Forum Math. **27** (2015), 2413–2452
DOI 10.1515/forum-2013-6029 **Forum Mathematicum**

K-groups for rings of finite Cohen–Macaulay type

Henrik Holm

Communicated by Andrew Ranicki

Abstract. For a local Cohen–Macaulay ring R of finite CM-type, Yoshino has applied methods of Auslander and Reiten to compute the Grothendieck group K_0 of the category $\mathsf{mod}\,R$ of finitely generated R-modules. For the same type of rings, we compute in this paper the first Quillen K-group $K_1(\mathsf{mod}\,R)$. We also describe the group homomorphism $R^* \to K_1(\mathsf{mod}\,R)$ induced by the inclusion functor $\mathsf{proj}\,R \to \mathsf{mod}\,R$ and illustrate our results with concrete examples.

Keywords. Auslander–Reiten sequence, Bass' universal determinant group, finite Cohen–Macaulay type, maximal Cohen–Macaulay module, Quillen's K-theory.

2010 Mathematics Subject Classification. 13C14, 13D15, 19B28.

1 Introduction

Throughout this introduction, R denotes a commutative noetherian local Cohen–Macaulay ring. The lower K-groups of R are known:

$$K_0(R) \cong \mathbb{Z} \quad \text{and} \quad K_1(R) \cong R^*.$$

For $n \in \{0,1\}$ the classical K-group $K_n(R)$ of the ring coincides with Quillen's K-group $K_n(\mathsf{proj}\,R)$ of the exact category of finitely generated projective R-modules; and if R is regular, then Quillen's resolution theorem shows that the inclusion functor $\mathsf{proj}\,R \to \mathsf{mod}\,R$ induces an isomorphism $K_n(\mathsf{proj}\,R) \cong K_n(\mathsf{mod}\,R)$. If R is non-regular, then these groups are usually not isomorphic. The groups $K_n(\mathsf{mod}\,R)$ are often denoted $G_n(R)$ and they are classical objects of study called the G-theory of R. A celebrated result of Quillen is that G-theory is well-behaved under (Laurent) polynomial extensions:

$$G_n(R[t]) \cong G_n(R) \quad \text{and} \quad G_n(R[t,t^{-1}]) \cong G_n(R) \oplus G_{n-1}(R).$$

Auslander and Reiten [4] and Butler [9] computed $K_0(\mathsf{mod}\,\Lambda)$ for an Artin algebra Λ of finite representation type. Using similar techniques, Yoshino [32] computed $K_0(\mathsf{mod}\,R)$ in the case where R has finite (as opposed to tame or wild) CM-type.

Theorem (Yoshino [32, Theorem (13.7)]). *Assume that R is henselian and that it has a dualizing module. If R has finite CM-type, then there is a group isomorphism*

$$\mathrm{K}_0(\mathsf{mod}\, R) \cong \operatorname{Coker} \Upsilon,$$

where $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ is the Auslander–Reiten homomorphism from Definition 2.3.

We mention that Yoshino's result is as much a contribution to algebraic K-theory as it is to the representation theory of the category $\mathsf{MCM}\, R$ of maximal Cohen–Macaulay R-modules. Indeed, the inclusion functor $\mathsf{MCM}\, R \to \mathsf{mod}\, R$ induces an isomorphism $\mathrm{K}_n(\mathsf{MCM}\, R) \cong \mathrm{K}_n(\mathsf{mod}\, R)$ for every n. The theory of maximal Cohen–Macaulay modules, which originates from algebraic geometry and integral representations of finite groups, is a highly active area of research.

In this paper, we build upon results and techniques of Auslander–Reiten [4], Bass [7], Lam [20], Leuschke [21], Quillen [24], Vaserstein [28, 29], Yoshino [32] to compute the group $\mathrm{K}_1(\mathsf{mod}\, R)$ when R has finite CM-type. Our main result is Theorem 2.12; it asserts that there is an isomorphism

$$\mathrm{K}_1(\mathsf{mod}\, R) \cong \operatorname{Aut}_R(M)_{\mathrm{ab}} / \Xi,$$

where M is any representation generator of the category of maximal Cohen–Macaulay R-modules and $\operatorname{Aut}_R(M)_{\mathrm{ab}}$ is the abelianization of its automorphism group. The subgroup Ξ is more complicated to describe; it is determined by the Auslander–Reiten sequences and defined in Definition 2.10. Observe that in contrast to $\mathrm{K}_0(\mathsf{mod}\, R)$, the group $\mathrm{K}_1(\mathsf{mod}\, R)$ is usually not finitely generated.

We also prove that if one writes $M = R \oplus M'$, then the group homomorphism $R^* \cong \mathrm{K}_1(\mathsf{proj}\, R) \to \mathrm{K}_1(\mathsf{mod}\, R)$ induced by the inclusion functor $\mathsf{proj}\, R \to \mathsf{mod}\, R$ can be identified with the map

$$\lambda\colon R^* \to \operatorname{Aut}_R(M)_{\mathrm{ab}} / \Xi \quad \text{given by} \quad r \mapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{M'} \end{pmatrix}.$$

The paper is organized as follows: In Section 2 we formulate our main result, Theorem 2.12. This theorem is not proved until Section 8, and the intermediate Section 3 (on the Gersten–Sherman transformation), Section 4 (on Auslander's and Reiten's theory for coherent pairs), Section 5 (on Vaserstein's result for semilocal rings), Section 6 (on certain equivalences of categories), and Section 7 (on Yoshino's results for the abelian category $\mathcal{Y}$) prepare the ground.

In Section 9 and Section 10 we apply our main theorem to compute the group $\mathrm{K}_1(\mathsf{mod}\, R)$ and the homomorphism $\lambda\colon R^* \to \mathrm{K}_1(\mathsf{mod}\, R)$ in some concrete examples. For example, for the simple curve singularity $R = k[\![T^2, T^3]\!]$ we obtain $\mathrm{K}_1(\mathsf{mod}\, R) \cong k[\![T]\!]^*$ and show that the homomorphism $\lambda\colon k[\![T^2, T^3]\!]^* \to k[\![T]\!]^*$

is the inclusion. It is well known that if R is artinian with residue field k, then one has $\mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$. We apply Theorem 2.12 to confirm this isomorphism for the ring $R = k[X]/(X^2)$ of dual numbers and to show that the homomorphism $\lambda\colon R^* \to k^*$ is given by $a + bX \mapsto a^2$.

We end this introduction by mentioning a related preprint [23] of Navkal. Although the present work and the paper of Navkal have been written completely independently (this fact is also pointed out in the latest version of [23]), there is a significant overlap between the two manuscripts: Navkal's main result [23, Theorem 1.2] is the existence of a long exact sequence involving the G-theory of the rings R and $\mathrm{End}_R(M)^{\mathrm{op}}$ (where M is a particular representation generator of the category of maximal Cohen–Macaulay R-modules) and the K-theory of certain division rings. In [23, Section 5], Navkal applies his main result to give some description of the group $\mathrm{K}_1(\mathsf{mod}\, R)$ for the ring $R = k[\![T^2, T^{2n+1}]\!]$ where $n \geqslant 1$. We point out that the techniques used in this paper and in Navkal's work are quite different.

2 Formulation of the main theorem

Let R be a commutative noetherian local Cohen–Macaulay ring. By $\mathsf{mod}\, R$ we denote the abelian category of finitely generated R-modules. The exact categories of finitely generated projective modules and of maximal Cohen–Macaulay modules over R are written $\mathsf{proj}\, R$ and $\mathsf{MCM}\, R$, respectively. The goal of this section is to state our main Theorem 2.12; its proof is postponed to Section 8.

Setup 2.1. Throughout this paper, $(R, \mathfrak{m}, k)$ is a commutative noetherian local Cohen–Macaulay ring satisfying the following assumptions.

(1) R is henselian.

(2) R admits a dualizing module.

(3) R has *finite CM-type*, that is, up to isomorphism, there are only finitely many non-isomorphic indecomposable maximal Cohen–Macaulay R-modules.

Note that (1) and (2) hold if R is $\mathfrak{m}$-adically complete. Since R is henselian, the category $\mathsf{mod}\, R$ is Krull–Schmidt by [32, Proposition (1.18)]; this fact will be important a number of times in this paper.

Set $M_0 = R$ and let $M_1, \dots, M_t$ be a set of representatives for the isomorphism classes of non-free indecomposable maximal Cohen–Macaulay R-modules. Let M be any *representation generator* of $\mathsf{MCM}\, R$, that is, a finitely generated R-module such that $\mathsf{add}_R M = \mathsf{MCM}\, R$ (where $\mathsf{add}_R M$ denotes the category of

R-modules that are isomorphic to a direct summand of some finite direct sum of copies of M). For example, M could be the square-free module

$$M = M_0 \oplus M_1 \oplus \cdots \oplus M_t. \tag{2.1.1}$$

We denote by $E = \mathrm{End}_R(M)$ the endomorphism ring of M.

It follows from [32, Theorem (4.22)] that R is an isolated singularity, and hence by [32, Theorem (3.2)] the category $\mathsf{MCM}\,R$ admits Auslander–Reiten sequences. Let

$$0 \to \tau(M_j) \to X_j \to M_j \to 0 \quad (1 \leqslant j \leqslant t) \tag{2.1.2}$$

be the Auslander–Reiten sequence in $\mathsf{MCM}\,R$ ending in M_j, where τ is the Auslander–Reiten translation.

Remark 2.2. The one-dimensional Cohen–Macaulay rings of finite CM-type are classified by Cimen [10, 11], Drozd and Roĭter [12], Green and Reiner [18], and Wiegand [30, 31]. The two-dimensional complete Cohen–Macaulay rings of finite CM-type that contains the complex numbers are classified by Auslander [2], Esnault [14], and Herzog [19]. They are the invariant rings $R = \mathbb{C}[\![X, Y]\!]^G$ where G is a non-trivial finite subgroup of $\mathrm{GL}_2(\mathbb{C})$. In this case, $M = \mathbb{C}[\![X, Y]\!]$ is a representation generator for $\mathsf{MCM}\,R$ which, unlike the one in (2.1.1), need not be square-free.

Definition 2.3. For each Auslander–Reiten sequence (2.1.2) we have

$$X_j \cong M_0^{n_{0j}} \oplus M_1^{n_{1j}} \oplus \cdots \oplus M_t^{n_{tj}}$$

for uniquely determined $n_{0j}, n_{1j}, \ldots, n_{tj} \geqslant 0$. Consider the element

$$\tau(M_j) + M_j - n_{0j} M_0 - n_{1j} M_1 - \cdots - n_{tj} M_t$$

in the free abelian group $\mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t$, and write this element as

$$y_{0j} M_0 + y_{1j} M_1 + \cdots + y_{tj} M_t,$$

where $y_{0j}, y_{1j}, \ldots, y_{tj} \in \mathbb{Z}$. Then define the *Auslander–Reiten matrix* Υ as the $(t+1) \times t$ matrix with entries in $\mathbb{Z}$ whose jth column is $(y_{0j}, y_{1j}, \ldots, y_{tj})$. When Υ is viewed as a homomorphism of abelian groups $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ (elements in $\mathbb{Z}^t$ and $\mathbb{Z}^{t+1}$ are viewed as column vectors), we refer to it as the *Auslander–Reiten homomorphism*.

Example 2.4. Let $R = \mathbb{C}[\![X, Y, Z]\!]/(X^3 + Y^4 + Z^2)$. Besides $M_0 = R$ there are exactly $t = 6$ non-isomorphic indecomposable maximal Cohen–Macaulay modules, and the Auslander–Reiten sequences have the following form,

$$
\begin{aligned}
&0 \to M_1 \to M_2 \to M_1 \to 0,\\
&0 \to M_2 \to M_1 \oplus M_3 \to M_2 \to 0,\\
&0 \to M_3 \to M_2 \oplus M_4 \oplus M_6 \to M_3 \to 0,\\
&0 \to M_4 \to M_3 \oplus M_5 \to M_4 \to 0,\\
&0 \to M_5 \to M_4 \to M_5 \to 0,\\
&0 \to M_6 \to M_0 \oplus M_3 \to M_6 \to 0;
\end{aligned}
$$

see [32, (13.9)]. The 7×6 Auslander–Reiten matrix Υ is therefore given by

$$
\Upsilon = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1\\
2 & -1 & 0 & 0 & 0 & 0\\
-1 & 2 & -1 & 0 & 0 & 0\\
0 & -1 & 2 & -1 & 0 & -1\\
0 & 0 & -1 & 2 & -1 & 0\\
0 & 0 & 0 & -1 & 2 & 0\\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}.
$$

In this case, the Auslander–Reiten homomorphism $\Upsilon\colon \mathbb{Z}^6 \to \mathbb{Z}^7$ is clearly injective.

One hypothesis in our main result, Theorem 2.12 below, is that the Auslander–Reiten homomorphism Υ over the ring R in question is injective. We are not aware of an example where Υ is not injective. The following lemma covers the situation of the rational double points, that is, the invariant rings $R = k[\![X, Y]\!]^G$, where k is an algebraically closed field of characteristic 0 and G is a non-trivial finite subgroup of $\mathrm{SL}_2(k)$; see [5].

Lemma 2.5. *Assume that R is complete, integrally closed, non-regular, Gorenstein, of Krull dimension 2, and that the residue field k is algebraically closed. Then the Auslander–Reiten homomorphism Υ is injective.*

Proof. Let $1 \leqslant j \leqslant t$ be given and consider the expression

$$\tau(M_j) + M_j - n_{0j} M_0 - n_{1j} M_1 - \cdots - n_{tj} M_t = y_{0j} M_0 + y_{1j} M_1 + \cdots + y_{tj} M_t$$

in the free abelian group $\mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t$, see Definition 2.3. Let Γ be the Auslander–Reiten quiver of $\mathsf{MCM}\, R$. We recall from [5, Theorem 1] that the arrows in Γ occur in pairs $\circ \rightleftarrows \circ$, and that collapsing each pair to an undirected edge gives an extended Dynkin diagram $\tilde{\Delta}$. Moreover, removing the vertex corresponding to $M_0 = R$ and any incident edges gives a Dynkin graph Δ.

Now, X_j has a direct summand M_k if and only if there is an arrow $M_k \to M_j$ in Γ. Also, the Auslander–Reiten translation τ satisfies $\tau(M_j) = M_j$ by [5, proof of Theorem 1]. Combined with the structure of the Auslander–Reiten quiver, this means that

$$y_{kj} = \begin{cases} 2 & \text{if } k = j, \\ -1 & \text{if there is an edge } M_k \text{ --- } M_j \text{ in } \tilde{\Delta}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the $t \times t$ matrix Υ_0 with $(y_{1j}, \ldots, y_{tj})$ as jth column, where $1 \leqslant j \leqslant t$, is the Cartan matrix of the Dynkin graph Δ; cf. [8, Definition 4.5.3]. This matrix is invertible by [15, Exercise (21.18)]. Deleting the first row $(y_{01}, \ldots, y_{0t})$ in the Auslander–Reiten matrix Υ, we get the invertible matrix Υ_0, and consequently, $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ determines an injective homomorphism. □

For a group G we denote by G_{ab} its *abelianization*, i.e., $G_{\mathrm{ab}} = G/[G,G]$, where $[G,G]$ is the commutator subgroup of G.

We refer to the following as the *tilde construction*. It associates to every automorphism $\alpha\colon X \to X$ of a maximal Cohen–Macaulay module X an automorphism $\tilde{\alpha}\colon M^q \to M^q$ of the smallest power q of the representation generator M such that X is a direct summand of M^q.

Construction 2.6. The chosen representation generator M for $\mathsf{MCM}\, R$ has the form $M = M_0^{m_0} \oplus \cdots \oplus M_t^{m_t}$ for uniquely determined integers $m_0, \ldots, m_t > 0$. For any module $X = M_0^{n_0} \oplus \cdots \oplus M_t^{n_t}$ in $\mathsf{MCM}\, R$, we define natural numbers,

$$q = q(X) = \min\{p \in \mathbb{N} \mid pm_j \geqslant n_j \text{ for all } 0 \leqslant j \leqslant t\},$$
$$v_j = v_j(X) = qm_j - n_j \geqslant 0,$$

and a module $Y = M_0^{v_0} \oplus \cdots \oplus M_t^{v_t}$ in $\mathsf{MCM}\, R$. Let $\psi\colon X \oplus Y \xrightarrow{\cong} M^q$ be the R-isomorphism that maps an element

$$((\underline{x}_0, \ldots, \underline{x}_t), (\underline{y}_0, \ldots, \underline{y}_t)) \in X \oplus Y = (M_0^{n_0} \oplus \cdots \oplus M_t^{n_t}) \oplus (M_0^{v_0} \oplus \cdots \oplus M_t^{v_t}),$$

where $\underline{x}_j \in M_j^{n_j}$ and $\underline{y}_j \in M_j^{v_j}$, to the element

$$((\underline{z}_{01}, \ldots, \underline{z}_{t1}), \ldots, (\underline{z}_{0q}, \ldots, \underline{z}_{tq})) \in M^q = (M_0^{m_0} \oplus \cdots \oplus M_t^{m_t})^q,$$

where $\underline{z}_{j1}, \ldots, \underline{z}_{jq} \in M_j^{m_j}$ are given by

$$(\underline{z}_{j1}, \ldots, \underline{z}_{jq}) = (\underline{x}_j, \underline{y}_j) \in M_j^{qm_j} = M_j^{n_j + v_j}.$$

Now, given α in $\mathrm{Aut}_R(X)$, we define $\tilde{\alpha}$ to be the uniquely determined element in $\mathrm{Aut}_R(M^q)$ that makes the following diagram commutative,

$$\begin{array}{ccc} X \oplus Y & \xrightarrow[\cong]{\psi} & M^q \\ {\scriptstyle \alpha\oplus 1_Y}\downarrow\cong & & \cong\downarrow{\scriptstyle \tilde{\alpha}} \\ X \oplus Y & \xrightarrow[\psi]{\cong} & M^q. \end{array}$$

The automorphism $\tilde{\alpha}$ of M^q has the form $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ for uniquely determined endomorphisms $\tilde{\alpha}_{ij}$ of M, that is, $\tilde{\alpha}_{ij} \in E = \mathrm{End}_R(M)$. Hence $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ can naturally be viewed as an invertible $q \times q$ matrix with entries in E.

Example 2.7. Let $M = M_0 \oplus \cdots \oplus M_t$ and $X = M_j$. Then $q = 1$ and

$$Y = M_0 \oplus \cdots \oplus M_{j-1} \oplus M_{j+1} \oplus \cdots \oplus M_t.$$

The isomorphism $\psi: X \oplus Y \to M$ maps

$$(x_j, (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_t)) \in X \oplus Y$$

to

$$(x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_t) \in M.$$

Therefore, for $\alpha \in \mathrm{Aut}_R(X) = \mathrm{Aut}_R(M_j)$, Construction 2.6 yields the following automorphism of M,

$$\tilde{\alpha} = \psi(\alpha \oplus 1_Y)\psi^{-1} = 1_{M_0} \oplus \cdots \oplus 1_{M_{j-1}} \oplus \alpha \oplus 1_{M_{j+1}} \oplus \cdots \oplus 1_{M_t},$$

which is an invertible 1×1 matrix with entry in $E = \mathrm{End}_R(M)$.

The following result on Auslander–Reiten sequences is quite standard. We provide a few proof details along with the appropriate references.

Proposition 2.8. *Let there be given Auslander–Reiten sequences in* MCM *R,*

$$0 \to \tau(M) \to X \to M \to 0 \quad \text{and} \quad 0 \to \tau(M') \to X' \to M' \to 0.$$

If $\alpha: M \to M'$ is a homomorphism, then there exist homomorphisms β and γ that make the following diagram commutative,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau(M) & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \tau(M') & \longrightarrow & X' & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Furthermore, if α is an isomorphism, then so are β and γ.

Proof. Write $\rho\colon X \to M$ and $\rho'\colon X' \to M'$. It suffices to prove the existence of β such that $\rho'\beta = \alpha\rho$, because then the existence of γ follows from diagram chasing.

As $0 \to \tau(M') \to X' \to M' \to 0$ is an Auslander–Reiten sequence, it suffices by [32, Lemma (2.9)] to show that $\alpha\rho\colon X \to M'$ is not a split epimorphism. Suppose that there do exist $\tau\colon M' \to X$ with $\alpha\rho\tau = 1_{M'}$. Hence α is a split epimorphism. As M is indecomposable, α must be an isomorphism. Thus

$$\rho\tau\alpha = \alpha^{-1}(\alpha\rho\tau)\alpha = 1_M,$$

which contradicts the fact that ρ is not a split epimorphism.

Finally, the fact that the maps β and γ are isomorphisms if α is so follows from [32, Lemma (2.4)]. □

The choice requested in Construction 2.9 is possible by Proposition 2.8.

Construction 2.9. Choose for each $1 \leqslant j \leqslant t$ and every $\alpha \in \mathrm{Aut}_R(M_j)$ elements $\beta_{j,\alpha} \in \mathrm{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \mathrm{Aut}_R(\tau(M_j))$ that make the next diagram commute,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau(M_j) & \longrightarrow & X_j & \longrightarrow & M_j & \longrightarrow & 0 \\ & & \cong \big\downarrow \gamma_{j,\alpha} & & \cong \big\downarrow \beta_{j,\alpha} & & \cong \big\downarrow \alpha & & \\ 0 & \longrightarrow & \tau(M_j) & \longrightarrow & X_j & \longrightarrow & M_j & \longrightarrow & 0; \end{array} \tag{2.9.1}$$

here the row(s) is the jth Auslander–Reiten sequence (2.1.2).

As shown in Lemma 5.1, the endomorphism ring $E = \mathrm{End}_R(M)$ of the chosen representation generator M is semilocal, that is, $E/\mathrm{J}(E)$ is semisimple. Thus, if the ground ring R, and hence also the endomorphism ring E, is an algebra over the residue field k and $\mathrm{char}(k) \neq 2$, then a result by Vaserstein [29, Theorem 2] yields that the canonical homomorphism $\theta_E\colon E^*_{\mathrm{ab}} \to \mathrm{K}^{\mathrm{C}}_1(E)$ is an isomorphism. Here $\mathrm{K}^{\mathrm{C}}_1(E)$ is the classical K_1-group of the ring E; see paragraph 3.1. Its inverse,

$$\theta_E^{-1} = \det{}_E\colon \mathrm{K}^{\mathrm{C}}_1(E) \to E^*_{\mathrm{ab}} = \mathrm{Aut}_R(M)_{\mathrm{ab}},$$

is called the *generalized determinant map*. The details are discussed in Section 5. We are now in a position to define the subgroup Ξ of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ that appears in our main Theorem 2.12 below.

Definition 2.10. Let $(R, \mathfrak{m}, k)$ be a ring satisfying the hypotheses in Setup 2.1. Assume, in addition, that R is an algebra over k and that one has $\mathrm{char}(k) \neq 2$. Define a subgroup Ξ of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ as follows.

- Choose for each $1 \leqslant j \leqslant t$ and each $\alpha \in \mathrm{Aut}_R(M_j)$ elements $\beta_{j,\alpha} \in \mathrm{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \mathrm{Aut}_R(\tau(M_j))$ as in Construction 2.9.

- Let $\tilde{\alpha}$, $\tilde{\beta}_{j,\alpha}$, and $\tilde{\gamma}_{j,\alpha}$ be the invertible matrices with entries in E obtained by applying the tilde Construction 2.6 to α, $\beta_{j,\alpha}$, and $\gamma_{j,\alpha}$.

Let Ξ be the subgroup of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ generated by the elements

$$(\det_E \tilde{\alpha})(\det_E \tilde{\beta}_{j,\alpha})^{-1}(\det_E \tilde{\gamma}_{j,\alpha}),$$

where j ranges over $\{1,\dots,t\}$ and α over $\mathrm{Aut}_R(M_j)$.

A priori the definition of the group Ξ involves certain choices. However, it follows from Proposition 8.8 that Ξ is actually independent of the choices made.

Remark 2.11. In specific examples it is convenient to consider the simplest possible representation generator

$$M = M_0 \oplus M_1 \oplus \cdots \oplus M_t.$$

In this case, Example 2.7 shows that $\tilde{\alpha}$ and $\tilde{\gamma}_{j,\alpha}$ are 1×1 matrices with entries in E, that is, $\tilde{\alpha}, \tilde{\gamma}_{j,\alpha} \in E^*$, and consequently $\det_E \tilde{\alpha} = \tilde{\alpha}$ and $\det_E \tilde{\gamma}_{j,\alpha} = \tilde{\gamma}_{j,\alpha}$ as elements in E^*_{ab}.

We are now in a position to state our main result.

Theorem 2.12. *Let $(R,\mathfrak{m},k)$ be a ring satisfying the hypotheses in Setup 2.1. Assume that R is an algebra over its residue field k with $\mathrm{char}(k) \neq 2$, and that the Auslander–Reiten homomorphism $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ from Definition 2.3 is injective.*

Let M be any representation generator of $\mathsf{MCM}\, R$. There is an isomorphism

$$\mathrm{K}_1(\mathsf{mod}\, R) \cong \mathrm{Aut}_R(M)_{\mathrm{ab}}/\,\Xi,$$

where Ξ is the subgroup of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ given in Definition 2.10.

Furthermore, if $\mathrm{inc}\colon \mathsf{proj}\, R \to \mathsf{mod}\, R$ is the inclusion functor and $M = R \oplus M'$, then

$$\mathrm{K}_1(\mathrm{inc})\colon \mathrm{K}_1(\mathsf{proj}\, R) \to \mathrm{K}_1(\mathsf{mod}\, R)$$

may be identified with the homomorphism

$$\lambda\colon R^* \to \mathrm{Aut}_R(M)_{\mathrm{ab}}/\,\Xi \quad \textit{given by} \quad r \mapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{M'} \end{pmatrix}.$$

As mentioned in the introduction, the proof of Theorem 2.12 spans Section 3 to Section 8. Applications and examples are presented in Sections 9 and 10. The interested reader could go ahead and read Sections 9–10 right away, since these sections are practically independent of Sections 3–8.

3 The Gersten–Sherman transformation

To prove Theorem 2.12, we need to compare and/or identify various K-groups. The relevant definitions and properties of these K-groups are recalled below. The (so-called) Gersten–Sherman transformation is our most valuable tool for comparing K-groups, and the main part of this section is devoted to this natural transformation. Readers who are familiar with K-theory may skip this section altogether.

In the following, the Grothendieck group functor is denoted by G.

3.1. Let A be a unital ring.

The *classical* K_0-*group* of A is defined as $\mathrm{K}_0^{\mathrm{C}}(A) = \mathrm{G}(\mathsf{proj}\, A)$, that is, the Grothendieck group of the category of finitely generated projective A-modules.

The *classical* K_1-*group* of A is defined as $\mathrm{K}_1^{\mathrm{C}}(A) = \mathrm{GL}(A)_{\mathrm{ab}}$, i.e., the abelianization of the infinite (or stable) general linear group; see, e.g., Bass [7, Chapter V].

3.2. Let $\mathcal{C}$ be any category. Its *loop category* $\Omega\mathcal{C}$ is the category whose objects are pairs (C,α) with $C \in \mathcal{C}$ and $\alpha \in \mathrm{Aut}_{\mathcal{C}}(C)$. A morphism $(C,\alpha) \to (C',\alpha')$ in $\Omega\mathcal{C}$ is a commutative diagram in $\mathcal{C}$,

$$\begin{array}{ccc} C & \xrightarrow{\psi} & C' \\ \alpha \big\downarrow \cong & & \cong \big\downarrow \alpha' \\ C & \xrightarrow{\psi} & C'. \end{array}$$

3.3. Let $\mathcal{C}$ be a skeletally small exact category. Its loop category $\Omega\mathcal{C}$ is also skeletally small, and it inherits a natural exact structure from $\mathcal{C}$. *Bass'* K_1-*group* (also called *Bass' universal determinant group*) of $\mathcal{C}$, which we denote by $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$, is the Grothendieck group of $\Omega\mathcal{C}$, that is, $\mathrm{G}(\Omega\mathcal{C})$, modulo the subgroup generated by all elements of the form

$$(C,\alpha) + (C,\beta) - (C,\alpha\beta),$$

where $C \in \mathcal{C}$ and $\alpha,\beta \in \mathrm{Aut}_{\mathcal{C}}(C)$; see the book of Bass [7, Chapter VIII, Section 1] or Rosenberg [25, Definition 3.1.6]. For (C,α) in $\Omega\mathcal{C}$ we denote by $[C,\alpha]$ its image in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$.

3.4. For every C in $\mathcal{C}$ one has

$$[C, 1_C] + [C, 1_C] = [C, 1_C 1_C] = [C, 1_C]$$

in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$. Consequently, $[C, 1_C]$ is the neutral element in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$.

3.5. For a unital ring A there is by [25, Theorem 3.1.7] a natural isomorphism

$$\eta_A \colon \mathrm{K}_1^{\mathrm{C}}(A) \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\operatorname{proj} A).$$

The isomorphism η_A maps $\xi \in \mathrm{GL}_n(A)$, to the class $[A^n, \xi] \in \mathrm{K}_1^{\mathrm{B}}(\operatorname{proj} A)$. Here ξ is viewed as an automorphism of the row space A^n (a free left A-module), that is, ξ acts by multiplication from the right.

The inverse map η_A^{-1} acts as follows. Let $[P, \alpha]$ be in $\mathrm{K}_1^{\mathrm{B}}(\operatorname{proj} A)$. Choose any Q in $\operatorname{proj} A$ and any isomorphism $\psi \colon P \oplus Q \to A^n$ with $n \in \mathbb{N}$. In $\mathrm{K}_1^{\mathrm{B}}(\operatorname{proj} A)$ one has

$$[P, \alpha] = [P, \alpha] + [Q, 1_Q] = [P \oplus Q, \alpha \oplus 1_Q] = [A^n, \psi(\alpha \oplus 1_Q)\psi^{-1}].$$

The automorphism $\psi(\alpha \oplus 1_Q)\psi^{-1}$ of (the row space) A^n can be identified with a matrix in $\beta \in \mathrm{GL}_n(A)$. The action of η_A^{-1} on $[P, \alpha]$ is now β's image in $\mathrm{K}_1^{\mathrm{C}}(A)$.

3.6. Quillen defines in [24] functors $\mathrm{K}_n^{\mathrm{Q}}$ from the category of skeletally small exact categories to the category of abelian groups. More precisely,

$$\mathrm{K}_n^{\mathrm{Q}}(\mathcal{C}) = \pi_{n+1}(\mathrm{BQ}\mathcal{C}, 0)$$

where Q is Quillen's Q-construction and B denotes the classifying space.

The functor $\mathrm{K}_0^{\mathrm{Q}}$ is naturally isomorphic to the Grothendieck group functor G; see [24, Section 2, Theorem 1]. For a ring A there is a natural isomorphism $\mathrm{K}_1^{\mathrm{Q}}(\operatorname{proj} A) \cong \mathrm{K}_1^{\mathrm{C}}(A)$; see for example Srinivas [27, Corollary (2.6) and Theorem (5.1)].

Gersten sketches in [17, Section 5] the construction of a natural transformation $\zeta \colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$ of functors on the category of skeletally small exact categories. The details of this construction were later given by Sherman [26, Section 3], and for this reason we refer to ζ as the *Gersten–Sherman transformation*[1]. Examples due to Gersten and Murthy [17, Propositions 5.1 and 5.2] show that for a general skeletally small exact category $\mathcal{C}$, the homomorphism $\zeta_{\mathcal{C}} \colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C})$ is neither injective nor surjective. For the exact category $\operatorname{proj} A$, where A is a ring, it is known that $\mathrm{K}_1^{\mathrm{B}}(\operatorname{proj} A)$ and $\mathrm{K}_1^{\mathrm{Q}}(\operatorname{proj} A)$ are isomorphic, indeed, they are both isomorphic to the classical K-group $\mathrm{K}_1^{\mathrm{C}}(A)$; see paragraphs 3.5 and 3.6. Therefore, a natural question arises: is $\zeta_{\operatorname{proj} A}$ an isomorphism? Sherman answers this question affirmatively in [26, pp. 231–232]; in fact, in [26, Theorem 3.3] it is proved that $\zeta_{\mathcal{C}}$ is an isomorphism for every semisimple exact category, that is, an exact category in which every short exact sequence splits. We note these results of Gersten and Sherman for later use.

[1] In the papers by Gersten [17] and Sherman [26], the functor $\mathrm{K}_1^{\mathrm{B}}$ is denoted by $\mathrm{K}_1^{\det}$.

Theorem 3.7. *There exists a natural transformation $\zeta\colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$, which we call the Gersten–Sherman transformation, of functors on the category of skeletally small exact categories such that $\zeta_{\mathsf{proj}\,A}\colon \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A) \to \mathrm{K}_1^{\mathrm{Q}}(\mathsf{proj}\,A)$ is an isomorphism for every ring A.* □

We will also need the next result on the Gersten–Sherman transformation. Recall that a *length category* is an abelian category in which every object has finite length.

Theorem 3.8. *If $\mathcal{A}$ is a skeletally small length category with only finitely many simple objects (up to isomorphism), then $\zeta_{\mathcal{A}}\colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{A}) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{A})$ is an isomorphism.*

Proof. We begin with a general observation. Given skeletally small exact categories $\mathcal{C}_1$ and $\mathcal{C}_2$, there are exact projection functors $p_j\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_j$ $(j = 1, 2)$. From the "elementary properties" of Quillen's K-groups listed in [24, Section 2], it follows that the homomorphism

$$(\mathrm{K}_1^{\mathrm{Q}}(p_1), \mathrm{K}_1^{\mathrm{Q}}(p_2))\colon \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1 \times \mathcal{C}_2) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1) \oplus \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_2)$$

is an isomorphism. A similar argument shows that $(\mathrm{K}_1^{\mathrm{B}}(p_1), \mathrm{K}_1^{\mathrm{B}}(p_2))$ is an isomorphism. Since $\zeta\colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$ is a natural transformation, it follows that $\zeta_{\mathcal{C}_1 \times \mathcal{C}_2}$ is an isomorphism if and only if $\zeta_{\mathcal{C}_1}$ and $\zeta_{\mathcal{C}_2}$ are isomorphisms.

Denote by $\mathcal{A}_{\mathrm{ss}}$ the full subcategory of $\mathcal{A}$ consisting of all semisimple objects. Note that $\mathcal{A}_{\mathrm{ss}}$ is a Serre subcategory of $\mathcal{A}$, and hence $\mathcal{A}_{\mathrm{ss}}$ is itself an abelian category. Let $i\colon \mathcal{A}_{\mathrm{ss}} \hookrightarrow \mathcal{A}$ be the (exact) inclusion and consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_1^{\mathrm{B}}(\mathcal{A}_{\mathrm{ss}}) & \xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(i)} & \mathrm{K}_1^{\mathrm{B}}(\mathcal{A}) \\
\Big\downarrow \zeta_{\mathcal{A}_{\mathrm{ss}}} & & \Big\downarrow \zeta_{\mathcal{A}} \\
\mathrm{K}_1^{\mathrm{Q}}(\mathcal{A}_{\mathrm{ss}}) & \xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{Q}}(i)} & \mathrm{K}_1^{\mathrm{Q}}(\mathcal{A}).
\end{array}$$

Since $\mathcal{A}$ is a length category, Bass' and Quillen's devissage theorems [7, Chapter VIII, Section 3, Theorem (3.4) (a)] and [24, Section 5, Theorem 4] show that $\mathrm{K}_1^{\mathrm{B}}(i)$ and $\mathrm{K}_1^{\mathrm{Q}}(i)$ are isomorphisms. Hence, it suffices to argue that $\zeta_{\mathcal{A}_{\mathrm{ss}}}$ is an isomorphism. By assumption there is a finite set $\{S_1, \dots, S_n\}$ of representatives of the isomorphism classes of simple objects in $\mathcal{A}$. Note that every object A in $\mathcal{A}_{\mathrm{ss}}$ has unique decomposition $A = S_1^{a_1} \oplus \cdots \oplus S_n^{a_n}$ where $a_1, \dots, a_n \in \mathbb{N}_0$; we used here the assumption that A has finite length to conclude that the cardinal numbers a_i must be finite. Since one has $\mathrm{Hom}_{\mathcal{A}}(S_i, S_j) = 0$ for $i \neq j$, it follows that there is an equivalence of abelian categories,

$$\mathcal{A}_{\mathrm{ss}} \simeq (\mathsf{add}\, S_1) \times \cdots \times (\mathsf{add}\, S_n).$$

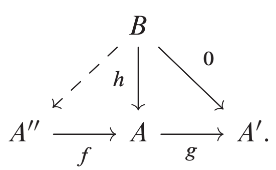
Consider the ring $D_i = \operatorname{End}_{\mathcal{A}}(S_i)^{\mathrm{op}}$. As S_i is simple, Schur's lemma gives that D_i is a division ring. It easy to see that the functor $\operatorname{Hom}_{\mathcal{A}}(S_i, -)\colon \mathcal{A} \to \mathsf{Mod}\, D_i$ induces an equivalence $\mathsf{add}\, S_i \simeq \mathsf{proj}\, D_i$. By Theorem 3.7, $\zeta_{\mathsf{proj}\, D_1}, \ldots, \zeta_{\mathsf{proj}\, D_n}$ are isomorphisms, so it follows from the equivalence above, and the general observation in the beginning af the proof, that $\zeta_{\mathcal{A}_{\mathrm{ss}}}$ is an isomorphism, as desired. □

Note that in this section, superscripts "C" (for classical), "B" (for Bass), and "Q" (for Quillen) have been used to distinguish between various K-groups. In the rest of the paper, K-groups without superscripts refer to Quillen's K-groups.

4 Coherent pairs

We recall a few results and notions from the paper [4] by Auslander and Reiten which are central in the proof of our main Theorem 2.12. Throughout this section, $\mathcal{A}$ denotes a skeletally small additive category.

Definition 4.1. A *pseudo* (or *weak*) kernel of a morphism $g\colon A \to A'$ in $\mathcal{A}$ is a morphism $f\colon A'' \to A$ in $\mathcal{A}$ such that $gf = 0$, and which satisfies that every diagram in $\mathcal{A}$ as below can be completed (but not necessarily in a unique way),



We say that $\mathcal{A}$ *has pseudo kernels* if every morphism in $\mathcal{A}$ has a pseudo kernel.

Observation 4.2. Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathcal{M}$. An $\mathcal{A}$*-precover* of an object $M \in \mathcal{M}$ is a morphism $u\colon A \to M$ with $A \in \mathcal{A}$ with the property that for every morphism $u'\colon A' \to M$ with $A' \in \mathcal{A}$ there exists a (not necessarily unique) morphism $v\colon A' \to A$ such that $uv = u'$. Following [13, Definition 5.1.1] we say that $\mathcal{A}$ is precovering (or contravariantly finite) in $\mathcal{M}$ if every object $M \in \mathcal{M}$ has an $\mathcal{A}$-precover. In this case, $\mathcal{A}$ has pseudo kernels. Indeed, if $i\colon K \to A$ is the kernel in $\mathcal{M}$ of $g\colon A \to A'$ in $\mathcal{A}$, and if $f\colon A'' \to K$ is an $\mathcal{A}$-precover of K, then $if\colon A'' \to A$ is a pseudo kernel of g.

Definition 4.3. Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Auslander–Reiten [4] call $(\mathcal{A}, \mathcal{B})$ a *coherent pair* if $\mathcal{A}$ has pseudo kernels in the sense of Definition 4.1, and $\mathcal{B}$ is precovering in $\mathcal{A}$.

If $(\mathcal{A}, \mathcal{B})$ is a coherent pair, then also $\mathcal{B}$ has pseudo kernels by [4, Proposition 1.4 (a)].

Definition 4.4. Write $\mathsf{Mod}\,\mathcal{A}$ for the abelian category of additive contravariant functors $\mathcal{A} \to \mathsf{Ab}$, where Ab is the category of abelian groups. Denote by $\mathsf{mod}\,\mathcal{A}$ the full subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of finitely presented functors.

4.5. If the category $\mathcal{A}$ has pseudo kernels, then $\mathsf{mod}\,\mathcal{A}$ is abelian, and the inclusion functor $\mathsf{mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$ is exact, see [4, Proposition 1.3].

If $(\mathcal{A}, \mathcal{B})$ is a coherent pair, see paragraph 4.3, then the exact restriction

$$\mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{B}$$

maps $\mathsf{mod}\,\mathcal{A}$ to $\mathsf{mod}\,\mathcal{B}$ by [4, Proposition 1.4 (b)]. In this case, there are functors

$$\operatorname{Ker} r \xrightarrow{i} \mathsf{mod}\,\mathcal{A} \xrightarrow{r} \mathsf{mod}\,\mathcal{B}, \tag{4.5.1}$$

where r is the restriction and i the inclusion functor. The kernel of r, that is,

$$\operatorname{Ker} r = \{F \in \mathsf{mod}\,\mathcal{A} \mid F(B) = 0 \text{ for all } B \in \mathcal{B}\},$$

is a Serre subcategory of the abelian category $\mathsf{mod}\,\mathcal{A}$. Moreover, the quotient $(\mathsf{mod}\,\mathcal{A})/(\operatorname{Ker} r)$, in the sense of Gabriel [16], is equivalent to the category $\mathsf{mod}\,\mathcal{B}$, and the canonical functor $\mathsf{mod}\,\mathcal{A} \to (\mathsf{mod}\,\mathcal{A})/(\operatorname{Ker} r)$ may be identified with r. These assertions are proved in [4, Proposition 1.5]. Therefore (4.5.1) induces by Quillen's localization theorem [24, Section 5, Theorem 5] a long exact sequence of K-groups,

$$\begin{array}{l} \cdots \longrightarrow \mathrm{K}_n(\operatorname{Ker} r) \xrightarrow{\mathrm{K}_n(i)} \mathrm{K}_n(\mathsf{mod}\,\mathcal{A}) \xrightarrow{\mathrm{K}_n(r)} \mathrm{K}_n(\mathsf{mod}\,\mathcal{B}) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{K}_0(\operatorname{Ker} r) \xrightarrow{\mathrm{K}_0(i)} \mathrm{K}_0(\mathsf{mod}\,\mathcal{A}) \xrightarrow{\mathrm{K}_0(r)} \mathrm{K}_0(\mathsf{mod}\,\mathcal{B}) \longrightarrow 0. \end{array} \tag{4.5.2}$$

5 Semilocal rings

A ring A is semilocal if $A/\mathrm{J}(A)$ is semisimple. Here $\mathrm{J}(A)$ is the Jacobson radical of A. If A is commutative, then this definition is equivalent to A having only finitely many maximal ideals; see Lam [20, Proposition (20.2)].

Lemma 5.1. *Let R be a commutative noetherian semilocal ring, and let $M \neq 0$ be a finitely generated R-module. Then the ring* $\operatorname{End}_R(M)$ *is semilocal.*

Proof. As the ring R is commutative and noetherian, $\operatorname{End}_R(M)$ is a module-finite R-algebra. Since R is semilocal, the assertion now follows from [20, Proposition (20.6)]. □

5.2. Denote by A^* the group of units in a ring A, and let $\vartheta_A\colon A^* \to \mathrm{K}_1^{\mathrm{C}}(A)$ be the composite of the group homomorphisms

$$A^* \cong \mathrm{GL}_1(A) \hookrightarrow \mathrm{GL}(A) \twoheadrightarrow \mathrm{GL}(A)_{\mathrm{ab}} = \mathrm{K}_1^{\mathrm{C}}(A). \tag{5.2.1}$$

Some authors refer to ϑ_A as the Whitehead determinant. If A is semilocal, then ϑ_A is surjective by Bass [7, Chapter V, Section 9, Theorem (9.1)]. As the group $\mathrm{K}_1^{\mathrm{C}}(A)$ is abelian, one has $[A^*, A^*] \subseteq \mathrm{Ker}\,\vartheta_A$, and we write $\theta_A\colon A^*_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{C}}(A)$ for the induced homomorphism.

Vaserstein [28] showed that the inclusion $[A^*, A^*] \subseteq \mathrm{Ker}\,\vartheta_A$ is strict for the semilocal ring $A = \mathrm{M}_2(\mathbb{F}_2)$ where $\mathbb{F}_2$ is the field with two elements. In [28, Theorem 3.6 (a)] it is shown that if A is semilocal, then $\mathrm{Ker}\,\vartheta_A$ is the subgroup of A^* generated by elements of the form $(1+ab)(1+ba)^{-1}$ where $a, b \in A$ and $1 + ab \in A^*$.

If A is semilocal, that is, $A/\mathrm{J}(A)$ is semisimple, then by the Artin–Wedderburn Theorem there is an isomorphism of rings

$$A/\mathrm{J}(A) \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_t}(D_t),$$

where $D_1, \dots, D_t$ are division rings, and $n_1, \dots, n_t$ are natural numbers all of which are uniquely determined by A. The next result is due to Vaserstein [29, Theorem 2].

Theorem 5.3. *Let A be semilocal and write $A/\mathrm{J}(A) \cong \mathrm{M}_{n_1}(D_1)\times\cdots\times\mathrm{M}_{n_t}(D_t)$. If none of the rings $\mathrm{M}_{n_i}(D_i)$ is $\mathrm{M}_2(\mathbb{F}_2)$, and at most one of the rings $\mathrm{M}_{n_i}(D_i)$ is $\mathrm{M}_1(\mathbb{F}_2) = \mathbb{F}_2$, then one has $\mathrm{Ker}\,\vartheta_A = [A^*, A^*]$. In particular, ϑ_A induces an isomorphism*

$$\theta_A\colon A^*_{\mathrm{ab}} \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{C}}(A).$$

Remark 5.4. Note that if A is a semilocal ring which is an algebra over a field k with characteristic $\neq 2$, then the hypothesis in Theorem 5.3 is satisfied.

If A is a commutative semilocal ring, then $\mathrm{Ker}\,\vartheta_A$ and the commutator subgroup $[A^*, A^*] = \{1\}$ are identical, i.e., the surjective homomorphism

$$\vartheta_A = \theta_A\colon A^* \to \mathrm{K}_1^{\mathrm{C}}(A)$$

is an isomorphism. Indeed, the determinant homomorphisms $\det_n\colon \mathrm{GL}_n(A) \to A^*$ induce a homomorphism $\det_A\colon \mathrm{K}_1^{\mathrm{C}}(A) \to A^*$ that evidently satisfies

$$\det_A \theta_A = 1_{A^*}.$$

Since θ_A is surjective, it follows that θ_A is an isomorphism with $\theta_A^{-1} = \det_A$.

Definition 5.5. Let A be a ring for which the homomorphism $\theta_A\colon A^*_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{C}}(A)$ from paragraph 5.2 is an isomorphism; for example, A could be a commutative semilocal ring or a noncommutative semilocal ring satisfying the assumptions in Theorem 5.3. The inverse θ_A^{-1} is denoted by $\det_A$, and we call it the *generalized determinant*.

Remark 5.6. Let ξ be an $m \times n$ and let χ be an $n \times p$ matrix with entries in a ring A. Denote by "$\cdot$" the product $\mathrm{M}_{m\times n}(A^{\mathrm{op}}) \times \mathrm{M}_{n\times p}(A^{\mathrm{op}}) \to \mathrm{M}_{m\times p}(A^{\mathrm{op}})$. Then

$$(\xi \cdot \chi)^T = \chi^T \xi^T,$$

where $\chi^T \xi^T$ is computed using the product $\mathrm{M}_{p\times n}(A) \times \mathrm{M}_{n\times m}(A) \to \mathrm{M}_{p\times m}(A)$. Thus, transposition $(-)^T\colon \mathrm{GL}_n(A^{\mathrm{op}}) \to \mathrm{GL}_n(A)$ is an anti-isomorphism (this is also noted in [7, Chapter V, Section 7]), which induces an isomorphism

$$(-)^T\colon \mathrm{K}_1^{\mathrm{C}}(A^{\mathrm{op}}) \to \mathrm{K}_1^{\mathrm{C}}(A).$$

Lemma 5.7. *Let A be a ring for which the generalized determinant $\det_A = \theta_A^{-1}$ exists; cf. Definition 5.5. For every invertible matrix ξ with entries in A one has an equality $\det_{A^{\mathrm{op}}}(\xi^T) = \det_A(\xi)$ in the abelian group $(A^{\mathrm{op}})^*_{\mathrm{ab}} = A^*_{\mathrm{ab}}$.*

Proof. Clearly, there is a commutative diagram

$$\begin{array}{ccc} A^*_{\mathrm{ab}} & = & (A^{\mathrm{op}})^*_{\mathrm{ab}} \\ {\scriptstyle\theta_A}\downarrow\cong & & \cong\downarrow{\scriptstyle\theta_{A^{\mathrm{op}}}} \\ \mathrm{K}_1^{\mathrm{C}}(A) & \xrightarrow[(-)^T]{\cong} & \mathrm{K}_1^{\mathrm{C}}(A^{\mathrm{op}}). \end{array}$$

It follows that one has $\theta_{A^{\mathrm{op}}}^{-1} \circ (-)^T = \theta_A^{-1}$, that is, $\det_{A^{\mathrm{op}}} \circ (-)^T = \det_A$. □

6 Some useful functors

Throughout this section, A is a ring and M is a fixed left A-module. We denote by $E = \mathrm{End}_A(M)$ the endomorphism ring of M. Note that $M = {}_{A,E}M$ has a natural left-A-left-E–bimodule structure.

6.1. There is a pair of adjoint functors

$$\mathsf{Mod}\,A \underset{-\otimes_E M}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathsf{Mod}(E^{\mathrm{op}}).$$

It is easily seen that they restrict to a pair of quasi-inverse equivalences,

$$\mathsf{add}_A M \underset{-\otimes_E M}{\overset{\operatorname{Hom}_A(M,-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{proj}(E^{\mathrm{op}}).$$

Auslander referred to this phenomenon as *projectivization*; see [6, Chapter I, Section 2].

Let $F \in \mathsf{Mod}(\mathsf{add}_A M)$, i.e., $F\colon \mathsf{add}_A M \to \mathsf{Ab}$ is a contravariant additive functor, see Definition 4.4. The compatible E-module structure on the given A-module M induces an E^{op}-module structure on the abelian group FM which is given by $z\alpha = (F\alpha)(z)$ for $\alpha \in E$ and $z \in FM$.

Proposition 6.2. *There are quasi-inverse equivalences of abelian categories*

$$\mathsf{Mod}(\mathsf{add}_A M) \underset{f_M}{\overset{e_M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{Mod}(E^{\mathrm{op}}),$$

where e_M (evaluation) and f_M (functorfication) are defined as follows,

$$e_M(F) = FM \quad \text{and} \quad f_M(Z) = Z \otimes_E \operatorname{Hom}_A(-, M)|_{\mathsf{add}_A M},$$

for F in $\mathsf{Mod}(\mathsf{add}_A M)$ and Z in $\mathsf{Mod}(E^{\mathrm{op}})$. They restrict to quasi-inverse equivalences between categories of finitely presented objects

$$\mathsf{mod}(\mathsf{add}_A M) \underset{f_M}{\overset{e_M}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{mod}(E^{\mathrm{op}}).$$

Proof. For Z in $\mathsf{Mod}(E^{\mathrm{op}})$ the canonical isomorphism

$$Z \xrightarrow{\cong} Z \otimes_E E = Z \otimes_E \operatorname{Hom}_A(M, M) = e_M f_M(Z)$$

is natural in Z. Thus, the functors $\mathrm{id}_{\mathsf{Mod}(E^{\mathrm{op}})}$ and $e_M f_M$ are naturally isomorphic. For F in $\mathsf{Mod}(\mathsf{add}_A M)$ there is a natural transformation

$$f_M e_M(F) = FM \otimes_E \operatorname{Hom}_A(-, M)|_{\mathsf{add}_A M} \xrightarrow{\delta} F; \tag{6.2.1}$$

for X in $\mathsf{add}_A M$ the homomorphism $\delta_X\colon FM \otimes_E \operatorname{Hom}_A(X, M) \to FX$ is given by $z \otimes \psi \mapsto (F\psi)(z)$. Note that δ_M is an isomorphism as it may be identified with the canonical isomorphism $FM \otimes_E {}_E E \xrightarrow{\cong} FM$ in Ab. As the functors in (6.2.1) are additive, it follows that δ_X is an isomorphism for every $X \in \mathsf{add}_A M$, that is, δ is a natural isomorphism. Since (6.2.1) is natural in F, the functors $f_M e_M$ and $\mathrm{id}_{\mathsf{Mod}(\mathsf{add}_A M)}$ are naturally isomorphic.

It is straightforward to verify that the functors e_M and f_M map finitely presented objects to finitely presented objects. □

Observation 6.3. In the case $M = A$ one has $E = \mathrm{End}_A(M) = A^{\mathrm{op}}$, and therefore Proposition 6.2 yields an equivalence $f_A \colon \mathsf{mod}\, A \to \mathsf{mod}(\mathsf{proj}\, A)$ given by

$$X \mapsto X \otimes_{A^{\mathrm{op}}} \mathrm{Hom}_A(-, A)|_{\mathsf{proj}\, A}.$$

It is easily seen that the functor f_A is naturally isomorphic to the functor given by

$$X \mapsto \mathrm{Hom}_A(-, X)|_{\mathsf{proj}\, A}.$$

We will usually identify f_A with this functor.

Definition 6.4. The functor $y_M \colon \mathsf{add}_A M \to \mathsf{mod}(\mathsf{add}_A M)$ which for $X \in \mathsf{add}_A M$ is given by $y_M(X) = \mathrm{Hom}_A(-, X)|_{\mathsf{add}_A M}$ is called the *Yoneda functor*.

Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathcal{M}$. If $\mathcal{A}$ is closed under extensions in $\mathcal{M}$, then $\mathcal{A}$ has a natural induced exact structure. However, one can always equip $\mathcal{A}$ with the *trivial exact structure*. In this structure, the "exact sequences" (sometimes called *conflations*) are only the split exact ones. When viewing $\mathcal{A}$ as an exact category with the trivial exact structure, we denote it $\mathcal{A}_0$.

Lemma 6.5. *Assume that A is commutative and noetherian and let $M \in \mathsf{mod}\, A$. Set $E = \mathrm{End}_A(M)$ and assume that E^{op} has finite global dimension. For the exact Yoneda functor $y_M \colon (\mathsf{add}_A M)_0 \to \mathsf{mod}(\mathsf{add}_A M)$, see Definition 6.4, the homomorphisms $\mathrm{K}_n(y_M)$, where $n \geqslant 0$, and $\mathrm{K}_1^{\mathrm{B}}(y_M)$ are isomorphisms.*

Proof. By application of K_n to the commutative diagram

$$\begin{array}{ccc}
(\mathsf{add}_A M)_0 & \xrightarrow[\simeq]{\mathrm{Hom}_A(M,-)} & \mathsf{proj}(E^{\mathrm{op}}) \\
{\scriptstyle y_M}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\
\mathsf{mod}(\mathsf{add}_A M) & \xrightarrow[e_M]{\simeq} & \mathsf{mod}(E^{\mathrm{op}})
\end{array}$$

it follows that $\mathrm{K}_n(y_M)$ is an isomorphism if and only if $\mathrm{K}_n(\mathrm{inc})$ is an isomorphism. The latter holds by Quillen's resolution theorem [24, Section 4, Theorem 3], since E^{op} has finite global dimension. A similar argument shows that $\mathrm{K}_1^{\mathrm{B}}(y_M)$ is an isomorphism. This time one needs to apply Bass' resolution theorem; see [7, Chapter VIII, Section 4, Theorem (4.6)]. □

Since K_0 may be identified with the Grothendieck group functor, cf. paragraph 3.6, the following result is well known. In any case, it is straightforward to verify.

Lemma 6.6. *Assume that* $\mathsf{mod}\, A$ *is Krull–Schmidt. Let* $N = N_1^{n_1} \oplus \cdots \oplus N_s^{n_s}$ *be a finitely generated* A*-module, where* $N_1, \ldots, N_s$ *are non-isomorphic indecomposable* A*-modules and* $n_1, \ldots, n_s > 0$*. The homomorphism of abelian groups*

$$\psi_N \colon \mathbb{Z}N_1 \oplus \cdots \oplus \mathbb{Z}N_s \to \mathrm{K}_0((\mathsf{add}_A N)_0)$$

given by $N_j \mapsto [N_j]$ *is an isomorphism.*

7 The abelian category $\mathcal{Y}$

By the assumptions in Setup 2.1, the ground ring R has a dualizing module. It follows from Auslander and Buchweitz [3, Theorem A] that $\mathsf{MCM}\, R$ is precovering in $\mathsf{mod}\, R$. Actually, in our case $\mathsf{MCM}\, R$ equals $\mathsf{add}_R M$ for some finitely generated R-module M (a representation generator), and it is easily seen that every category of this form is precovering in $\mathsf{mod}\, R$. By Observation 4.2 we have a coherent pair $(\mathsf{MCM}\, R, \mathsf{proj}\, R)$, which by paragraph 4.5 yields a Gabriel localization sequence

$$\mathcal{Y} = \operatorname{Ker} r \xrightarrow{i} \mathsf{mod}(\mathsf{MCM}\, R) \xrightarrow{r} \mathsf{mod}(\mathsf{proj}\, R). \tag{7.0.1}$$

Here r is the restriction functor, $\mathcal{Y} = \operatorname{Ker} r$, and i is the inclusion. Since an additive functor vanishes on $\mathsf{proj}\, R$ if and only if it vanishes on R, one has

$$\mathcal{Y} = \{F \in \mathsf{mod}(\mathsf{MCM}\, R) \mid F(R) = 0\}.$$

The following two results about the abelian category $\mathcal{Y}$ are due to Yoshino. The first result is [32, (13.7.4)]; the second is (proofs of) [32, Lemma (4.12) and Proposition (4.13)].

Theorem 7.1. *Every object in* $\mathcal{Y}$ *has finite length, i.e.,* $\mathcal{Y}$ *is a length category.*

Theorem 7.2. *Consider for* $1 \leqslant j \leqslant t$ *the Auslander–Reiten sequence* (2.1.2) *ending in* M_j*. The functor* F_j *defined by the following exact sequence in* $\mathsf{mod}(\mathsf{MCM}\, R)$*,*

$$0 \to \operatorname{Hom}_R(-, \tau(M_j)) \to \operatorname{Hom}_R(-, X_j) \to \operatorname{Hom}_R(-, M_j) \to F_j \to 0,$$

is a simple object in $\mathcal{Y}$*. Conversely, every simple functor in* $\mathcal{Y}$ *is naturally isomorphic to* F_j *for some* $1 \leqslant j \leqslant t$*.* □

Proposition 7.3. *Let* $i \colon \mathcal{Y} \to \mathsf{mod}(\mathsf{MCM}\, R)$ *be the inclusion functor from* (7.0.1) *and* $\Upsilon \colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ *be the Auslander–Reiten homomorphism; see Definition* 2.3*. The homomorphisms* $\mathrm{K}_0(i)$ *and* Υ *are isomorphic.*

Proof. We claim that the following diagram of abelian groups is commutative,

$$
\begin{array}{ccc}
\mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t & \xrightarrow{\Upsilon} & \mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t \\
 & & \cong \big\downarrow \psi_M \\
\varphi \big\downarrow \cong & & \mathrm{K}_0((\mathsf{MCM}\,R)_0) \\
 & & \cong \big\downarrow \mathrm{K}_0(y_M) \\
\mathrm{K}_0(\mathcal{Y}) & \xrightarrow{\mathrm{K}_0(i)} & \mathrm{K}_0(\mathsf{mod}(\mathsf{MCM}\,R)).
\end{array}
$$

The homomorphism φ is defined by $M_j \mapsto [F_j]$ where $F_j \in \mathcal{Y}$ is described in Theorem 7.2. From Theorems 7.1 and 7.2 and the proof of Rosenberg [25, Theorem 3.1.8 (1)] (or the proof of Theorem 3.8), it follows that φ is an isomorphism. The module M is a representation generator for $\mathsf{MCM}\,R$, see Setup 2.1, and ψ_M is the isomorphism given in Lemma 6.6. Finally, y_M is the Yoneda functor from Definition 6.4. By Leuschke [21, Theorem 6] the ring E^{op}, where $E = \mathrm{End}_R(M)$, has finite global dimension, and thus Lemma 6.5 implies that $\mathrm{K}_0(y_M)$ is an isomorphism.

From the definitions of the relevant homomorphims, it is straightforward to see that the diagram is commutative; indeed, both $\mathrm{K}_0(i)\varphi$ and $\mathrm{K}_0(y_M)\psi_M\Upsilon$ map a generator M_j to the element $[F_j] \in \mathrm{K}_0(\mathsf{mod}(\mathsf{MCM}\,R))$. □

8 Proof of the main theorem

Throughout this section, we fix the notation in Setup 2.1. Thus, R is a commutative noetherian local Cohen–Macaulay ring satisfying conditions Setup 2.1 (1)–(3), M is any representation generator of $\mathsf{MCM}\,R$, and E is its endomorphism ring.

We shall frequently make use of the Gabriel localization sequence (7.0.1), and i and r always denote the inclusion and the restriction functor in this sequence.

Remark 8.1. Let $\mathcal{C}$ be an exact category. As in the paragraph preceding Lemma 6.5, we denote by $\mathcal{C}_0$ the category $\mathcal{C}$ equipped with the trivial exact structure. Note that the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C}_0 \to \mathcal{C}$ is exact and the induced homomorphism $\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathcal{C}})\colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}_0) \to \mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$ is surjective, indeed, one has

$$\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathcal{C}})([C,\alpha]) = [C,\alpha].$$

Lemma 8.2. *Consider the restriction functor* $r\colon \mathsf{mod}(\mathsf{MCM}\,R) \to \mathsf{mod}(\mathsf{proj}\,R)$ *and the identity functor* $\mathrm{id}_{\mathsf{MCM}\,R}\colon (\mathsf{MCM}\,R)_0 \to \mathsf{MCM}\,R$. *The homomorphisms* $\mathrm{K}_1^{\mathrm{B}}(r)$ *and* $\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathsf{MCM}\,R})$ *are isomorphic, in particular,* $\mathrm{K}_1^{\mathrm{B}}(r)$ *is surjective by Remark* 8.1.

Proof. Consider the commutative diagram of exact categories and exact functors

$$\begin{array}{ccc} (\mathsf{MCM}\,R)_0 & \xrightarrow{\mathrm{id}_{\mathsf{MCM}\,R}} & \mathsf{MCM}\,R \\ & & \big\downarrow j \\ \big\downarrow y_M & & \mathsf{mod}\,R \\ & & \simeq \big\downarrow f_R \\ \mathsf{mod}(\mathsf{MCM}\,R) & \xrightarrow{r} & \mathsf{mod}(\mathsf{proj}\,R), \end{array}$$

where y_M is the Yoneda functor from Definition 6.4, j is the inclusion, and f_R is the equivalence from Observation 6.3. We will prove the lemma by arguing that the vertical functors induce isomorphisms on the level of $\mathrm{K}_1^{\mathrm{B}}$.

The ring E^{op} has finite global dimension by Leuschke [21, Theorem 6], and hence Lemma 6.5 gives that $\mathrm{K}_1^{\mathrm{B}}(y_M)$ is an isomorphism. Since f_R is an equivalence, $\mathrm{K}_1^{\mathrm{B}}(f_R)$ is obviously an isomorphism. To argue that $\mathrm{K}_1^{\mathrm{B}}(j)$ is an isomorphism, we apply Bass' resolution theorem [25, Theorem 3.1.14]. We must check that the subcategory $\mathsf{MCM}\,R$ of $\mathsf{mod}\,R$ satisfies conditions (1)–(3) of [25, Theorem 3.1.14]. Condition (1) follows as $\mathsf{MCM}\,R$ is precovering in $\mathsf{mod}\,R$. As R is Cohen–Macaulay, every module in $\mathsf{mod}\,R$ has a resolution of finite length by modules in $\mathsf{MCM}\,R$, see [32, Proposition (1.4)]; thus condition (2) holds. Condition (3) requires that $\mathsf{MCM}\,R$ is closed under kernels of epimorphisms; this is well known from, e.g., [32, Proposition (1.3)]. □

Next we show some results on the Gersten–Sherman transformation; see Section 3.

Lemma 8.3. *The map*

$$\zeta_{\mathcal{C}} \colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}) \to \mathrm{K}_1(\mathcal{C})$$

is an isomorphism for $\mathcal{C} = \mathsf{mod}(\mathsf{MCM}\,R)$.

Proof. As ζ is a natural transformation, there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}})) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}})) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(f_M)} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) \\ \big\downarrow \zeta_{\mathsf{proj}(E^{\mathrm{op}})} & & \big\downarrow \zeta_{\mathsf{mod}(E^{\mathrm{op}})} & & \big\downarrow \zeta_{\mathsf{mod}(\mathsf{MCM}\,R)} \\ \mathrm{K}_1(\mathsf{proj}(E^{\mathrm{op}})) & \xrightarrow{\mathrm{K}_1(\mathrm{inc})} & \mathrm{K}_1(\mathsf{mod}(E^{\mathrm{op}})) & \xrightarrow{\mathrm{K}_1(f_M)} & \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R)), \end{array}$$

where $f_M \colon \mathsf{mod}(E^{\mathrm{op}}) \to \mathsf{mod}(\mathsf{MCM}\,R)$ is the equivalence from Proposition 6.2 and inc is the inclusion of $\mathsf{proj}(E^{\mathrm{op}})$ into $\mathsf{mod}(E^{\mathrm{op}})$.

From Leuschke [21, Theorem 6], the noetherian ring E^{op} has finite global dimension. Hence Bass' and Quillen's resolution theorems, [7, Chapter VIII, Section 4, Theorem (4.6)] (see also Rosenberg [25, Theorem 3.1.14]) and [24, Section 4, Theorem 3], imply that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ and $\mathrm{K}_1(\mathrm{inc})$ are isomorphisms. Since f_M is an equivalence, $\mathrm{K}_1^{\mathrm{B}}(f_M)$ and $\mathrm{K}_1(f_M)$ are isomorphisms as well. Consequently, $\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}$ is an isomorphism if and only if $\zeta_{\mathsf{proj}(E^{\mathrm{op}})}$ is an isomorphism, and the latter holds by Theorem 3.7. □

The goal is to compute Quillen's K-group $\mathrm{K}_1(\mathsf{mod}\,R)$ for the ring R in question. For our proof of Theorem 2.12, it is crucial that this group can be naturally identified with Bass' K-group $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$. To put Proposition 8.4 in perspective, we remind the reader that the Gersten–Sherman transformation $\zeta_{\mathsf{mod}\,A}$ is not surjective for the ring $A = \mathbb{Z}C_2$; see [17, Proposition 5.1].

Proposition 8.4. *If the Auslander–Reiten homomorphism from Definition* 2.3 *is injective, then the following assertions hold:*

(a) *The homomorphism* $\zeta_{\mathsf{mod}\,R}\colon \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R) \to \mathrm{K}_1(\mathsf{mod}\,R)$ *is an isomorphism.*

(b) *There is an exact sequence*

$$\mathrm{K}_1^{\mathrm{B}}(\mathcal{Y}) \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(i)} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(r)} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{proj}\,R)) \longrightarrow 0.$$

Proof. The Gabriel localization sequence (7.0.1) induces by paragraph 4.5 a long exact sequence of Quillen K-groups,

$$\begin{aligned}\cdots \longrightarrow \mathrm{K}_1(\mathcal{Y}) &\xrightarrow{\mathrm{K}_1(i)} \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R)) \xrightarrow{\mathrm{K}_1(r)} \mathrm{K}_1(\mathsf{mod}(\mathsf{proj}\,R)) \\ &\longrightarrow \mathrm{K}_0(\mathcal{Y}) \xrightarrow{\mathrm{K}_0(i)} \cdots.\end{aligned}$$

By Proposition 7.3, we may identify $\mathrm{K}_0(i)$ with the Auslander–Reiten homomorphism, which is assumed to be injective. Therefore, the bottom row in the following commutative diagram of abelian groups is exact,

$$\begin{array}{ccccccc} \mathrm{K}_1^{\mathrm{B}}(\mathcal{Y}) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(i)} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(r)} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{proj}\,R)) & \longrightarrow & 0 \\ \cong\downarrow \zeta_{\mathcal{Y}} & & \cong\downarrow \zeta_{\mathsf{mod}(\mathsf{MCM}\,R)} & & \downarrow \zeta_{\mathsf{mod}(\mathsf{proj}\,R)} & & \\ \mathrm{K}_1(\mathcal{Y}) & \xrightarrow{\mathrm{K}_1(i)} & \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R)) & \xrightarrow{\mathrm{K}_1(r)} & \mathrm{K}_1(\mathsf{mod}(\mathsf{proj}\,R)) & \longrightarrow & 0. \end{array}$$

The vertical homomorphisms are given by the Gersten–Sherman transformation; see Section 3. It follows from Theorems 7.1 and 7.2 that $\mathcal{Y}$ is a length category with only finitely many simple objects; thus $\zeta_{\mathcal{Y}}$ is an isomorphism by Theorem 3.8. And $\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}$ is an isomorphism by Lemma 8.3. Since $ri = 0$, it follows that $\mathrm{K}_1^{\mathrm{B}}(r)\mathrm{K}_1^{\mathrm{B}}(i) = 0$ holds, and a diagram chase now shows that

$$\operatorname{Im} \mathrm{K}_1^{\mathrm{B}}(i) = \operatorname{Ker} \mathrm{K}_1^{\mathrm{B}}(r).$$

Furthermore $\mathrm{K}_1^{\mathrm{B}}(r)$ is surjective by Lemma 8.2. This proves part (b).

The Five Lemma now implies that $\zeta_{\mathsf{mod}(\mathsf{proj}\,R)}$ is an isomorphism. Since the category $\mathsf{mod}(\mathsf{proj}\,R)$ is equivalent to $\mathsf{mod}\,R$, see Observation 6.3, it follows that $\zeta_{\mathsf{mod}\,R}$ is an isomorphism as well. This proves (a). □

We will also need the following classical notion.

Definition 8.5. Let $\mathcal{M}$ be an abelian category, and let M be an object in $\mathcal{M}$. A *projective cover* of M is an epimorphism $\varepsilon\colon P \twoheadrightarrow M$ in $\mathcal{M}$, where P is projective, such that every endomorphism $\alpha\colon P \to P$ satisfying $\varepsilon\alpha = \varepsilon$ is an automorphism.

Lemma 8.6. *Let there be given a commutative diagram*

$$\begin{array}{ccc} P & \overset{\varepsilon}{\twoheadrightarrow} & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi} \\ P & \overset{\varepsilon}{\twoheadrightarrow} & M \end{array}$$

in an abelian category $\mathcal{M}$, where $\varepsilon\colon P \twoheadrightarrow M$ is a projective cover of M. If φ is an automorphism, then α is an automorphism.

Proof. As P is projective and ε is an epimorphism, there exists $\beta\colon P \to P$ such that $\varepsilon\beta = \varphi^{-1}\varepsilon$. By assumption one has $\varepsilon\alpha = \varphi\varepsilon$. Hence

$$\varepsilon\alpha\beta = \varphi\varepsilon\beta = \varphi\varphi^{-1}\varepsilon = \varepsilon,$$

and similarly, $\varepsilon\beta\alpha = \varepsilon$. As ε is a projective cover, we conclude that $\alpha\beta$ and $\beta\alpha$ are automorphisms of P, and thus α must be an automorphism. □

The following lemma explains the point of the tilde Construction 2.6.

Lemma 8.7. *Consider the isomorphism $\eta_{E^{\mathrm{op}}}\colon \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}}) \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$ in paragraph* 3.5. *Let $X \in \mathsf{MCM}\,R$ and $\alpha \in \operatorname{Aut}_R(X)$ be given, and $\tilde{\alpha}$ be the invertible matrix with entries in E obtained by applying Construction* 2.6 *to α. There is an equality*

$$\eta_{E^{\mathrm{op}}}(\tilde{\alpha}^T) = [\operatorname{Hom}_R(M, X), \operatorname{Hom}_R(M, \alpha)].$$

Proof. Write $(M,-)$ for $\operatorname{Hom}_R(M,-)$, and let $\psi\colon X\oplus Y \xrightarrow{\cong} M^q$ be as in Construction 2.6. The R-module isomorphism ψ induces an isomorphism of E^{op}-modules

$$(M,X)\oplus(M,Y)=(M,X\oplus Y)\xrightarrow[\cong]{(M,\psi)}(M,M^q)\cong E^q.$$

Consider the automorphism of the free E^{op}-module E^q given by

$$(M,\psi)((M,\alpha)\oplus 1_{(M,Y)})(M,\psi)^{-1}=(M,\psi(\alpha\oplus 1_Y)\psi^{-1})=(M,\tilde{\alpha}).$$

We view elements in the R-module M^q as columns and elements in E^q as rows. The isomorphism $E^q\cong(M,M^q)$ identifies a row vector $\beta=(\beta_1,\dots,\beta_q)\in E^q$ with the R-linear map $\beta^T\colon M\to M^q$ whose coordinate functions are $\beta_1,\dots,\beta_q$. Then the coordinate functions of $(M,\tilde{\alpha})(\beta^T)=\tilde{\alpha}\circ\beta^T$ are the entries in the column $\tilde{\alpha}\beta^T$, where the matrix product used is

$$\mathrm{M}_{q\times q}(E)\times\mathrm{M}_{q\times 1}(E)\to\mathrm{M}_{q\times 1}(E).$$

Thus, the action of $(M,\tilde{\alpha})$ on a row $\beta\in E^q$ is the row $(\tilde{\alpha}\beta^T)^T\in E^q$. In view of Remark 5.6 one has $(\tilde{\alpha}\beta^T)^T=\beta\cdot\tilde{\alpha}^T$, where "$\cdot$" is the product

$$\mathrm{M}_{1\times q}(E^{\mathrm{op}})\times\mathrm{M}_{q\times q}(E^{\mathrm{op}})\to\mathrm{M}_{1\times q}(E^{\mathrm{op}}).$$

Consequently, over the ring E^{op}, the automorphism $(M,\tilde{\alpha})$ of the E^{op}-module E^q acts on row vectors by multiplication with $\tilde{\alpha}^T$ from the right. These arguments show that $\eta^{-1}_{E^{\mathrm{op}}}$ applied to $[(M,X),(M,\alpha)]$ is $\tilde{\alpha}^T$; see paragraph 3.5. □

Proposition 8.8. *Suppose, in addition to the blanket assumptions for this section, that R is an algebra over its residue field k and that* $\operatorname{char}(k)\neq 2$. *Then there is a group isomorphism*

$$\sigma\colon \operatorname{Aut}_R(M)_{\mathrm{ab}}\xrightarrow{\cong}\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$$

given by

$$\alpha\mapsto[\operatorname{Hom}_R(-,M)|_{\mathsf{MCM}\,R},\operatorname{Hom}_R(-,\alpha)|_{\mathsf{MCM}\,R}].$$

Furthermore, there is an equality,

$$\sigma(\Xi)=\operatorname{Im}\mathrm{K}_1^{\mathrm{B}}(i).$$

Here Ξ is the subgroup of $\operatorname{Aut}_R(M)_{\mathrm{ab}}$ *given in Definition* 2.10, *and*

$$i\colon\mathcal{Y}\to\mathsf{mod}(\mathsf{MCM}\,R)$$

is the inclusion functor from the Gabriel localization sequence (7.0.1).

Proof. We define σ to be the composite of the following isomorphisms,

$$\begin{aligned}\mathrm{Aut}_R(M)_{\mathrm{ab}} = E^*_{\mathrm{ab}} = (E^{\mathrm{op}})^*_{\mathrm{ab}} &\xrightarrow[\cong]{\theta_{E^{\mathrm{op}}}} \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}}) \\ &\xrightarrow[\cong]{\eta_{E^{\mathrm{op}}}} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}})) \\ &\xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(\jmath)} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}})) \\ &\xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(f_M)} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)).\end{aligned} \tag{8.8.1}$$

The ring E, and hence also its opposite ring E^{op}, is semilocal by Lemma 5.1. By assumption, R is a k-algebra, and hence so is E^{op}. Thus, in view of Remark 5.4 and the assumption $\mathrm{char}(k) \neq 2$, we get the isomorphism $\theta_{E^{\mathrm{op}}}$ from Theorem 5.3. It maps $\alpha \in \mathrm{Aut}_R(M)_{\mathrm{ab}}$ to the image of the 1×1 matrix $(\alpha) \in \mathrm{GL}(E^{\mathrm{op}})$ in $\mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}})$.

The isomorphism $\eta_{E^{\mathrm{op}}}$ is described in paragraph 3.5; it maps $\xi \in \mathrm{GL}_n(E^{\mathrm{op}})$ to the class $[(E_E)^n, \xi] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$.

The third map in (8.8.1) is induced by the inclusion $\jmath\colon \mathsf{proj}(E^{\mathrm{op}}) \to \mathsf{mod}(E^{\mathrm{op}})$. By Leuschke [21, Theorem 6] the noetherian ring E^{op} has finite global dimension and hence Bass' resolution theorem [7, Chapter VIII, Section 4, Theorem (4.6)], or Rosenberg [25, Theorem 3.1.14], implies that $\mathrm{K}_1^{\mathrm{B}}(\jmath)$ is an isomorphism. It maps an element $[P, \alpha] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$ to $[P, \alpha] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))$.

The fourth and last isomorphism $\mathrm{K}_1^{\mathrm{B}}(f_M)$ in (8.8.1) is induced by the equivalence $f_M\colon \mathsf{mod}(E^{\mathrm{op}}) \to \mathsf{mod}(\mathsf{MCM}\,R)$ from Proposition 6.2.

Thus, σ is an isomorphism that maps an element $\alpha \in \mathrm{Aut}_R(M)_{\mathrm{ab}}$ to the class

$$[E_E \otimes_E \mathrm{Hom}_R(-, M)|_{\mathsf{MCM}\,R}, (\alpha\,\cdot) \otimes_E \mathrm{Hom}_R(-, M)|_{\mathsf{MCM}\,R}],$$

which is evidently the same as the class

$$[\mathrm{Hom}_R(-, M)|_{\mathsf{MCM}\,R}, \mathrm{Hom}_R(-, \alpha)|_{\mathsf{MCM}\,R}].$$

It remains to show the equality $\sigma(\Xi) = \mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i)$. By the definition (8.8.1) of σ this is tantamount to showing that $\mathrm{K}_1^{\mathrm{B}}(\jmath)\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi) = \mathrm{K}_1^{\mathrm{B}}(f_M)^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$. As e_M is a quasi-inverse of f_M, see Proposition 6.2, we have

$$\mathrm{K}_1^{\mathrm{B}}(f_M)^{-1} = \mathrm{K}_1^{\mathrm{B}}(e_M),$$

and hence we need to show the equality

$$\mathrm{K}_1^{\mathrm{B}}(\jmath)\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi) \;=\; \mathrm{K}_1^{\mathrm{B}}(e_M)(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i)). \tag{8.8.2}$$

By Definition 2.10, the group Ξ is generated by all elements of the form

$$\xi_{j,\alpha} := (\det_E \tilde{\alpha})(\det_E \tilde{\beta}_{j,\alpha})^{-1}(\det_E \tilde{\gamma}_{j,\alpha}) \in E^*_{\mathrm{ab}}$$

for $j \in \{1,\dots,t\}$ and $\alpha \in \mathrm{Aut}_R(M_j)$; here the automorphisms $\beta_{j,\alpha} \in \mathrm{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \mathrm{Aut}_R(\tau(M_j))$ are choices such that the diagram (2.9.1) is commutative. It follows from Lemma 5.7 that

$$\xi_{j,\alpha} = (\det_{E^{\mathrm{op}}} \tilde{\alpha}^T)(\det_{E^{\mathrm{op}}} \tilde{\beta}^T_{j,\alpha})^{-1}(\det_{E^{\mathrm{op}}} \tilde{\gamma}^T_{j,\alpha}) \in (E^{\mathrm{op}})^*_{\mathrm{ab}}.$$

By Definition 5.5 the homomorphism $\det_{E^{\mathrm{op}}}$ is the inverse of $\theta_{E^{\mathrm{op}}}$, and consequently the group $\theta_{E^{\mathrm{op}}}(\Xi)$ is generated by the elements

$$\xi'_{j,\alpha} := \theta_{E^{\mathrm{op}}}(\xi_{j,\alpha}) = \tilde{\alpha}^T(\tilde{\beta}^T_{j,\alpha})^{-1}\tilde{\gamma}^T_{j,\alpha} \in \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}}).$$

Thus $\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi)$ is generated by the elements

$$\xi''_{j,\alpha} := \eta_{E^{\mathrm{op}}}(\xi'_{j,\alpha}) \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}})),$$

and it follows from Lemma 8.7 that

$$\begin{aligned}\xi''_{j,\alpha} = {} & [\mathrm{Hom}_R(M,M_j),\mathrm{Hom}_R(M,\alpha)] - [\mathrm{Hom}_R(M,X_j),\mathrm{Hom}_R(M,\beta_{j,\alpha})] \\ & + [\mathrm{Hom}_R(M,\tau(M_j)),\mathrm{Hom}_R(M,\gamma_{j,\alpha})].\end{aligned}$$

Thus, the group $\mathrm{K}_1^{\mathrm{B}}(\jmath)\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi)$ on the left-hand side in (8.8.2) is generated by the elements $\mathrm{K}_1^{\mathrm{B}}(\jmath)(\xi''_{j,\alpha})$. Note that $\mathrm{K}_1^{\mathrm{B}}(\jmath)(\xi''_{j,\alpha})$ is nothing but $\xi''_{j,\alpha}$ viewed as an element in $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))$. We have reached the following conclusion:

> The group $\mathrm{K}_1^{\mathrm{B}}(\jmath)\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi)$ is generated by the elements $\xi''_{j,\alpha}$, where j ranges over $\{1,\dots,t\}$ and α over all automorphisms of M_j.

To give a useful set of generators of the group $\mathrm{K}_1^{\mathrm{B}}(e_M)(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$ on the right-hand side in (8.8.2), recall from Theorems 7.1 and 7.2 that every element in $\mathcal{Y}$ has finite length and that the simple objects in $\mathcal{Y}$ are, up to isomorphism, exactly the functors $F_1,\dots,F_t$. Thus, by [25, (proof of) Theorem 3.1.8 (2)] the group $\mathrm{K}_1^{\mathrm{B}}(\mathcal{Y})$ is generated by all elements of the form $[F_j,\varphi]$, where $j \in \{1,\dots,t\}$ and φ is an automorphism of F_j. It follows that the group $\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i)$ is generated by the elements $\mathrm{K}_1^{\mathrm{B}}(i)([F_j,\varphi])$. Note that $\mathrm{K}_1^{\mathrm{B}}(i)([F_j,\varphi])$ is nothing but $[F_j,\varphi]$ viewed as an element in $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$. By definition of the functor e_M, see Proposition 6.2, one has

$$\lambda_{j,\varphi} := \mathrm{K}_1^{\mathrm{B}}(e_M)([F_j,\varphi]) = [F_j M, \varphi_M].$$

We have reached the following conclusion:

> The group $\mathrm{K}_1^{\mathrm{B}}(e_M)(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$ is generated by the elements $\lambda_{j,\varphi}$, where j ranges over $\{1,\dots,t\}$ and φ over all automorphisms of F_j.

With the descriptions of the generators $\xi''_{j,\alpha}$ and $\lambda_{j,\varphi}$ at hand, we are now in a position to prove the identity (8.8.2).

Consider an arbitrary generator $\xi''_{j,\alpha}$ in the group $\mathrm{K}_1^{\mathrm{B}}(\jmath)\eta_{E^{\mathrm{op}}}\theta_{E^{\mathrm{op}}}(\Xi)$. Recall from Theorem 7.2 that there is an exact sequence in $\mathsf{mod}(\mathsf{MCM}\,R)$,

$$0 \to \mathrm{Hom}_R(-, \tau(M_j)) \to \mathrm{Hom}_R(-, X_j) \to \mathrm{Hom}_R(-, M_j) \to F_j \to 0.$$

Thus, the commutative diagram (2.9.1) in $\mathsf{MCM}\,R$ induces a commutative diagram in $\mathsf{mod}(\mathsf{MCM}\,R)$ with exact row(s),

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Hom}_R(-, \tau(M_j)) & \longrightarrow & \mathrm{Hom}_R(-, X_j) & \longrightarrow & \mathrm{Hom}_R(-, M_j) & \longrightarrow & F_j & \longrightarrow 0 \\
& \cong \big\downarrow \mathrm{Hom}_R(-,\gamma_{j,\alpha}) & & \cong \big\downarrow \mathrm{Hom}_R(-,\beta_{j,\alpha}) & & \cong \big\downarrow \mathrm{Hom}_R(-,\alpha) & & \cong \,\big\downarrow\, \varphi & \\
0 \longrightarrow & \mathrm{Hom}_R(-, \tau(M_j)) & \longrightarrow & \mathrm{Hom}_R(-, X_j) & \longrightarrow & \mathrm{Hom}_R(-, M_j) & \longrightarrow & F_j & \longrightarrow 0,
\end{array}$$

where φ is the uniquely determined natural endotransformation of F_j that makes this diagram commutative. Note that the map φ is an automorphism by the Five Lemma, and thus $[F_j, \varphi]$ is a well-defined element in $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$. The diagram above is an exact sequence in the loop category $\Omega(\mathsf{mod}(\mathsf{MCM}\,R))$, see paragraphs 3.2 and 3.3, so in the group $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$ there is an equality:

$$\begin{aligned}
[F_j, \varphi] = [\mathrm{Hom}_R(-, M_j), \mathrm{Hom}_R(-, \alpha)] - [\mathrm{Hom}_R(-, X_j), \mathrm{Hom}_R(-, \beta_{j,\alpha})] \\
+ [\mathrm{Hom}_R(-, \tau(M_j)), \mathrm{Hom}_R(-, \gamma_{j,\alpha})].
\end{aligned}$$

Applying the homomorphism $\mathrm{K}_1^{\mathrm{B}}(e_M)$ to this equality, we get $\lambda_{j,\varphi} = \xi''_{j,\alpha}$. These arguments show that every generator $\xi''_{j,\alpha}$ has the form $\lambda_{j,\varphi}$ for some φ, and hence the inclusion "$\subseteq$" in (8.8.2) is established.

Conversely, we shall now consider an arbitrary generator $\lambda_{j,\varphi}$ in the group $\mathrm{K}_1^{\mathrm{B}}(e_M)(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$. As the category $\mathsf{MCM}\,R$ is a Krull–Schmidt variety in the sense of Auslander [1, Chapter II, Section 2], it follows by [1, Chapter II, Proposition 2.1 (b, c)] and [1, Chapter I, Proposition 4.7] that $\mathrm{Hom}_R(-, M_j) \twoheadrightarrow F_j$ is a projective cover in $\mathsf{mod}(\mathsf{MCM}\,R)$ in the sense of Definition 8.5. In particular, φ lifts to a natural transformation ψ of $\mathrm{Hom}_R(-, M_j)$, which must be an automorphism by Lemma 8.6. Thus we have a commutative diagram in $\mathsf{mod}(\mathsf{MCM}\,R)$,

$$\begin{array}{ccc}
\mathrm{Hom}_R(-, M_j) & \longrightarrow\!\!\!\!\!\rightarrow & F_j \\
\psi \,\big\downarrow\, \cong & & \cong \big\downarrow \varphi \\
\mathrm{Hom}_R(-, M_j) & \longrightarrow\!\!\!\!\!\rightarrow & F_j.
\end{array}$$

As the Yoneda functor

$$y_M \colon \mathsf{MCM}\,R \to \mathsf{mod}(\mathsf{MCM}\,R)$$

is fully faithful, see [32, Lemma (4.3)], there exists a unique automorphism α of M_j such that $\psi = \mathrm{Hom}_R(-,\alpha)$. For this particular α, the arguments above show that $\lambda_{j,\varphi} = \xi''_{j,\alpha}$. Thus every generator $\lambda_{j,\varphi}$ has the form $\xi''_{j,\alpha}$ for some α, and hence the inclusion "$\supseteq$" in (8.8.2) holds. □

Observation 8.9. For any commutative noetherian local ring R, there is an isomorphism $\rho_R \colon R^* \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R)$ given by the composite of

$$R^* \underset{\cong}{\xrightarrow{\theta_R}} \mathrm{K}_1^{\mathrm{C}}(R) \underset{\cong}{\xrightarrow{\eta_R}} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R).$$

The first map is described in paragraph 5.2; it is an isomorphism by Srinivas [27, Example (1.6)]. The second isomorphism is discussed in paragraph 3.5. Thus, ρ_R maps $r \in R^*$ to $[R, r1_R]$.

We are finally in a position to prove the main result.

Proof of Theorem 2.12. By Proposition 8.4 we can identify $\mathrm{K}_1(\mathsf{mod}\,R)$ with the group $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$. Recall that i and r denote the inclusion and restriction functors from the localization sequence (7.0.1). By the relations that define $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$, see paragraph 3.3, there is a homomorphism $\pi_0 \colon \mathrm{Aut}_R(M) \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ given by $\alpha \mapsto [M,\alpha]$. Since $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ is abelian, π_0 induces a homomorphism π, which is displayed as the upper horizontal map in the following diagram,

$$\begin{array}{ccc} \mathrm{Aut}_R(M)_{\mathrm{ab}} & \xrightarrow{\quad\pi\quad} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R) \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \mathrm{K}_1^{\mathrm{B}}(f_R)} \\ \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(r)} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{proj}\,R)). \end{array} \tag{8.9.1}$$

Here σ is the isomorphism from Proposition 8.8, and the isomorphism $\mathrm{K}_1^{\mathrm{B}}(f_R)$ is induced by the equivalence f_R from Observation 6.3. The diagram (8.9.1) is commutative, indeed, $\mathrm{K}_1^{\mathrm{B}}(r)\sigma$ and $\mathrm{K}_1^{\mathrm{B}}(f_R)\pi$ both map $\alpha \in \mathrm{Aut}_R(M)_{\mathrm{ab}}$ to the class

$$[\mathrm{Hom}_R(-,M)|_{\mathsf{proj}\,R}, \mathrm{Hom}_R(-,\alpha)|_{\mathsf{proj}\,R}].$$

By Lemma 8.2 the homomorphism $\mathrm{K}_1^{\mathrm{B}}(r)$ is surjective, and hence so is π. Exactness of the sequence in Proposition 8.4 (b) and commutativity of the diagram (8.9.1) show that $\mathrm{Ker}\,\pi = \sigma^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$. Therefore Proposition 8.8 implies that there is an equality $\mathrm{Ker}\,\pi = \Xi$, and it follows that π induces an isomorphism

$$\widehat{\pi} \colon \mathrm{Aut}_R(M)_{\mathrm{ab}}/\,\Xi \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R).$$

This proves the first assertion in Theorem 2.12.

To prove the second assertion, let inc: $\mathsf{proj}\, R \to \mathsf{mod}\, R$ denote the inclusion functor. Note that the Gersten–Sherman transformation identifies the homomorphisms $\mathrm{K}_1(\mathrm{inc})$ and $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$; indeed $\zeta_{\mathsf{proj}\, R}$ is an isomorphism by Theorem 3.7 and $\zeta_{\mathsf{mod}\, R}$ is an isomorphism by Proposition 8.4 (a). Thus, we must show that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ can be identified with the homomorphism $\lambda\colon R^* \to \mathrm{Aut}_R(M)_{\mathrm{ab}}/\,\Xi$ given by $r \mapsto r1_R \oplus 1_{M'}$ (recall that we have written $M = R \oplus M'$). To this end, let us consider the isomorphism $\rho_R\colon R^* \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\, R)$ from Observation 8.9 given by $r \mapsto [R, r1_R]$. The fact that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ and λ are isomorphic maps now follows from the diagram

$$\begin{array}{ccc} R^* & \xrightarrow{\quad\lambda\quad} & \mathrm{Aut}_R(M)_{\mathrm{ab}}/\,\Xi \\ {\scriptstyle\rho_R}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\widehat{\pi}} \\ \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\, R) & \xrightarrow{\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})} & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\, R), \end{array}$$

which is commutative. Indeed, for $r \in R^*$ one has

$$\begin{aligned} (\widehat{\pi}\lambda)(r) &= [M, r1_R \oplus 1_{M'}] = [R, r1_R] + [M', 1_{M'}] = [R, r1_R] \\ &= (\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})\rho_R)(r), \end{aligned}$$

where the penultimate equality is by paragraph 3.4. □

9 Abelianization of automorphism groups

To apply Theorem 2.12, one must compute $\mathrm{Aut}_R(M)_{\mathrm{ab}}$, i.e., the abelianization of the automorphism group of the representation generator M. In Proposition 9.6 we compute $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ for the R-module $M = R \oplus \mathfrak{m}$, which is a representation generator for $\mathsf{MCM}\, R$ if $\mathfrak{m}$ happens to be the only non-free indecomposable maximal Cohen–Macaulay module over R. Specific examples of rings for which this is the case will be studied in Section 10. Throughout this section, A denotes any ring.

Definition 9.1. Let $N_1, \dots, N_s$ be A-modules, and set $N = N_1 \oplus \cdots \oplus N_s$. We view elements in N as column vectors.

For $\varphi \in \mathrm{Aut}_A(N_i)$ we denote by $d_i(\varphi)$ the automorphism of N which has as its diagonal $1_{N_1}, \dots, 1_{N_{i-1}}, \varphi, 1_{N_{i+1}}, \dots, 1_{N_s}$ and 0 in all other entries.

For $i \neq j$ and $\mu \in \mathrm{Hom}_A(N_j, N_i)$ we denote by $e_{ij}(\mu)$ the automorphism of N with diagonal $1_{N_1}, \dots, 1_{N_s}$, and whose only non-trivial off-diagonal entry is μ in position (i, j).

Lemma 9.2. *Let $N_1, \dots, N_s$ be A-modules and set $N = N_1 \oplus \cdots \oplus N_s$. If $2 \in A$ is a unit, $i \neq j$ and $\mu \in \mathrm{Hom}_A(N_j, N_i)$, then $e_{ij}(\mu)$ is a commutator in* $\mathrm{Aut}_A(N)$.

Proof. The commutator of φ and ψ in $\mathrm{Aut}_A(N)$ is

$$[\varphi,\psi]=\varphi\psi\varphi^{-1}\psi^{-1}.$$

It is easily verified that $e_{ij}(\mu)=[e_{ij}(\frac{\mu}{2}),d_j(-1_{N_j})]$ if $i\neq j$. □

The idea in the proof above is certainly not new. It appears, for example, already in Litoff [22, proof of Theorem 2] in the case $s=2$. Of course, if $s\geqslant 3$, then $e_{ij}(\mu)$ is a commutator even without the assumption that 2 is a unit; see, e.g., [25, Lemma 2.1.2 (c)].

Lemma 9.3. *Let X and Y be non-isomorphic A-modules with local endomorphism rings. Let $\varphi,\psi\in\mathrm{End}_A(X)$ and assume that ψ factors through Y. Then one has $\psi\notin\mathrm{Aut}_A(X)$. Furthermore, $\varphi\in\mathrm{Aut}_A(X)$ if and only if $\varphi+\psi\in\mathrm{Aut}_A(X)$.*

Proof. Write $\psi=\psi''\psi'$ with $\psi'\colon X\to Y$ and $\psi''\colon Y\to X$. If ψ is an automorphism, then ψ'' is a split epimorphism and hence an isomorphism as Y is indecomposable. This contradicts the assumption that X and Y are not isomorphic. The second assertion now follows as $\mathrm{Aut}_A(X)$ is the set of units in the local ring $\mathrm{End}_A(X)$. □

Proposition 9.4. *Let $N_1,\dots,N_s$ be pairwise non-isomorphic A-modules with local endomorphism rings. An endomorphism*

$$\alpha=(\alpha_{ij})\in\mathrm{End}_A(N_1\oplus\cdots\oplus N_s)\quad\textit{with}\quad\alpha_{ij}\in\mathrm{Hom}_A(N_j,N_i)$$

is an automorphism if and only if $\alpha_{11},\alpha_{22},\dots,\alpha_{ss}$ are automorphisms.

Furthermore, every α in $\mathrm{Aut}_A(N)$ can be written as a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$, cf. Definition 9.1.

Proof. "Only if" part: Assume that $\alpha=(\alpha_{ij})$ is an automorphism with inverse $\beta=(\beta_{ij})$ and let $i=1,\dots,s$ be given. In the local ring $\mathrm{End}_A(N_i)$ one has

$$1_{N_i}=\sum_{j=1}^{s}\alpha_{ij}\beta_{ji},$$

and hence one of the terms $\alpha_{ij}\beta_{ji}$ must be an automorphism. As $\alpha_{ij}\beta_{ji}$ is not an automorphism for $j\neq i$, see Lemma 9.3, it follows that $\alpha_{ii}\beta_{ii}$ is an automorphism. In particular, α_{ii} has a right inverse and β_{ii} has a left inverse, and since the ring $\mathrm{End}_A(N_i)$ is local, this means that α_{ii} and β_{ii} are both automorphisms.

"If" part: By induction on $s\geqslant 1$. The assertion is trivial for $s=1$. Now let $s>1$. Assume that $\alpha_{11},\alpha_{22},\dots,\alpha_{ss}$ are automorphisms. Recall the notation from Definition 9.1. By composing α with $e_{s1}(-\alpha_{s1}\alpha_{11}^{-1})\cdots e_{31}(-\alpha_{31}\alpha_{11}^{-1})e_{21}(-\alpha_{21}\alpha_{11}^{-1})$

from the left and with $e_{12}(-\alpha_{11}^{-1}\alpha_{12})e_{13}(-\alpha_{11}^{-1}\alpha_{13})\cdots e_{1s}(-\alpha_{11}^{-1}\alpha_{1s})$ from the right, one gets an endomorphism of the form

$$\alpha' = \left(\begin{array}{c|c} \alpha_{11} & 0 \\ \hline 0 & \beta \end{array}\right) = d_1(\alpha_{11})\left(\begin{array}{c|c} 1_{N_1} & 0 \\ \hline 0 & \beta \end{array}\right),$$

where $\beta \in \operatorname{End}_A(N_2 \oplus \cdots \oplus N_s)$ is an $(s-1)\times(s-1)$ matrix with diagonal entries given by $\alpha_{jj} - \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$ for $j = 2,\ldots,s$. By applying Lemma 9.3 to the situation $\varphi = \alpha_{jj} - \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$ and $\psi = \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$, it follows that the diagonal entries in β are all automorphisms. By the induction hypothesis, β is now an automorphism and can be written as a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$. Consequently, the same is true for α', and hence also for α. □

Corollary 9.5. *Assume that $2 \in A$ is a unit and let $N_1,\ldots,N_s$ be pairwise non-isomorphic A-modules with local endomorphism rings. The homomorphism*

$$\Delta\colon \operatorname{Aut}_A(N_1)\times\cdots\times\operatorname{Aut}_A(N_s) \to \operatorname{Aut}_A(N_1\oplus\cdots\oplus N_s)$$

given by $\Delta(\varphi_1,\ldots,\varphi_s) = d_1(\varphi_1)\cdots d_s(\varphi_s)$ induces a surjective homomorphism

$$\Delta_{\mathrm{ab}}\colon \operatorname{Aut}_A(N_1)_{\mathrm{ab}}\oplus\cdots\oplus\operatorname{Aut}_A(N_s)_{\mathrm{ab}} \to \operatorname{Aut}_A(N_1\oplus\cdots\oplus N_s)_{\mathrm{ab}}.$$

Proof. By Proposition 9.4 every element in $\operatorname{Aut}_A(N_1\oplus\cdots\oplus N_s)$ is a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$. As $2 \in A$ is a unit, Lemma 9.2 yields that every element of the form $e_{ij}(\cdot)$ is a commutator; thus in the group $\operatorname{Aut}_A(N_1\oplus\cdots\oplus N_s)_{\mathrm{ab}}$ every element is a product of elements of the form $d_i(\cdot)$, so Δ_{ab} is surjective. □

As noted above, Lemma 9.2, and consequently also Corollary 9.5, holds without the assumption that $2 \in A$ is a unit provided that $s \geqslant 3$.

In the following, we write $[\cdot]_{\mathfrak{m}}\colon R \twoheadrightarrow R/\mathfrak{m} = k$ for the quotient homomorphism.

Proposition 9.6. *Let $(R,\mathfrak{m},k)$ be any commutative local ring such that $2 \in R$ is a unit. Assume that $\mathfrak{m}$ is not isomorphic to R and that the endomorphism ring $\operatorname{End}_R(\mathfrak{m})$ is commutative and local. There is an isomorphism of abelian groups*

$$\delta\colon \operatorname{Aut}_R(R\oplus\mathfrak{m})_{\mathrm{ab}} \xrightarrow{\cong} k^*\oplus\operatorname{Aut}_R(\mathfrak{m})$$

given by

$$\begin{pmatrix} \alpha_{11} & \alpha_{22} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \longmapsto \big([\alpha_{11}(1)]_{\mathfrak{m}}, \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}\big).$$

Proof. First note that the image of any homomorphism $\alpha\colon \mathfrak{m} \to R$ is contained in the module $\mathfrak{m}$. Indeed if $\operatorname{Im}\alpha \not\subseteq \mathfrak{m}$, then $u = \alpha(a)$ is a unit for some $a \in \mathfrak{m}$, and thus $\alpha(u^{-1}a) = 1$. It follows that α is surjective, and hence a split epimorphism as R is free. Since $\mathfrak{m}$ is indecomposable, α must be an isomorphism, which is a contradiction.

Therefore, given an endomorphism

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \operatorname{End}_R(R \oplus \mathfrak{m}) = \begin{pmatrix} \operatorname{Hom}_R(R,R) & \operatorname{Hom}_R(\mathfrak{m},R) \\ \operatorname{Hom}_R(R,\mathfrak{m}) & \operatorname{Hom}_R(\mathfrak{m},\mathfrak{m}) \end{pmatrix},$$

we may by (co)restriction view the entries α_{ij} as elements in the endomorphism ring $\operatorname{End}_R(\mathfrak{m})$. As this ring is assumed to be commutative, the determinant map

$$\operatorname{End}_R(R \oplus \mathfrak{m}) \to \operatorname{End}_R(\mathfrak{m}) \quad \text{given by} \quad (\alpha_{ij}) \mapsto \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$$

preserves multiplication. If $(\alpha_{ij}) \in \operatorname{Aut}_R(R \oplus \mathfrak{m})$, then Proposition 9.4 implies that $\alpha_{11} \in \operatorname{Aut}_R(R)$ and $\alpha_{22} \in \operatorname{Aut}_R(\mathfrak{m})$, and thus $\alpha_{11}\alpha_{22} \in \operatorname{Aut}_R(\mathfrak{m})$. By applying Lemma 9.3 to $\varphi = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$ and $\psi = \alpha_{21}\alpha_{12}$ we get $\varphi \in \operatorname{Aut}_R(\mathfrak{m})$, and hence the determinant map is a group homomorphism

$$\operatorname{Aut}_R(R \oplus \mathfrak{m}) \to \operatorname{Aut}_R(\mathfrak{m}).$$

The map $\operatorname{Aut}_R(R \oplus \mathfrak{m}) \to k^*$ defined by $(\alpha_{ij}) \mapsto [\alpha_{11}(1)]_{\mathfrak{m}}$ is also a group homomorphism. Indeed, entry $(1,1)$ in the product $(\alpha_{ij})(\beta_{ij})$ is $\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21}$. Here α_{12} is a homomorphism $\mathfrak{m} \to R$, and hence $\alpha_{12}\beta_{21}(1) \in \mathfrak{m}$ by the arguments in the beginning of the proof. Consequently one has

$$\begin{aligned} [(\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21})(1)]_{\mathfrak{m}} &= [(\alpha_{11}\beta_{11})(1)]_{\mathfrak{m}} = [\alpha_{11}(1)\beta_{11}(1)]_{\mathfrak{m}} \\ &= [\alpha_{11}(1)]_{\mathfrak{m}}[\beta_{11}(1)]_{\mathfrak{m}}. \end{aligned}$$

These arguments and the fact that the groups k^* and $\operatorname{Aut}_R(\mathfrak{m})$ are abelian show that the map δ described in the proposition is a well-defined group homomorphism. Evidently, δ is surjective; indeed, for $[r]_{\mathfrak{m}} \in k^*$ and $\varphi \in \operatorname{Aut}_R(\mathfrak{m})$ one has

$$\delta \begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}\varphi \end{pmatrix} = ([r]_{\mathfrak{m}}, \varphi).$$

To show that δ is injective, assume that $\alpha \in \operatorname{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ with the property that $\delta(\alpha) = ([1]_{\mathfrak{m}}, 1_{\mathfrak{m}})$. By Corollary 9.5 we can assume that $\alpha = (\alpha_{ij})$ is a diagonal matrix. We write $\alpha_{11} = r1_R$ for some unit $r \in R$. Since $\delta(\alpha) = ([r]_{\mathfrak{m}}, r\alpha_{22})$, we conclude that $r \in 1 + \mathfrak{m}$ and $\alpha_{22} = r^{-1}1_{\mathfrak{m}}$, that is, α has the form

$$\alpha = \begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}1_{\mathfrak{m}} \end{pmatrix} \quad \text{with} \quad r \in 1 + \mathfrak{m}.$$

Thus, proving injectivity of δ amounts to showing that every automorphism α of the form above belongs to the commutator subgroup of $\mathrm{Aut}_R(R \oplus \mathfrak{m})$. As $r - 1 \in \mathfrak{m}$, the map $(r-1)1_R$ gives a homomorphism $R \to \mathfrak{m}$. Since one has $r(r^{-1}-1) = 1-r \in \mathfrak{m}$ and $r \notin \mathfrak{m}$, it follows that $r^{-1}-1 \in \mathfrak{m}$. Thus $(r^{-1}-1)1_R$ gives another homomorphism $R \to \mathfrak{m}$. If $\iota\colon \mathfrak{m} \hookrightarrow R$ denotes the inclusion, then one has[2]

$$\begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}1_{\mathfrak{m}} \end{pmatrix} = \begin{pmatrix} 1_R & 0 \\ (r^{-1}-1)1_R & 1_{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1_R & \iota \\ 0 & 1_{\mathfrak{m}} \end{pmatrix} \times \begin{pmatrix} 1_R & 0 \\ (r-1)1_R & 1_{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1_R & -r^{-1}\iota \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

The right-hand of this equality is a product of matrices of the form $e_{ij}(\cdot)$, and since $2 \in R$ is a unit the desired conclusion now follows from Lemma 9.2. □

10 Examples

We begin with a trivial example.

Example 10.1. If R is regular, then there are isomorphisms

$$\mathrm{K}_1(\mathsf{mod}\,R) \cong \mathrm{K}_1(\mathsf{proj}\,R) \cong \mathrm{K}_1^{\mathrm{C}}(R) \cong R^*.$$

The first isomorphism is by Quillen's resolution theorem [24, Section 4, Theorem 3], the second one is mentioned in paragraph 3.6, and the third one is well known; see, e.g., [27, Example (1.6)]. Theorem 2.12 confirms this result, indeed, as $M = R$ is a representation generator for $\mathsf{MCM}\,R = \mathsf{proj}\,R$ one has

$$\mathrm{Aut}_R(M)_{\mathrm{ab}} = R^*.$$

As there are no Auslander–Reiten sequences in this case, the subgroup Ξ is generated by the empty set, so $\Xi = 0$.

We now illustrate how Theorem 2.12 applies to compute $\mathrm{K}_1(\mathsf{mod}\,R)$ for the ring $R = k[X]/(X^2)$. The answer is well known to be k^*, indeed, for any commutative artinian local ring R with residue field k one has $\mathrm{K}_1(\mathsf{mod}\,R) \cong k^*$ by [24, Section 5, Corollary 1].

Example 10.2. Let $R = k[X]/(X^2)$ be the ring of dual numbers over a field k with $\mathrm{char}(k) \neq 2$. Denote by $\mathrm{inc}\colon \mathsf{proj}\,R \to \mathsf{mod}\,R$ the inclusion functor. The ho-

[2] The identity comes from the standard proof of Whitehead's lemma; see, e.g., [27, Lemma (1.4)].

momorphism $K_1(\mathrm{inc})$ may be identified with the map

$$\mu\colon R^* \to k^* \quad \text{given by} \quad a + bX \mapsto a^2.$$

Proof. The maximal ideal $\mathfrak{m} = (X)$ is the only non-free indecomposable maximal Cohen–Macaulay R-module, so $M = R \oplus \mathfrak{m}$ is a representation generator for $\mathsf{MCM}\, R$; see (2.1.1). There is an isomorphism $k \to \mathrm{End}_R(\mathfrak{m})$ of R-algebras given by $a \mapsto a1_{\mathfrak{m}}$, in particular, $\mathrm{End}_R(\mathfrak{m})$ is commutative. Via this isomorphism, k^* corresponds to $\mathrm{Aut}_R(\mathfrak{m})$. The Auslander–Reiten sequence ending in $\mathfrak{m}$ is

$$0 \to \mathfrak{m} \xrightarrow{\iota} R \xrightarrow{X} \mathfrak{m} \to 0,$$

where ι is the inclusion. The Auslander–Reiten homomorphism

$$\Upsilon = \begin{pmatrix} -1 \\ 2 \end{pmatrix} : \mathbb{Z} \to \mathbb{Z}^2$$

is injective, so Theorem 2.12 can be applied. Note that for every $a1_{\mathfrak{m}} \in \mathrm{Aut}_R(\mathfrak{m})$, where $a \in k^*$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m} & \xrightarrow{\iota} & R & \xrightarrow{X} & \mathfrak{m} & \longrightarrow & 0 \\ & & \cong \big\downarrow a1_{\mathfrak{m}} & & \cong \big\downarrow a1_R & & \cong \big\downarrow a1_{\mathfrak{m}} & & \\ 0 & \longrightarrow & \mathfrak{m} & \xrightarrow{\iota} & R & \xrightarrow{X} & \mathfrak{m} & \longrightarrow & 0. \end{array}$$

Applying the tilde Construction 2.6 to the automorphisms $a1_{\mathfrak{m}}$ and $a1_R$, one gets

$$\widetilde{a1_{\mathfrak{m}}} = \begin{pmatrix} 1_R & 0 \\ 0 & a1_{\mathfrak{m}} \end{pmatrix} \quad \text{and} \quad \widetilde{a1_R} = \begin{pmatrix} a1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix};$$

see Example 2.7. In view of Definition 2.10 and Remark 2.11, the subgroup Ξ of $\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ is therefore generated by all elements of the form

$$\xi_a := (\widetilde{a1_{\mathfrak{m}}})(\widetilde{a1_R})^{-1}(\widetilde{a1_{\mathfrak{m}}}) = \begin{pmatrix} a^{-1}1_R & 0 \\ 0 & a^2 1_{\mathfrak{m}} \end{pmatrix} \quad \text{where} \quad a \in k^*.$$

Denote by ω the composite of the isomorphisms

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}} \underset{\cong}{\xrightarrow{\delta}} k^* \oplus \mathrm{Aut}_R(\mathfrak{m}) \underset{\cong}{\longrightarrow} k^* \oplus k^*,$$

where δ is the isomorphism from Proposition 9.6. As $\omega(\xi_a) = (a^{-1}, a)$, we get that $\omega(\Xi) = \{(a^{-1}, a) \,|\, a \in k^*\}$ and thus ω induces the first group isomorphism below,

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\,\Xi \underset{\cong}{\xrightarrow{\omega}} (k^* \oplus k^*)/\omega(\Xi) \underset{\cong}{\xrightarrow{\chi}} k^*;$$

the second isomorphism is induced by the surjective homomorphism

$$k^* \oplus k^* \to k^* \quad \text{given by} \quad (b, a) \mapsto ba,$$

whose kernel is exactly $\omega(\Xi)$. In view of Theorem 2.12 and the isomorphisms $\overline{\omega}$ and χ above, it follows that $\mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$.

Theorem 2.12 asserts that $\mathrm{K}_1(\mathrm{inc})$ may be identified with the homomorphism

$$\lambda\colon R^* \to \mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\,\Xi \quad \text{given by} \quad r \mapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

It remains to note that the isomorphism $\chi\overline{\omega}$ identifies λ with the homomorphism μ described in the example, indeed, one has $\chi\overline{\omega}\lambda = \mu$. □

Example 10.2 shows that for $R = k[X]/(X^2)$ the canonical homomorphism

$$R^* \cong \mathrm{K}_1(\mathsf{proj}\, R) \xrightarrow{\mathrm{K}_1(\mathrm{inc})} \mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$$

is not an isomorphism. It turns out that if k is algebraically closed with characteristic zero, then there exists a non-canonical isomorphism between R^* and k^*.

Proposition 10.3. *Let $R = k[X]/(X^2)$ where k is an algebraically closed field with characteristic $p \geqslant 0$. The following assertions hold.*

(a) *If $p > 0$, then the groups R^* and k^* are not isomorphic.*

(b) *If $p = 0$, then there exists a (non-canonical) group isomorphism $R^* \cong k^*$.*

Proof. There is a group isomorphism $R^* \to k^* \oplus k^+$ given by $a + bX \mapsto (a, b/a)$ where k^+ denotes the underlying abelian group of the field k.

(a) Let $\varphi = (\varphi_1, \varphi_2)\colon k^* \to k^* \oplus k^+$ be any group homomorphism. As k is algebraically closed, every element in $x \in k^*$ has the form $x = y^p$ for some $y \in k^*$. Therefore

$$\varphi(x) = \varphi(y^p) = \varphi(y)^p = (\varphi_1(y), \varphi_2(y))^p = (\varphi_1(y)^p, p\varphi_2(y)) = (\varphi_1(x), 0),$$

which shows that φ is not surjective.

(b) Since $p = 0$, the abelian group k^+ is divisible and torsion free. Therefore $k^+ \cong \mathbb{Q}^{(I)}$ for some index set I. There exist algebraic field extensions of $\mathbb{Q}$ of any finite degree, and these are all contained in the algebraically closed field k. Thus $|I| = \dim_{\mathbb{Q}} k$ must be infinite, and it follows that $|I| = |k|$.

The abelian group k^* is also divisible, but it has torsion. Write

$$k^* \cong T \oplus (k^*/T),$$

where $T = \{x \in k^* \mid \exists\, n \in \mathbb{N}\colon x^n = 1\}$ is the torsion subgroup of k^*. For the divisible torsion free abelian group k^*/T one has $k^*/T \cong \mathbb{Q}^{(J)}$ for some index set J. It is not hard to see that $|J|$ must be infinite, and hence $|J| = |k^*/T|$. As $|T| = \aleph_0$, it follows that $|k| = |k^*| = \aleph_0 + |J| = |J|$.

Since $|J| = |k| = |I|$, one gets

$$k^* \cong T \oplus \mathbb{Q}^{(J)} \cong T \oplus \mathbb{Q}^{(J)} \oplus \mathbb{Q}^{(I)} \cong k^* \oplus k^+. \qquad \square$$

The artinian ring $R = k[X]/(X^2)$ from Example 10.2 has length $\ell = 2$ and this power is also involved in the description of the homomorphism $\mu = \mathrm{K}_1(\mathrm{inc})$. The next result shows that this is no coincidence. As Proposition 10.4 might be well known to experts, and since we do not really need it, we do not give a proof.

Proposition 10.4. *Let $(R, \mathfrak{m}, k)$ be a commutative artinian local ring of length ℓ. The group homomorphism $R^* \cong \mathrm{K}_1(\mathsf{proj}\, R) \to \mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$ induced by the inclusion* $\mathrm{inc}\colon \mathsf{proj}\, R \to \mathsf{mod}\, R$ *is the composition of the homomorphisms*

$$R^* \xrightarrow{\ \pi\ } k^* \xrightarrow{\ (\cdot)^\ell\ } k^*,$$

where $\pi\colon R \twoheadrightarrow R/\mathfrak{m} = k$ is the canonical quotient map and $(\cdot)^\ell$ is the ℓth power.

Our next example is a non-artinian ring, namely the simple curve singularity of type (A_2) studied by, e.g., Herzog [19, Satz 1.6] and Yoshino [32, Proposition (5.11)].

Example 10.5. Let $R = k[\![T^2, T^3]\!]$ where k is an algebraically closed field with $\mathrm{char}(k) \neq 2$. Denote by $\mathrm{inc}\colon \mathsf{proj}\, R \to \mathsf{mod}\, R$ the inclusion functor. The homomorphism $\mathrm{K}_1(\mathrm{inc})$ may be identified with the inclusion map

$$\mu\colon R^* = k[\![T^2, T^3]\!]^* \hookrightarrow k[\![T]\!]^*.$$

Proof. The maximal ideal $\mathfrak{m} = (T^2, T^3)$ is the only non-free indecomposable maximal Cohen–Macaulay R-module, so $M = R \oplus \mathfrak{m}$ is a representation generator for $\mathsf{MCM}\, R$; see (2.1.1). Even though T is not an element in $R = k[\![T^2, T^3]\!]$, multiplication by T is a well-defined endomorphism of $\mathfrak{m}$. Thus there is a ring homomorphism

$$\chi\colon k[\![T]\!] \to \mathrm{End}_R(\mathfrak{m}) \quad \text{given by} \quad h \mapsto h1_{\mathfrak{m}}.$$

It is not hard to see that χ is injective. To prove that it is surjective, i.e., that one has $\mathrm{End}_R(\mathfrak{m})/k[\![T]\!] = 0$, note that there is a short exact sequence of R-modules,

$$0 \to k[\![T]\!]/R \to \mathrm{End}_R(\mathfrak{m})/R \to \mathrm{End}_R(\mathfrak{m})/k[\![T]\!] \to 0.$$

To see that

$$\mathrm{End}_R(\mathfrak{m})/k[\![T]\!] = 0,$$

it suffices to argue that the R-module $\mathrm{End}_R(\mathfrak{m})/R$ is simple. As noted in the beginning of the proof of Proposition 9.6, the inclusion $\mathfrak{m} \hookrightarrow R$ induces an isomorphism $\mathrm{End}_R(\mathfrak{m}) \cong \mathrm{Hom}_R(\mathfrak{m}, R)$, so by applying $\mathrm{Hom}_R(-, R)$ to the short exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, it follows that

$$\mathrm{End}_R(\mathfrak{m})/R \cong \mathrm{Ext}^1_R(k, R).$$

The latter module is isomorphic to k since R is a one-dimensional Gorenstein ring.

Note that via the isomorphism χ, the group $k[\![T]\!]^*$ corresponds to $\mathrm{Aut}_R(\mathfrak{m})$.

The Auslander–Reiten sequence ending in $\mathfrak{m}$ is

$$0 \longrightarrow \mathfrak{m} \xrightarrow{(1\ \ -T)^t} R \oplus \mathfrak{m} \xrightarrow{(T^2\ \ T)} \mathfrak{m} \longrightarrow 0.$$

Since the Auslander–Reiten homomorphism $\Upsilon = \binom{-1}{1}\colon \mathbb{Z} \to \mathbb{Z}^2$ is injective, Theorem 2.12 can be applied. We regard elements in $R \oplus \mathfrak{m}$ as column vectors. Let $\alpha = h1_{\mathfrak{m}} \in \mathrm{Aut}_R(\mathfrak{m})$, where $h \in k[\![T]\!]^*$, be given. Write $h = f + gT$ for some $f \in R^*$ and $g \in R$. It is straightforward to verify that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m} & \xrightarrow{(1\ \ -T)^t} & R \oplus \mathfrak{m} & \xrightarrow{(T^2\ \ T)} & \mathfrak{m} & \longrightarrow & 0 \\
 & & \cong \big\downarrow \gamma = (f - gT)1_{\mathfrak{m}} & & \cong \big\downarrow \beta = \begin{pmatrix} f & g \\ gT^2 & f \end{pmatrix} & & \cong \big\downarrow \alpha = (f + gT)1_{\mathfrak{m}} & & \\
0 & \longrightarrow & \mathfrak{m} & \xrightarrow[(1\ \ -T)^t]{} & R \oplus \mathfrak{m} & \xrightarrow[(T^2\ \ T)]{} & \mathfrak{m} & \longrightarrow & 0.
\end{array}$$

Note that β really is an automorphism; indeed, its inverse is given by

$$\beta^{-1} = (f^2 - g^2T^2)^{-1} \begin{pmatrix} f & -g \\ -gT^2 & f \end{pmatrix}.$$

We now apply the tilde Construction 2.6 to α, β, and γ; by Example 2.7 we get

$$\tilde{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & f + gT \end{pmatrix}, \quad \tilde{\beta} = \beta, \quad \text{and} \quad \tilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & f - gT \end{pmatrix}.$$

In view of Definition 2.10 and Remark 2.11, the subgroup Ξ of $\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ is therefore generated by all the elements

$$\xi_h := \tilde{\alpha}\tilde{\beta}^{-1}\tilde{\gamma} = (f^2 - g^2T^2)^{-1} \begin{pmatrix} f & -g(f - gT) \\ -gT^2(f + gT) & f(f^2 - g^2T^2) \end{pmatrix}.$$

Denote by ω the composite of the isomorphisms,

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}} \xrightarrow[\cong]{\delta} k^* \oplus \mathrm{Aut}_R(\mathfrak{m}) \xrightarrow[\cong]{1 \oplus \chi^{-1}} k^* \oplus k[\![T]\!]^*,$$

where δ is the isomorphism from Proposition 9.6. Note that

$$\delta(\xi_h) = ([f]_{\mathfrak{m}}, 1_{\mathfrak{m}}) = (h(0), 1_{\mathfrak{m}})$$

and hence $\omega(\xi_h) = (h(0), 1)$. It follows that $\omega(\Xi) = k^* \oplus \{1\}$ and thus ω induces a group isomorphism

$$\overline{\omega}\colon \mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\,\Xi \xrightarrow{\cong} (k^* \oplus k[\![T]\!]^*)/\omega(\Xi) = k[\![T]\!]^*.$$

In view of this isomorphism, Theorem 2.12 shows that $\mathrm{K}_1(\mathsf{mod}\,R) \cong k[\![T]\!]^*$. Theorem 2.12 also asserts that $\mathrm{K}_1(\mathrm{inc})$ may be identified with the homomorphism

$$\lambda\colon R^* \to \mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\,\Xi \quad \text{given by} \quad f \mapsto \begin{pmatrix} f1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

It remains to note that the isomorphism $\overline{\omega}$ identifies λ with the inclusion map μ described in the example, indeed, one has $\overline{\omega}\lambda = \mu$. □

Acknowledgments. It is a pleasure to thank Peter Jørgensen without whom this paper could not have been written. However, Peter did not wish to be a coauthor of this manuscript. We are also grateful to Marcel Bökstedt and Charles A. Weibel for valuable input on the Gersten–Sherman transformation, and to Christian U. Jensen for pointing out Proposition 10.3. Finally, we thank Viraj Navkal and the anonymous referee for their thoughtful comments and for making us aware of the paper [23].

Bibliography

[1] M. Auslander, Representation theory of Artin algebras. I, II, *Comm. Algebra* **1** (1974), 177–268, *ibid.* **1** (1974), 269–310.

[2] M. Auslander, Rational singularities and almost split sequences, *Trans. Amer. Math. Soc.* **293** (1986), no. 2, 511–531.

[3] M. Auslander and R.-O. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, *Mém. Soc. Math. France (N.S.)* **38** (1989), 5–37.

[4] M. Auslander and I. Reiten, Grothendieck groups of algebras and orders, *J. Pure Appl. Algebra* **39** (1986), no. 1–2, 1–51.

[5] M. Auslander and I. Reiten, Almost split sequences for rational double points, *Trans. Amer. Math. Soc.* **302** (1987), no. 1, 87–97.

[6] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge University Press, Cambridge, 1995.

[7] H. Bass, *Algebraic K-Theory*, W. A. Benjamin, New York, 1968.

[8] D. J. Benson, *Representations and Cohomology. I*, 2nd ed., Cambridge Stud. Adv. Math. 30, Cambridge University Press, Cambridge, 1998.

[9] M. C. R. Butler, Grothendieck groups and almost split sequences, in: *Integral Representations and Applications* (Oberwolfach 1980), Lecture Notes in Math. 882, Springer-Verlag, Berlin (1981), 357–368.

[10] N. Cimen, *One-dimensional rings of finite Cohen–Macaulay type*, ProQuest LLC, Ann Arbor, Ph.D. thesis, The University of Nebraska, Lincoln, 1994.

[11] N. Cimen, One-dimensional rings of finite Cohen–Macaulay type, *J. Pure Appl. Algebra* **132** (1998), no. 3, 275–308.

[12] J. A. Drozd and A. V. Roĭter, Commutative rings with a finite number of indecomposable integral representations, *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967), 783–798.

[13] E. E. Enochs and P. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Exp. Math. 30, Walter de Gruyter, Berlin, 2000.

[14] H. Esnault, Reflexive modules on quotient surface singularities, *J. Reine Angew. Math.* **362** (1985), 63–71.

[15] W. Fulton and J. Harris, *Representation Theory*, Grad. Texts in Math. 129, Springer-Verlag, New York, 1991.

[16] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[17] S. M. Gersten, Higher K-theory of rings, in: *Algebraic K-Theory. I: Higher K-Theories* (Seattle 1972), Lecture Notes in Math. 341, Springer-Verlag, Berlin (1973), 3–42.

[18] E. L. Green and I. Reiner, Integral representations and diagrams, *Michigan Math. J.* **25** (1978), no. 1, 53–84.

[19] J. Herzog, Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen–Macaulay-Moduln, *Math. Ann.* **233** (1978), no. 1, 21–34.

[20] T.-Y. Lam, *A First Course in Noncommutative Rings*, 2nd ed., Grad. Texts in Math. 131, Springer-Verlag, New York, 2001.

[21] G. J. Leuschke, Endomorphism rings of finite global dimension, *Canad. J. Math.* **59** (2007), no. 2, 332–342.

[22] O. Litoff, On the commutator subgroup of the general linear group, *Proc. Amer. Math. Soc.* **6** (1955), 465–470.

[23] V. Navkal, K'-theory of a local ring of finite Cohen–Macaulay type, preprint (2012), `http://arxiv.org/abs/1108.2000v2`; see `http://www.math.ucla.edu/~viraj/` for the latest version.

[24] D. Quillen, Higher algebraic K-theory. I, in: *Algebraic K-Theory. I: Higher K-Theories* (Seattle 1972), Lecture Notes in Math. 341, Springer-Verlag, Berlin (1973), 85–147.

[25] J. Rosenberg, *Algebraic K-Theory and Its Applications*, Grad. Texts in Math. 147, Springer-Verlag, New York, 1994.

[26] C. Sherman, Group representations and algebraic K-theory, in: *Algebraic K-Theory. Part I* (Oberwolfach 1980), Lecture Notes in Math. 966, Springer-Verlag, Berlin (1982), 208–243.

[27] V. Srinivas, *Algebraic K-Theory*, 2nd ed., Progr. Math. 90, Birkhäuser-Verlag, Boston, 1996.

[28] L. N. Vaserstein, On the stabilization of the general linear group over a ring, *Math. USSR-Sb.* **8** (1969), 383–400.

[29] L. N. Vaserstein, On the Whitehead determinant for semi-local rings, *J. Algebra* **283** (2005), no. 2, 690–699.

[30] R. Wiegand, Noetherian rings of bounded representation type, in: *Commutative Algebra* (Berkeley 1987), Math. Sci. Res. Inst. Publ. 15, Springer-Verlag, New York (1989), 497–516.

[31] R. Wiegand, One-dimensional local rings with finite Cohen-Macaulay type, in: *Algebraic Geometry and Its Applications* (West Lafayette 1990), Springer-Verlag, New York (1994), 381–389.

[32] Y. Yoshino, *Cohen–Macaulay Modules over Cohen–Macaulay Rings*, London Math. Soc. Lecture Note Ser. 146, Cambridge University Press, Cambridge, 1990.

Received February 28, 2013; revised July 4, 2013.

Author information

Henrik Holm, Department of Mathematical Sciences, Faculty of Science,
University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark.
E-mail: holm@math.ku.dk