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Construction of totally reflexive modules from an exact pair of zero divisors

Henrik Holm

Abstract

Let A be a local ring that admits an exact pair x, y of zero divisors as defined by Henriques and Şega. Assuming that this pair is orthogonal and that there exists a regular element on the A-module $A/(x, y)$, we explicitly construct an infinite family of non-isomorphic indecomposable A-modules whose minimal free resolutions are periodic of period 2, and which are totally reflexive. In this setting, our construction provides an answer to a question by Christensen, Piepmeyer, Striuli, and Takahashi. Furthermore, we compute the module of homomorphisms between any two given modules from the infinite family mentioned above.

1. *Introduction*

Throughout this paper, A is a commutative noetherian local ring.

This paper is concerned with explicit constructions of indecomposable A-modules whose minimal free resolutions are periodic. More precisely, assume that there exist non-units $x, y \in A$ such that $\mathrm{Ann}_A(x) = (y)$ and $\mathrm{Ann}_A(y) = (x)$. Then

$$\cdots \xrightarrow{y} A \xrightarrow{x} A \xrightarrow{y} A \xrightarrow{x} A \longrightarrow 0,$$

is a (non-augmented) periodic minimal free resolution of the indecomposable module $A/(x)$. In general, $A/(x)$ may be the only non-free indecomposable A-module having a periodic minimal free resolution; for example, this is the case if A is the ring of dual numbers $k[X]/(X^2)$, and $x = y \in A$ denotes the coset of $X \in k[X]$ with respect to the ideal (X^2). The main result of this paper implies that under additional assumptions on the pair x, y, it is possible to find infinitely many non-isomorphic indecomposable A-modules whose minimal free resolutions are periodic.

Given x and y as above, we construct in Section 3, for each element $a \in A$, two A-modules G_a and H_a whose minimal free resolutions are periodic of period 2. Our construction is quite elementary, and it makes explicit computations of relevant Hom-modules possible. A special case of our main result, Corollary 5.5, is given below; it applies to large classes of examples; cf. Example 2.4 and Lemma 2.5.

Theorem. *Let $x, y \in A$ be non-units that satisfy the conditions $\mathrm{Ann}_A(x) = (y)$, $\mathrm{Ann}_A(y) = (x)$, and $(x) \cap (y) = 0$, and let $a \in A$ be an element that is regular on the A-module $A/(x, y)$. Then*

$$G_a, G_{a^2}, G_{a^3}, \ldots, H_a, H_{a^2}, H_{a^3}, \ldots$$

is a (double) infinite family of non-isomorphic indecomposable A-modules whose minimal free resolutions are periodic of period 2. Furthermore, one has

$$\mathrm{Hom}_A(H_{a^m}, G_{a^n}) \cong G_{a^{m+n}} \quad \text{for all} \quad m, n \geqslant 1,$$
$$\mathrm{Hom}_A(H_{a^n}, H_{a^m}) \cong \mathrm{Hom}_A(G_{a^m}, G_{a^n}) \cong H_{a^{m-n}} \quad \text{for all} \quad m > n \geqslant 1.$$

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The modules G_{a^n} and H_{a^n} in the theorem above are also of *G-dimension zero*, a notion defined by Auslander [**1**] in 1967. Avramov and Martsinkovsky [**3**] called such modules *totally reflexive*, and we shall adopt their terminology throughout. A pair of non-units $x, y \in A$ satisfying $\mathrm{Ann}_A(x) = (y)$ and $\mathrm{Ann}_A(y) = (x)$ is called an *exact pair of zero divisors* by Henriques and Şega [**7**], and in view of this, the theorem above explains the title of the paper.

If A is Gorenstein, then the totally reflexive A-modules are exactly the maximal Cohen–Macaulay modules, and their representation theory is a classical field of study. A main result of Christensen, Piepmeyer, Striuli, and Takahashi [**4**, Theorem B] asserts that if A is not Gorenstein, then existence of one non-free totally reflexive A-module implies the existence of infinitely many non-isomorphic indecomposable totally reflexive A-modules. The known proof of this interesting result is not constructive, which is why the authors of [**4**] raise the following question.

QUESTION [**4**, (4.8)]. Assume that A is not Gorenstein, and that there exists a non-free totally reflexive A-module. Are there constructions that produce infinite families of non-isomorphic indecomposable totally reflexive A-modules?

Since the modules G_a and H_a are constructed very explicitly, namely as cokernels of certain 2×2 matrices with entries in A, our main result also provides a partial answer to the question above. Other partial answers to this question are given by Avramov and Iyengar [**2**] and by Takahashi [**9**]. While the approach taken in [**2**] is too techincal to explain in this introduction, the result in [**9**] is easily described.

Assume that A is complete or has an uncountable residue field. Assume, furthermore, that there exist a prime ideal $\mathfrak{p}$ in A with $\operatorname{grade} \mathfrak{p} > 0$ and $\dim A/\mathfrak{p} > 1$, and a totally reflexive A-module M such that $M_{\mathfrak{p}}$ is not $A_{\mathfrak{p}}$-free. Under these assumptions, Takahashi [**9**] proves the existence of uncountably many non-isomorphic indecomposable totally reflexive A-modules. The proof is not constructive; however, in [**9**, Example 4.3] one does find an example ($A = Q[\![x, y, z]\!]/(x^2)$, where Q is a complete local domain which is not a field) where uncountably many non-isomorphic indecomposable totally reflexive A-modules are explicitly constructed.

The paper is organized as follows. In Section 2, we introduce exact pairs of zero divisors and discuss orthogonality. In Section 3, the modules G_a and H_a are defined. In the technical Section 4, we compute the module of homomorphisms between various combinations of G_a and H_b. Finally, Section 5 contains our main results. In this section, we study the isomorphism classes and indecomposability of the modules G_a and H_a.

2. *Exact pairs of zero divisors*

In this section, we consider an exact pair of zero divisors as defined by Henriques and Şega [**7**], one definition is given in Section 1, and below we mention an alternative one. We also introduce a notion of orthogonality for such a pair and give some examples.

DEFINITION 2.1. Two non-units $x, y \in A$ are called an *exact pair of zero divisors* if the following sequence of A-modules is exact:

$$\cdots \longrightarrow A \xrightarrow{x} A \xrightarrow{y} A \xrightarrow{x} A \longrightarrow \cdots .$$

Let M be a finitely generated A-module and let $a \in A$ be an element. Recall that a is *weakly regular on M* if multiplication by a on M is a monomorphism. If, in addition, the element a is not a unit, then a is called *regular on M*.

A (weakly) regular element on the A-module $M = A$ is simply referred to as a (weakly) regular element in the ring A.

Lemma 2.2. *Let $x, y \in A$ be an exact pair of zero divisors. Then the following conditions are equivalent*:

(i) *x is regular on $A/(y)$;*
(ii) *y is regular on $A/(x)$;*
(iii) $\operatorname{Tor}_1^A(A/(x), A/(y)) = 0$;
(iv) $(x) \cap (y) = 0$.

Proof. Since $\operatorname{Ann}_A(x) = (y)$ there is a monomorphism $\chi : A/(y) \hookrightarrow A$, defined by $[a]_{(y)} \mapsto ax$, which fits into the following short exact sequence:

$$0 \longrightarrow A/(y) \xrightarrow{\chi} A \longrightarrow A/(x) \longrightarrow 0.$$

As $\chi \otimes_A A/(y)$ may be identified with multiplication by x on $A/(y)$, the short exact sequence above induces the (long) exact sequence

$$0 = \operatorname{Tor}_1^A(A, A/(y)) \longrightarrow \operatorname{Tor}_1^A(A/(x), A/(y)) \longrightarrow A/(y) \xrightarrow{x} A/(y),$$

from which it follows that conditions (i) and (iii) are equivalent. Thus, by symmetry, conditions (ii) and (iii) are also equivalent.

Consider the ideal $(y : x) = \{a \in A \mid ax \in (y)\}$ in A. The kernel of multiplication by x on $A/(y)$ is the quotient $(y : x)/(y)$, and hence (i) is equivalent to $(y : x) \subseteq (y)$. The latter condition is easily seen to be equivalent to (iv). □

Definition 2.3. An exact pair of zero divisors $x, y \in A$ that satisfies the equivalent conditions in Lemma 2.2 is called an *orthogonal* exact pair of zero divisors.

We shall adopt a standard convention from commutative algebra. A local ring is tacitly assumed to be commutative and noetherian as well.

Example 2.4. Let Q be a local ring and let f, g be regular elements in Q. Set $A = Q/(fg)$ and let $x, y \in A$ denote the cosets of f, g with respect to the ideal (fg). Then x, y is an exact pair of zero divisors in A; and this pair is orthogonal if and only if $(f) \cap (g) = (fg)$ in Q.

In particular, if f and g are non-zero and non-units in a local UFD Q, then x, y is an exact pair of zero divisors in A; and this pair is orthogonal if and only if f and g are relatively prime.

Note that in some sense, local UFDs are abundant. By Heitmann [**6**, Theorem 8] every complete local ring of depth greater than 1, in which no integer is a zero divisor, is the completion of a local UFD.

Not every exact pair of zero divisors arises in the way described above.

Several results in this paper refer to a weakly regular element a on the A-module $A/(x, y)$. For example, a could be a unit in A. However, only in the case where a is regular on the A-module $A/(x, y)$ do we get interesting applications of our results. For the existence of regular elements on $A/(x, y)$, we give the following lemma.

Lemma 2.5. *Let Q be a local ring, and let f, g be regular elements in Q such that $(f) \cap (g) = (fg)$. Consider the orthogonal exact pair of zero divisors x, y in $A = Q/(fg)$ constructed in Example* 2.4.

If $\operatorname{depth} Q > 2$, *then there exist regular elements on the A-module $A/(x, y)$.*

Proof. Immediate from the ring isomorphism $A/(x,y) \cong Q/(f,g)$. □

3. *Two families of totally reflexive modules*

Assuming that the ring A admits an exact pair of zero divisors, we introduce in this section two families $(G_a)_{a\in A}$ and $(H_a)_{a\in A}$ of totally reflexive A-modules.

We begin with the following definition due to Auslander [**1**, § 3.2.2]. In Auslander's terminology, the modules defined in (3.1) are said to be of *G-dimension zero*, however, we prefer the terminology *totally reflexive*, which was introduced by Avramov and Martsinkovsky [**3**, § 2], and has grown to be standard in several later papers on the subject.

Definition 3.1. A finitely generated A-module G is *totally reflexive* if it satisfies the following three conditions:

(1) $\operatorname{Ext}_A^i(G, A) = 0$ for all $i > 0$;

(2) $\operatorname{Ext}_A^i(\operatorname{Hom}_A(G, A), A) = 0$ for all $i > 0$;

(3) the biduality map $G \longrightarrow \operatorname{Hom}_A(\operatorname{Hom}_A(G, A), A)$ is an isomorphism.

Definition 3.2. Let $x, y \in A$ be an exact pair of zero divisors and let $a \in A$. We define two 2×2 matrices

$$\gamma_a = \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} \quad \text{and} \quad \eta_a = \begin{pmatrix} y & -a \\ 0 & x \end{pmatrix}.$$

Furthermore, considering γ_a and η_a as A-linear maps $A^2 \to A^2$ acting on column vectors by multiplication from the left, we define two finitely generated A-modules

$$G_a = \operatorname{Coker}\gamma_a \quad \text{and} \quad H_a = \operatorname{Coker}\eta_a.$$

Remark 3.3. Let $x, y \in A$ be an exact pair of zero divisors, let a be any element in A, and let u be a unit in A. It is straightforward to see that there are isomorphisms, $G_{ua} \cong G_a$ and $H_{ua} \cong H_a$. In particular, $G_{-a} \cong G_a$ and $H_{-a} \cong H_a$.

Observation 3.4. Of course, the modules G_a and H_a depend not only on the ring element a, but also on the exact pair of zero divisors x, y. A more precise notation would therefore be $G_a = G_a^{x,y}$ and $H_a = H_a^{x,y}$.

Note that if x, y is an (orthogonal) exact pair of zero divisors, then so is y, x. Thus, Definition 3.2 and Remark 3.3 imply that $G_a^{y,x} \cong H_a^{x,y}$ and $H_a^{y,x} \cong G_a^{x,y}$. It is useful to keep this observation in mind when reading results in this paper. For example, it applies together with Theorem 3.5 below to show that $\operatorname{Ext}_A^i(G_a, H_b) \cong \operatorname{Ext}_A^i(G_b, H_a)$ for all $a, b \in A$ and $i \in \mathbb{Z}$.

Theorem 3.5. *Let $x, y \in A$ be an exact pair of zero divisors and let $a \in A$. The complex of A-modules*

$$\boldsymbol{F} = \cdots \longrightarrow A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} \cdots,$$

is totally acyclic. In particular, the modules $G_a^{x,y}$ and $H_a^{x,y}$ are totally reflexive. Moreover, $G_a^{x,y} \cong \operatorname{Hom}_A(H_a^{x,y}, A)$ and $H_a^{x,y} \cong \operatorname{Hom}_A(G_a^{x,y}, A)$, and for any exact pair of zero divisors $u, v \in A$ and element $b \in A$, one has

$$\operatorname{Ext}_A^i(G_a^{x,y}, G_b^{u,v}) \cong \operatorname{Ext}_A^i(H_b^{u,v}, H_a^{x,y}),$$

for all $i \in \mathbb{Z}$.

Proof. To prove the exactness of the complex $\boldsymbol{F}$, it suffices to argue that $\operatorname{Ker}\gamma_a \subseteq \operatorname{Im}\eta_a$. To this end, assume that

$$\gamma_a \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1x + b_2a \\ b_2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As $b_2y = 0$, we get that $b_2 = c_2x$ for some c_2 in A. Hence, $(b_1 + c_2a)x = b_1x + b_2a = 0$, so there exists c_1 in A with $b_1 + c_2a = c_1y$. Consequently,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_1y - c_2a \\ c_2x \end{pmatrix} = \eta_a \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \operatorname{Im}\eta_a.$$

There is an isomorphism of complexes $\operatorname{Hom}_A(\boldsymbol{F}, A) \xrightarrow{\cong} \boldsymbol{F}$ given by

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\eta_a^t} & A^2 & \xrightarrow{\gamma_a^t} & A^2 & \xrightarrow{\eta_a^t} & A^2 & \xrightarrow{\gamma_a^t} & \cdots \\
 & & \cong\big\downarrow \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & & \cong\big\downarrow \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & & \cong\big\downarrow \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & & , \\
\cdots & \xrightarrow{\gamma_a} & A^2 & \xrightarrow{\eta_a} & A^2 & \xrightarrow{\gamma_a} & A^2 & \xrightarrow{\eta_a} & \cdots
\end{array}$$

where $(-)^t$ means transposition of matrices. It follows that also $\operatorname{Hom}_A(\boldsymbol{F}, A)$ is exact and hence, by definition, $\boldsymbol{F}$ is totally acyclic. By [**1**, Proposition 8 in § 3.2.2], the modules $G_a^{x,y}$ and $H_a^{x,y}$ are totally reflexive. That $G_a^{x,y}$ and $H_a^{x,y}$ are each other's dual is immediate from the isomorphism $\operatorname{Hom}_A(\boldsymbol{F}, A) \cong \boldsymbol{F}$.

For the last assertion, note that $\operatorname{Ext}_A^i(H_b^{u,v}, A) = 0$ for all $i > 0$, since $H_b^{u,v}$ is totally reflexive. As $\operatorname{Hom}_A(H_b^{u,v}, A) \cong G_b^{u,v}$, it follows that $\mathbf{R}\operatorname{Hom}_A(H_b^{u,v}, A) \cong G_b^{u,v}$ in the derived category $\mathsf{D}(A)$. Thus, one has the following isomorphisms in $\mathsf{D}(A)$:

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_A(G_a^{x,y}, G_b^{u,v}) &\cong \mathbf{R}\operatorname{Hom}_A(G_a^{x,y}, \mathbf{R}\operatorname{Hom}_A(H_b^{u,v}, A)) \\
&\cong \mathbf{R}\operatorname{Hom}_A(H_b^{u,v}, \mathbf{R}\operatorname{Hom}_A(G_a^{x,y}, A)) \\
&\cong \mathbf{R}\operatorname{Hom}_A(H_b^{u,v}, H_a^{x,y}),
\end{aligned}$$

where the second isomorphism is the so-called swap isomorphism. The claimed isomorphism of Ext-modules is now obtained by taking homology. □

Under additional assumptions on the ring element a, more can be said about the modules G_a and H_a from Definition 3.2.

PROPOSITION 3.6. *Let $x, y \in A$ be an exact pair of zero divisors and let $a \in A$.*

(a) *If a is weakly regular on $A/(y)$, then G_a is isomorphic to the ideal (y, a).*

(b) *If a is a unit, then $G_a \cong H_a \cong A$.*

(c) *If a is not a unit, then neither G_a nor H_a is free.*

Proof. (a) Since $G_a = \operatorname{Coker}\gamma_a$, it suffices to prove the exactness of the sequence

$$A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{(y\ \ -a)} (y, a) \longrightarrow 0.$$

Clearly, the image of $(y\ \ -a)$ is all of (y, a), and its composite with γ_a is zero. To prove $\operatorname{Ker}(y\ \ -a) \subseteq \operatorname{Im}\gamma_a$, assume that $(b_1, b_2) \in A^2$ satisfies

$$(y\ \ -a) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1y - b_2a = 0.$$

As $b_2a = b_1y \in (y)$ and since a is weakly regular on $A/(y)$, it follows that $b_2 \in (y)$, that is, $b_2 = c_2y$ for some c_2. Now $(b_1 - c_2a)y = b_1y - b_2a = 0$, and as $\operatorname{Ann}_A(y) = (x)$, we conclude that $b_1 - c_2 a = c_1 x$ for some c_1. Consequently,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_1 x + c_2 a \\ c_2 y \end{pmatrix} = \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \gamma_a \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \operatorname{Im} \gamma_a.$$

(b) Immediate from Part (a) and Observation 3.4.

(c) If a is not a unit, then all the entries in γ_a and η_a belong to the unique maximal ideal $\mathfrak{m}$ in A. Consequently, if $k = A/\mathfrak{m}$ denotes the residue field of A, then the homomorphisms $k \otimes_A \gamma_a$ and $k \otimes_A \eta_a$ are zero, and it follows that

$$\cdots \longrightarrow A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} A^2 \xrightarrow{\gamma_a} A^2 \longrightarrow G_a \longrightarrow 0$$

is an augmented minimal free resolution of G_a. Hence, G_a has infinite projective dimension. Analogously one sees that H_a has infinite projective dimension. □

4. *Computation of Hom-modules*

In this section, we explicitly compute certain Hom-modules. For example, we will prove that $\operatorname{Hom}_A(H_b, G_a)$ is isomorphic to G_{ab}. In order to carry out such computations, we need a concrete way of representing the module of homomorphisms between two given finitely generated A-modules. The representation we use can be found in, for example, Greuel and Pfister [**5**, Example 2.1.26]; we give a recap in (4.1).

REMARK 4.1. Let ρ_i be an $m_i \times n_i$ matrix with entries in A, where $i = 1, 2$. Then

$$Q_{\rho_1,\rho_2} = \{ \psi \in \mathrm{M}_{m_2 \times m_1}(A) \mid \psi \rho_1 = \rho_2 \xi \text{ for some } \xi \in \mathrm{M}_{n_2 \times n_1}(A) \}$$

is an A-submodule of $\mathrm{M}_{m_2 \times m_1}(A)$ that contains $\rho_2 \, \mathrm{M}_{n_2 \times m_1}(A)$ as a submodule. Furthermore, there is an isomorphism of A-modules

$$\operatorname{Hom}_A(\operatorname{Coker} \rho_1, \operatorname{Coker} \rho_2) \cong Q_{\rho_1,\rho_2} / (\rho_2 \, \mathrm{M}_{n_2 \times m_1}(A)).$$

LEMMA 4.2. *Let $x, y \in A$ be an orthogonal exact pair of zero divisors. Furthermore, let $a, b \in A$ be elements such that a or b is weakly regular on the A-module $A/(x, y)$. Then the A-submodule*

$$Q_{\eta_b,\gamma_a} = \{ \psi \in \mathrm{M}_2(A) \mid \psi \eta_b = \gamma_a \xi \text{ for some } \xi \in \mathrm{M}_2(A) \}$$

of $\mathrm{M}_2(A)$ is generated by the following five matrices

$$\psi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 & 0 \\ x & b \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}.$$

Proof. First note that if we define

$$\xi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \quad \xi_5 = \begin{pmatrix} 0 & 0 \\ y & -b \end{pmatrix},$$

then $\psi_i \eta_b = \gamma_a \xi_i$, and hence ψ_i belongs to Q_{η_b,γ_a}.

Next we show that $\psi_1, \ldots, \psi_5$ generate all of Q_{η_b,γ_a}. To this end, let $\psi = (b_{ij})$ be any matrix in Q_{η_b,γ_a}, that is, there exists $\xi = (c_{ij})$ such that $\psi \eta_b = \gamma_a \xi$; that is,

$$\begin{pmatrix} b_{11} y & -b_{11} b + b_{12} x \\ b_{21} y & -b_{21} b + b_{22} x \end{pmatrix} = \begin{pmatrix} c_{11} x + c_{21} a & c_{12} x + c_{22} a \\ c_{21} y & c_{22} y \end{pmatrix}. \qquad (*)$$

We must prove the existence of $f_1,\dots,f_5$ in A such that $\psi=\sum_{i=1}^{5} f_i\psi_i$, that is,

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} f_4x+f_5a & f_1 \\ f_2x+f_5y & f_2b+f_3y \end{pmatrix}. \tag{$**$}$$

Of course, if we define $f_1=b_{12}$, then entry $(1,2)$ in $(**)$ holds. Also note that, from entry $(2,1)$ in $(*)$, we get $(b_{21}-c_{21})y=0$, and hence $b_{21}-c_{21}=qx$ for some q.

Below, we construct f_2 and f_5 such that entry $(2,1)$ in $(**)$ holds, that is,

$$b_{21}=f_2x+f_5y. \tag{$\dagger$}$$

However, first we demonstrate how f_3 and f_4 can be constructed from $(*)$ and $(\dagger)$ such that entries $(1,1)$ and $(2,2)$ in $(**)$ hold, respectively:

Existence of f_3: From entry $(2,2)$ in $(*)$ we get $-b_{21}b+b_{22}x=c_{22}y$. Combining this with $(\dagger)$, we obtain $(b_{22}-f_2b)x=(c_{22}+f_5b)y$. As the exact pair of zero divisors x,y is orthogonal (see Definition 2.3), it follows that $b_{22}-f_2b=f_3y$ for some f_3. Thus, entry $(2,2)$ in $(**)$ holds.

Existence of f_4: From entry $(1,1)$ in $(*)$, from the equation $b_{21}-c_{21}=qx$ found above, and from $(\dagger)$, we get the following equalities:

$$b_{11}y=c_{11}x+c_{21}a=c_{11}x+(b_{21}-qx)a=(c_{11}+f_2a-qa)x+f_5ay.$$

Consequently, $(b_{11}-f_5a)y\in(x)$. As the pair x,y is orthogonal (see Definition 2.3), it follows that $b_{11}-f_5a=f_4x$ for some f_4. Therefore, entry $(1,1)$ in $(**)$ holds.

It remains to prove that $(\dagger)$ holds, that is, that b_{21} belongs to (x,y). The proof is divided into the following two cases:

Existence of $(\dagger)$ *in the case where* a *is weakly regular on* $A/(x,y)$:

From entry $(1,1)$ in $(*)$, we have $c_{21}a=-c_{11}x+b_{11}y\in(x,y)$. As a is weakly regular on $A/(x,y)$, it follows that $c_{21}\in(x,y)$. Combining this with the equation $b_{21}-c_{21}=qx$ found above, it follows that $b_{21}\in(x,y)$.

Existence of $(\dagger)$ *in the case where* b *is weakly regular on* $A/(x,y)$:

From entry $(2,2)$ in $(*)$, we have $b_{21}b=b_{22}x-c_{22}y\in(x,y)$. As b is weakly regular on $A/(x,y)$, it follows that $b_{21}\in(x,y)$ as desired. □

THEOREM 4.3. *Let $x,y\in A$ be an orthogonal exact pair of zero divisors. Furthermore, let $a,b\in A$ be elements such that a or b is weakly regular on the A-module $A/(x,y)$. Then there is an isomorphism of A-modules*

$$\operatorname{Hom}_A(H_a,G_b)\cong\operatorname{Hom}_A(H_b,G_a)\cong G_{ab}.$$

Proof. The first isomorphism is by Theorem 3.5 and Observation 3.4. To prove the second isomorphism, we first apply Remark 4.1 to get the representation

$$\operatorname{Hom}_A(H_b,G_a)\cong Q_{\eta_b,\gamma_a}/(\gamma_a\,\mathrm{M}_2(A)).$$

Using this identification, we can consider the A-linear map

$$A^2 \xrightarrow{\ \pi\ } \operatorname{Hom}_A(H_b,G_a),\qquad \begin{pmatrix} s\\ t\end{pmatrix}\longmapsto s[\psi_1]+t[\psi_2].$$

Here $[\,\cdot\,]$ denotes a coset with respect to the submodule $\gamma_a\,\mathrm{M}_2(A)$ of Q_{η_b,γ_a}, and ψ_i are the matrices introduced in Lemma 4.2. Recall that $G_{ab}=\operatorname{Coker}\gamma_{ab}$; thus, to prove the theorem, it suffices to show the exactness of the sequence

$$A^2\xrightarrow{\ \gamma_{ab}\ }A^2\xrightarrow{\ \pi\ }\operatorname{Hom}_A(H_b,G_a)\longrightarrow 0.$$

To show that π is surjective, it suffices, by Lemma 4.2, to see that each $[\psi_i]$ is in the image of π. However, this is clear since $[\psi_3]=-a[\psi_1]$ and $[\psi_4]=[\psi_5]=[0]$.

To see that $\pi\gamma_{ab} = 0$, we must show that if $(s,t) = \gamma_{ab}(u,v)$, where $(u,v) \in A^2$, then $s\psi_1 + t\psi_2$ is in $\gamma_a \operatorname{M}_2(A)$. But as

$$\begin{pmatrix} s \\ t \end{pmatrix} = \gamma_{ab} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x & ab \\ 0 & y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ux + vab \\ vy \end{pmatrix}, \qquad (\dagger)$$

the desired conclusion follows since

$$s\psi_1 + t\psi_2 = (ux + vab) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + vy \begin{pmatrix} 0 & 0 \\ x & b \end{pmatrix} = \begin{pmatrix} 0 & ux + vab \\ 0 & vyb \end{pmatrix} = \gamma_a \begin{pmatrix} 0 & u \\ 0 & vb \end{pmatrix}.$$

It remains to prove that $\operatorname{Ker}\pi \subseteq \operatorname{Im}\gamma_{ab}$. To this end, assume that $(s,t) \in A^2$ satisfies $\pi(s,t) = 0$. This means that $s\psi_1 + t\psi_2$ belongs to $\gamma_a \operatorname{M}_2(A)$, that is, there exists $\xi = (c_{ij})$ in $\operatorname{M}_2(A)$ such that $s\psi_1 + t\psi_2 = \gamma_a\xi$; equivalently,

$$\begin{pmatrix} 0 & s \\ tx & tb \end{pmatrix} = \begin{pmatrix} c_{11}x + c_{21}a & c_{12}x + c_{22}a \\ c_{21}y & c_{22}y \end{pmatrix}. \qquad (\ddagger)$$

From entry $(2,1)$ in ($\ddagger$) we get that $tx = c_{21}y$. Since the exact pair of zero divisors x, y is orthogonal (see Definition 2.3), it follows that $t = vy$ for some v.

From entry $(2,2)$ in ($\ddagger$) we get that $tb = c_{22}y$. Combining this with $t = vy$, one gets $(vb - c_{22})y = 0$. Consequently, $vb - c_{22} = px$ for some p. Now, inserting $c_{22} = vb - px$ in the equation $s = c_{12}x + c_{22}a$ coming from entry $(1,2)$ in ($\ddagger$), we get that $s = (c_{12} - pa)x + vab = ux + vab$, where $u = c_{12} - pa$.

Since $s = ux + vab$ and $t = vy$, we see from ($\dagger$) that $(s,t) = \gamma_{ab}(u,v) \in \operatorname{Im}\gamma_{ab}$. □

It is also desirable to know what the module $\operatorname{Hom}_A(G_s, G_t) \cong \operatorname{Hom}_A(H_t, H_s)$ (see Theorem 3.5 for this isomorphism) looks like for every combination of $s, t \in A$. Since the author was not able to figure this out, we restrict ourselves to the case where s is in the ideal generated by t; see Theorem 4.5 below.

LEMMA 4.4. *Let $x, y \in A$ be an orthogonal exact pair of zero divisors. Furthermore, let $a \in A$ be weakly regular on the A-module $A/(x,y)$ and let $b \in A$ be any element. Then the A-submodule*

$$Q_{\gamma_{ab},\gamma_a} = \{\, \psi \in \operatorname{M}_2(A) \,|\, \psi\gamma_{ab} = \gamma_a\xi \text{ for some } \xi \in \operatorname{M}_2(A) \,\}$$

of $\operatorname{M}_2(A)$ is generated by the following five matrices

$$\psi_1 = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} 0 & a \\ 0 & y \end{pmatrix}.$$

Proof. Similar to (but easier than) the proof of Lemma 4.2. □

THEOREM 4.5. *Let $x, y \in A$ be an orthogonal exact pair of zero divisors. Furthermore, let $a \in A$ be weakly regular on the A-module $A/(x,y)$ and let $b \in A$ be any element. Then there is an isomorphism of A-modules*

$$\operatorname{Hom}_A(H_a, H_{ab}) \cong \operatorname{Hom}_A(G_{ab}, G_a) \cong H_b.$$

Proof. The first isomorphism is by Theorem 3.5 and Observation 3.4. To prove the second isomorphism, we apply Remark 4.1 to get the representation

$$\operatorname{Hom}_A(G_{ab}, G_a) \cong Q_{\gamma_{ab},\gamma_a}/(\gamma_a \operatorname{M}_2(A)).$$

Using this identification, we can consider the A-linear map

$$A^2 \xrightarrow{\pi} \mathrm{Hom}_A(G_{ab}, G_a), \quad \begin{pmatrix} s \\ t \end{pmatrix} \longmapsto s[\psi_1] + t[\psi_2].$$

Here $[\,\cdot\,]$ denotes a coset with respect to the submodule $\gamma_a \, \mathrm{M}_2(A)$ of Q_{γ_{ab},γ_a}, and ψ_i are the matrices introduced in Lemma 4.4. Recall that $H_b = \mathrm{Coker}\,\eta_b$ (see Definition 3.2); thus, to prove the result, it suffices to show exactness of the sequence

$$A^2 \xrightarrow{\eta_b} A^2 \xrightarrow{\pi} \mathrm{Hom}_A(G_{ab}, G_a) \longrightarrow 0.$$

To see that π is surjective, it suffices by Lemma 4.4 to see that each $[\psi_i]$ belongs to $\mathrm{Im}\,\pi$. However, this is clear since $[\psi_3] = [\psi_4] = [\psi_5] = [0]$.

It remains to prove $\mathrm{Im}\,\eta_b = \mathrm{Ker}\,\pi$. The arguments needed to show this are similar to the ones found in the proof of Theorem 4.3. □

5. *Isomorphism classes and indecomposability*

In this section, we address two natural questions about the family of totally reflexive modules $(G_a)_{a\in A} \cup (H_a)_{a\in A}$ defined in (3.2), namely: are the modules in this family indecomposable, and are they pairwise non-isomorphic?

In general, both of these questions have a negative answer; see Example 5.4, Remark 3.3, and Proposition 3.6(b). Under suitable assumptions on $a \in A$, we will prove that $\mathrm{End}_A(G_a)$ and $\mathrm{End}_A(H_a)$ are both isomorphic to A and, in particular, these endomorphism rings are (commutative, noetherian, and) local. Hence, G_a and H_a are indecomposable in quite a strong sense. We shall also give a condition on ring elements $a, b \in A$ which ensures that, for example, G_a and H_b are not isomorphic.

Remark 5.1. Let M be a reflexive A-module. Clearly, M is indecomposable if and only if its dual $\mathrm{Hom}_A(M, A)$ is indecomposable. It is not hard to see that something stronger holds, namely, there is an isomorphism of A-algebras

$$\mathrm{End}_A(M)^{\mathrm{op}} \cong \mathrm{End}_A(\mathrm{Hom}_A(M, A)).$$

In particular, G_a is indecomposable if and only if H_a is indecomposable.

The following result is probably folklore. However, since the author was not able to find a reference, a short proof has been included.

Lemma 5.2. *Let M be a finitely generated A-module. If A and $\mathrm{Hom}_A(M, M)$ are isomorphic as A-modules, then the canonical homomorphism of A-algebras, given by $\chi : A \to \mathrm{End}_A(M)$, $\chi(a) = a1_M$, is also an isomorphism.*

Proof. Let $\zeta : A \to \mathrm{Hom}_A(M, M)$ be an isomorphism of A-modules. As ζ is surjective, there exists b in A such that $b\zeta(1) = \zeta(b) = 1_M$. It follows that $bM = M$. Furthermore, $M \neq 0$ since $\mathrm{Hom}_A(M, M)$ is non-zero. Hence, Nakayama's lemma [**8**, Theorem 2.2] implies that b is a unit in A, and thus $b\zeta$ is also an isomorphism. It remains to note that $b\zeta = \chi$ since $b\zeta(a) = ba\zeta(1) = a\zeta(b) = a1_M = \chi(a)$. □

Theorem 5.3. *Let $x, y \in A$ be an orthogonal exact pair of zero divisors and let a, b be arbitrary elements in A. Then*:

(a) *If a is weakly regular on $A/(x, y)$, then there are isomorphisms of A-algebras*

$$\mathrm{End}_A(G_a) \cong A \cong \mathrm{End}_A(H_a).$$

In particular, the A-modules G_a and H_a are indecomposable.

(b) *Assume that a or b is weakly regular on $A/(x,y)$. Assume furthermore that a and b are not both units. Then*

$$G_a \ncong H_b.$$

(c) *If a is weakly regular on $A/(x,y)$, and b is not a unit, then*

$$G_a \ncong G_{ab} \quad \text{and} \quad H_a \ncong H_{ab}.$$

Proof. (a) Taking $b=1$ in Theorem 4.5, we get the first two isomorphisms of A-modules in the chain $\operatorname{Hom}_A(H_a,H_a) \cong \operatorname{Hom}_A(G_a,G_a) \cong H_1 \cong A$. The last isomorphism is by Proposition 3.6(b). Now Lemma 5.2 completes the proof.

(b) Since a or b is weakly regular on $A/(x,y)$, Part (a) gives $\operatorname{Hom}_A(G_a,G_a) \cong A$ or $\operatorname{Hom}_A(H_b,H_b) \cong A$. Thus, to prove that G_a and H_b are not isomorphic, it suffices to argue that $\operatorname{Hom}_A(H_b,G_a) \ncong A$. By Theorem 4.3 there is an isomorphism, $\operatorname{Hom}_A(H_b,G_a) \cong G_{ab}$. Since both a and b are not units, ab is not a unit, and hence G_{ab} is not free by Proposition 3.6(c).

(c) In view of Observation 3.4, we need only prove $G_a \ncong G_{ab}$. By Part (a) there is an isomorphism, $\operatorname{Hom}_A(G_a,G_a) \cong A$. Thus, to prove that G_a and G_{ab} are not isomorphic, it is enough to show $\operatorname{Hom}_A(G_{ab},G_a) \ncong A$. By Theorem 4.5 one has $\operatorname{Hom}_A(G_{ab},G_a) \cong H_b$, which is not free as b is not a unit; see Proposition 3.6(c). □

EXAMPLE 5.4. For a general ring element a, the conclusion in Theorem 5.3(a) fails. For example, if a is in (x), then $G_a \cong A/(x) \oplus A/(y)$ is not indecomposable.

For the last assertion in the following corollary, it is useful to keep in mind that $H_1 \cong A$; cf. Proposition 3.6(b).

COROLLARY 5.5. *Let $x, y \in A$ be an orthogonal exact pair of zero divisors, and let $(b_n)_{n\geqslant 1}$ be a sequence of (not necessarily distinct) elements in A that are regular on the A-module $A/(x,y)$. If we define $a_n = b_1 \dots b_n$, then*

$$G_{a_1}, G_{a_2}, G_{a_3}, \dots, H_{a_1}, H_{a_2}, H_{a_3}, \dots$$

is a (double) infinite family of indecomposable, non-free, totally reflexive A-modules, which are pairwise non-isomorphic. Furthermore, one has the following identities:

$$\operatorname{Hom}_A(H_{a_m}, G_{a_n}) \cong G_{a_m a_n} \quad \text{for all } m,n \geqslant 1,$$
$$\operatorname{Hom}_A(H_{a_n}, H_{a_m}) \cong \operatorname{Hom}_A(G_{a_m}, G_{a_n}) \cong H_{a_m/a_n} \quad \text{for all } m \geqslant n \geqslant 1.$$

Proof. By Theorem 3.5, the modules G_{a_n} and H_{a_n} are totally reflexive. As a_n is not a unit, G_{a_n} and H_{a_n} are non-free by Proposition 3.6(c). Since a_n is regular on $A/(x,y)$, it follows from Theorem 5.3(a) that G_{a_n} and H_{a_n} are indecomposable.

By Theorem 5.3(b), the modules G_{a_m} and H_{a_n} are not isomorphic. If $m > n$, then $a_m = a_n b$, where $b = b_{n+1} \dots b_m$ (which is not a unit). Thus, Theorem 5.3(c) gives that G_{a_m} and G_{a_n} are not isomorphic and, furthermore, that H_{a_m} and H_{a_n} are not isomorphic. Hence, the modules in the given list are pairwise non-isomorphic.

The Hom-identities are immediate from Theorems 4.3 and 4.5. □

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Henrik Holm
Department of Basic Sciences and Environment
Faculty of Life Sciences
University of Copenhagen
Thorvaldsensvej 40
DK-1871 Frederiksberg C
Denmark

hholm@life.ku.dk
http://www.matdat.life.ku.dk/∼hholm/