X \to K$. The endomorphism $\iota\sigma$ of X satisfies $\iota\sigma\mu = \iota\sigma\iota\mu_0 = \iota\mu_0 = \mu$, and since μ is an envelope, we conclude that $\iota\sigma$ is an automorphism. In particular, ι is surjective, and hence F is zero. □

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