REND. SEM. MAT. UNIV. PADOVA, Vol. 117 (2007)

Cohen-Macaulay Homological Dimensions.

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ABSTRACT - We define three new homological dimensions: Cohen-Macaulay injective, projective, and flat dimension, and show that they inhabit a theory similar to that of classical injective, projective, and flat dimension. In particular, we show that finiteness of the new dimensions characterizes Cohen-Macaulay rings with dualizing modules.

1. Introduction.

The classical theory of homological dimensions is very important to commutative algebra. In particular, it is useful that there are a number of finiteness conditions on these dimensions which characterize regular rings. For instance, if the projective dimension of each finitely generated A-module is finite, then A is a regular ring.

Several attempts have been made to mimic this success by defining homological dimensions whose finiteness would characterize other rings than the regular ones. These efforts have given us complete intersection dimension [2], Gorenstein dimension [1], and Cohen-Macaulay dimension [8].

The normal practice has not been to mimic the full classical theory, which comprises both injective, projective, and flat dimension for arbitrary modules, but rather to focus on projective dimension for finitely generated modules. Hence complete intersection dimension and Cohen-Macaulay dimension only exist in this restricted sense, and the same used to be the case for Gorenstein dimension.

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2000 *Mathematics Subject Classification*. 13D05, 13D25.

However, recent years have seen much work on the Gorenstein theory which now contains both Gorenstein injective, projective, and flat dimension. These dimensions inhabit a nice theory similar to classical homological algebra. A summary is given in [3], although this reference is already a bit dated.

The purpose of this paper is to develop a similar theory in the Cohen-Macaulay case. So we will define Cohen-Macaulay injective, projective, and flat dimension, denoted CMid, CMpd, and CMfd, and show that they inhabit a theory with a number of desirable features. The main result is the following, whereby finiteness of the Cohen-Macaulay dimensions characterizes Cohen-Macaulay rings with dualizing modules.

THEOREM. *Let A be a local commutative noetherian ring. Then the following conditions are equivalent.*

(i) *A is a Cohen-Macaulay ring with a dualizing module.*
(ii) $\mathrm{CMid}_A M < \infty$ *for each A-module M.*
(iii) $\mathrm{CMpd}_A M < \infty$ *for each A-module M.*
(iv) $\mathrm{CMfd}_A M < \infty$ *for each A-module M.*

Another main result is a bound for the Cohen-Macaulay dimensions in terms of the Krull dimension $\dim A$ of the ring.

THEOREM. *Let A be a local commutative noetherian ring and let M be an A-module. Then*

$$\mathrm{CMid}_A M < \infty \;\Leftrightarrow\; \mathrm{CMid}_A M \leq \dim A,$$

$$\mathrm{CMpd}_A M < \infty \;\Leftrightarrow\; \mathrm{CMpd}_A M \leq \dim A,$$

$$\mathrm{CMfd}_A M < \infty \;\Leftrightarrow\; \mathrm{CMfd}_A M \leq \dim A.$$

These results are proved as theorems 5.1 and 5.4; in fact, theorem 5.1 is somewhat more general than the first of the above theorems.

There are also some other results: Theorem 5.6 compares our Cohen-Macaulay projective dimension to Gerko's Cohen-Macaulay dimension from [8], and to the Gorenstein projective dimension, see [3, chp. 4]. And propositions 5.7 and 5.8 show Auslander-Buchsbaum and Bass formulae for the Cohen-Macaulay dimensions.

As tools to define the Cohen-Macaulay dimensions, we use change of rings. If A is a ring with a semi-dualizing module C (as defined in [4]), then we can consider the trivial extension ring $A \ltimes C$, and if M is a complex of A-modules, then we can consider M as a complex of $(A \ltimes C)$-modules and take

Gorenstein homological dimensions of M over $A \ltimes C$. The infima of these over all semi-dualizing modules C define the Cohen-Macaulay dimensions of M. Our use of trivial extension rings was in part inspired by [8], and there are also some similarities to [7].

It is worth noting that while finiteness of our Cohen-Macaulay dimensions characterises Cohen-Macaulay rings with dualizing modules, there are Cohen-Macaulay rings without dualizing modules, see [6]. It would be of interest to develop similar dimensions whose finiteness characterised all Cohen-Macaulay rings, but we do not yet know how to do that.

The paper is organized as follows. Section 2 defines the Cohen-Macaulay dimensions and gives an example of their computation. Sections 3 and 4 establish a number of technical tools by studying the trivial extension ring $A \ltimes C$ when C is a semi-dualizing module for A, and giving some bounds on the injective dimension of C. And finally, section 5 proves all the main results as described.

Let us end the introduction with a few blanket items. Throughout, A is a commutative noetherian ring, and complexes of A-modules have the form

$$\cdots \to M_{i+1} \to M_i \to M_{i-1} \to \cdots.$$

Isomorphisms in categories are denoted by $\cong$ and equivalences of functors by $\simeq$.

2. Cohen-Macaulay dimensions.

This section introduces Cohen-Macaulay injective, projective, and flat dimension, and gives an example of their computation.

Definition 2.1. Let C be an A-module. The direct sum $A \oplus C$ can be equipped with the product

$$(a, c) \cdot (a_1, c_1) = (aa_1, ac_1 + ca_1).$$

This turns $A \oplus C$ into a ring which is called the trivial extension of A by C and denoted $A \ltimes C$.

There are ring homomorphisms

$$\begin{array}{ccccc} A & \to & A \ltimes C & \to & A, \\ a & \longmapsto & (a, 0), & & \\ & & (a, c) & \longmapsto & a \end{array}$$

whose composition is the identity on A. These homomorphisms allow us to view any A-module as an $(A \ltimes C)$-module and any $(A \ltimes C)$-module as an A-module, and we shall do so freely.

In particular, if M is a complex of A-modules with suitably bounded homology, then we can consider the Gorenstein homological dimensions over $A \ltimes C$,

$$\mathrm{Gid}_{A \ltimes C} M, \quad \mathrm{Gpd}_{A \ltimes C} M, \quad \text{and} \quad \mathrm{Gfd}_{A \ltimes C} M,$$

where Gid, Gpd, and Gfd denote the Gorenstein injective, projective, and flat dimensions as described e.g. in [3].

Let us briefly recall the definitions of the Gorenstein dimensions. A complex N with $\mathrm{H}_i N = 0$ for $i \ll 0$ has $\mathrm{Gpd} N \leq n$ if it is quasi-isomorphic to a complex G which has the form

$$\cdots \to 0 \to G_n \to G_{n-1} \to \cdots$$

and consists of Gorenstein projective modules. The Gorenstein projective modules are the kernels of differentials in complete projective resolutions, that is, exact complexes P of projective modules for which $\mathrm{Hom}(P, Q)$ is also exact for each projective module Q.

Dualizing this gives the definition of Gorenstein injective dimension. Finally, Gorenstein flat dimension is defined as follows. A complex N with $\mathrm{H}_i N = 0$ for $i \ll 0$ has $\mathrm{Gfd} N \leq n$ if it is quasi-isomorphic to a complex H which has the form

$$\cdots \to 0 \to H_n \to H_{n-1} \to \cdots$$

and consists of Gorenstein flat modules. The Gorenstein flat modules are the kernels of differentials in exact complexes of flat modules which remain exact when tensored with an injective module.

DEFINITION 2.2. A semi-dualizing module C for A is a finitely generated module for which the canonical map $A \to \mathrm{Hom}_A(C, C)$ is an isomorphism, while $\mathrm{Ext}_A^i(C, C) = 0$ for each $i \geqslant 1$.

A semi-dualizing module is called dualizing if it has finite injective dimension.

Note that equivalently, a finitely generated module C is semi-dualizing if the canonical morphism $A \to \mathrm{RHom}_A(C, C)$ is an isomorphism in the derived category $\mathrm{D}(A)$. An example of a semi-dualizing module is A itself, so A always has at least one semi-dualizing module. On the other hand, if A has a dualizing module, then it is necessarily Cohen-Macaulay, as follows for instance by [3, (A.8.5.3)].

The theory of semi-dualizing modules (and complexes) is developed in [4].

Definition 2.3. Let M and N be complexes of A-modules such that $\mathrm{H}_i M = 0$ for $i \gg 0$ and $\mathrm{H}_i N = 0$ for $i \ll 0$.

The Cohen-Macaulay injective, projective, and flat dimensions of M and N over A are

$$\mathrm{CMid}_A M = \inf \{ \mathrm{Gid}_{A \ltimes C} M \mid C \text{ is a semi-dualizing module} \},$$
$$\mathrm{CMpd}_A N = \inf \{ \mathrm{Gpd}_{A \ltimes C} N \mid C \text{ is a semi-dualizing module} \},$$
$$\mathrm{CMfd}_A N = \inf \{ \mathrm{Gfd}_{A \ltimes C} N \mid C \text{ is a semi-dualizing module} \}.$$

Remark 2.4. From the definition of the Cohen-Macaulay homological dimensions, the connection to Cohen-Macaulayness is less than obvious, but as described in the introduction, we shall prove in section 5 that the Cohen-Macaulay dimensions have the same relation to Cohen-Macaulay rings with dualizing modules as classical injective, projective, and flat dimension have to regular rings.

Example 2.5. Let the ring A be local and artinian. Then it is easy directly to compute the Cohen-Macaulay dimensions of any module.

Namely, for a local artinian ring, the injective hull $E(k)$ of the residue class field is a dualizing module. By [7, thm. 5.6], the trivial extension $A \ltimes E(k)$ is Gorenstein, and it is clearly also local and artinian.

But over a local artinian Gorenstein ring, each module is both Gorenstein injective, projective, and flat. So if M is an A-module and we view it as an $A \ltimes E(k)$-module, then

$$\mathrm{Gid}_{A \ltimes E(k)} M = 0, \quad \mathrm{Gpd}_{A \ltimes E(k)} M = 0, \quad \text{and} \quad \mathrm{Gfd}_{A \ltimes E(k)} M = 0,$$

and hence

$$\mathrm{CMid}_A M = 0, \quad \mathrm{CMpd}_A M = 0, \quad \text{and} \quad \mathrm{CMfd}_A M = 0.$$

Remark 2.6. The result in example 2.5 is a special case of the theorems in the introduction, since the local artinian ring A is Cohen-Macaulay and has Krull dimension zero.

3. Lemmas on the trivial extension

This section studies the homological properties of the trivial extension $A \ltimes C$. This is essential for subsequent developments since our Cohen-Macaulay dimensions are defined in terms of $A \ltimes C$.

Some of the material in this section is related to [7]. See in particular [7, prop. 4.35 and thm. 5.2].

LEMMA 3.1. *Let C be an A-module and F a faithfully flat $(A \ltimes C)$-module.*

(i) *If I is an injective A-module, then* $\operatorname{Hom}_A(F, I)$, *with the* $(A \ltimes C)$-*structure coming from the first variable, is an injective* $(A \ltimes C)$-*module. If I is faithfully injective, then so is* $\operatorname{Hom}_A(F, I)$.

(ii) *Each injective $(A \ltimes C)$-module is a direct summand in a module* $\operatorname{Hom}_A(A \ltimes C, I)$, *with the* $(A \ltimes C)$-*structure coming from the first variable, where I is an injective A-module.*

PROOF. (i) This holds because adjunction gives

$$\operatorname{Hom}_{A\ltimes C}(-, \operatorname{Hom}_A(F, I)) \simeq \operatorname{Hom}_A(F \otimes_{A\ltimes C} -, I) \tag{1}$$

on $(A \ltimes C)$-modules.

(ii) Note first the handy special case of equation (1) with $F = A \ltimes C$,

$$\operatorname{Hom}_{A\ltimes C}(-, \operatorname{Hom}_A(A \ltimes C, I)) \simeq \operatorname{Hom}_A(-, I). \tag{2}$$

To see that an injective $(A \ltimes C)$-module J is a direct summand in a module of the form $\operatorname{Hom}_A(A \ltimes C, I)$, it is enough to embed it into such a module. For this, first view J as an A-module and choose an embedding $J \hookrightarrow I$ into an injective A-module I. Then use equation (2) to convert the morphism of A-modules $J \hookrightarrow I$ to a morphism of $(A \ltimes C)$-modules $J \to \operatorname{Hom}_A(A \ltimes C, I)$. It is not hard to check that this last morphism is in fact injective. □

The following lemma is closely related to [8, sec. 3], although that paper was not phrased in terms of derived categories. The lemma and its proof use the right derived Hom functor, RHom, and the left derived tensor functor, $\otimes^{\mathrm{L}}$, which are defined on derived categories.

LEMMA 3.2. *Let C be a semi-dualizing module for A.*

(i) *There is an isomorphism*

$$A \ltimes C \cong \operatorname{RHom}_A(A \ltimes C, C)$$

in the derived category $\mathrm{D}(A \ltimes C)$, *where the* $(A \ltimes C)$-*structure of the* RHom *comes from the first variable.*

(ii) *There is a natural equivalence*

$$\operatorname{RHom}_{A\ltimes C}(-, A \ltimes C) \simeq \operatorname{RHom}_A(-, C)$$

of functors from $\mathrm{D}(A)$ *to* $\mathrm{D}(A)$.

(iii) *If M is in* $\mathrm{D}(A)$ *then the biduality morphisms*

$$M \to \mathrm{RHom}_A(\mathrm{RHom}_A(M, C), C),$$
$$M \to \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_{A\ltimes C}(M, A \ltimes C), A \ltimes C)$$

are identified under the equivalence from part (ii).

(iv) *There is an isomorphism*

$$\mathrm{RHom}_{A\ltimes C}(A, A \ltimes C) \cong C$$

in $\mathrm{D}(A \ltimes C)$.

Proof. (i) Since C is semi-dualizing, there is a canonical isomorphism in $\mathrm{D}(A)$,

$$A \oplus C \to \mathrm{RHom}_A(C \oplus A, C). \tag{3}$$

Let us write out more carefully how this arises. Let $C \to I$ be an injective resolution of C. Then the canonical map $A \to \mathrm{Hom}(C, I)$ is a quasi-isomorphism, and there is clearly a quasi-isomorphism $C \to \mathrm{Hom}_A(A, I)$. Combining these maps gives a quasi-isomorphism

$$A \oplus C \to \mathrm{Hom}_A(C, I) \oplus \mathrm{Hom}_A(A, I),$$

that is, a quasi-isomorphism

$$A \oplus C \to \mathrm{Hom}_A(C \oplus A, I). \tag{4}$$

The right-hand side here is

$$\mathrm{Hom}_A(C \oplus A, I) \cong \mathrm{RHom}_A(C \oplus A, C),$$

so the quasi-isomorphism (4) represents the isomorphism (3) in $\mathrm{D}(A)$.

But it is easy to check that in fact, the morphism (4) respects the action of $A \ltimes C$ on both sides, so

$$A \ltimes C \cong \mathrm{RHom}_A(A \ltimes C, C)$$

in $\mathrm{D}(A \ltimes C)$.

(ii) This is a computation,

$$\begin{aligned} \mathrm{RHom}_{A\ltimes C}(-, A \ltimes C) &\overset{(a)}{\simeq} \mathrm{RHom}_{A\ltimes C}(-, \mathrm{RHom}_A(A \ltimes C, C)) \\ &\overset{(b)}{\simeq} \mathrm{RHom}_A((A \ltimes C) \otimes^{\mathrm{L}}_{A\ltimes C} -, C) \\ &\simeq \mathrm{RHom}_A(-, C), \end{aligned}$$

where (a) is by part (i) and (b) is by adjunction.

(iii) and (iv) These are easy to obtain from (ii). □

LEMMA 3.3. *Let C be a semi-dualizing module for A and let I be an injective A-module.*

(i) *A and C are Gorenstein projective over $A \ltimes C$.*
(ii) $\mathrm{Hom}_A(A, I) \cong I$ *and* $\mathrm{Hom}_A(C, I)$ *are Gorenstein injective over $A \ltimes C$.*

PROOF. (i) To see that A is Gorenstein projective over $A \ltimes C$, by [3, prop. (2.2.2)] we need to see that the homology of

$$\mathrm{RHom}_{A\ltimes C}(A, A \ltimes C)$$

is concentrated in degree zero and that the biduality morphism

$$A \to \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_{A\ltimes C}(A, A \ltimes C), A \ltimes C)$$

is an isomorphism. The first of these conditions follows from lemma 3.2(ii). The second condition holds because the biduality morphism can be identified with

$$A \to \mathrm{RHom}_A(\mathrm{RHom}_A(A, C), C)$$

by lemma 3.2(iii), and this in turn can be identified with the canonical morphism

$$A \to \mathrm{RHom}_A(C, C)$$

which is an isomorphism.

To see that C is Gorenstein projective over $A \ltimes C$ can be done by the same method.

(ii) We will prove that $\mathrm{Hom}_A(C, I)$ is Gorenstein injective over $A \ltimes C$, the case of $\mathrm{Hom}_A(A, I) \cong I$ being similar.

Since C is Gorenstein projective over $A \ltimes C$, it has a complete projective resolution P; see [3, def. (4.2.1)]. In particular, P is exact and the zeroth cycle module $\mathrm{Z}_0(P)$ is isomorphic to C. Since C is finitely generated, we can assume that P consists of finitely generated $(A \ltimes C)$-modules by [3, thms. (4.1.4) and (4.2.6)].

Lemma 3.1(i) shows that $J = \mathrm{Hom}_A(P, I)$ is a complex of injective $(A \ltimes C)$-modules. It is clear that J is exact and that $\mathrm{Z}_0(J) \cong \mathrm{Hom}_A(\mathrm{Z}_0(P), I) \cong$ $\cong \mathrm{Hom}_A(C, I)$, so if we can prove that $\mathrm{Hom}_{A\ltimes C}(K, J)$ is exact for each injective $(A \ltimes C)$-module K, it will follow that J is a complete injective resolution of $\mathrm{Hom}_A(C,I)$ over $A \ltimes C$ whence $\mathrm{Hom}_A(C,I)$ is Gorenstein injective over $A \ltimes C$; see [3, def. (6.1.1)].

But

$$\mathrm{Hom}_{A\ltimes C}(K,J) = \mathrm{Hom}_{A\ltimes C}(K,\mathrm{Hom}_A(P,I)) \cong \mathrm{Hom}_A(K \otimes_{A\ltimes C} P, I),$$

so it is enough to see that $K \otimes_{A\ltimes C} P$ is exact. And by lemma 3.1(ii), the module K is a direct summand in $\mathrm{Hom}_A(A \ltimes C, L)$ for some injective A-module L, and hence it is enough to see that $\mathrm{Hom}_A(A \ltimes C, L) \otimes_{A\ltimes C} P$ is exact. But P consists of finitely generated projective $(A \ltimes C)$-modules which by [3, (A.2.11)] implies

$$\mathrm{Hom}_A(A \ltimes C, L) \otimes_{A\ltimes C} P \cong \mathrm{Hom}_A(\mathrm{Hom}_{A\ltimes C}(P, A \ltimes C), L),$$

and this is exact because L is an injective A-module while the complex $\mathrm{Hom}_{A\ltimes C}(P, A \ltimes C)$ is exact because P is a complete projective resolution.

LEMMA 3.4. *Let C be a semi-dualizing module for A and let I be an injective A-module. Then there is an equivalence of functors on* $\mathrm{D}(A \ltimes C)$,

$$\mathrm{RHom}_{A\ltimes C}(\mathrm{Hom}_A(A \ltimes C, I), -) \simeq \mathrm{RHom}_A(\mathrm{Hom}_A(C,I), -),$$

where the $(A \ltimes C)$*-structure of* $\mathrm{Hom}_A(A \ltimes C, I)$ *comes from the first variable.*

PROOF. First note that

$$\begin{aligned}\mathrm{Hom}_A(A \ltimes C, I) &\simeq \mathrm{RHom}_A(A \ltimes C, I)\\ &\overset{(a)}{\simeq} \mathrm{RHom}_A(\mathrm{RHom}_A(A \ltimes C, C), I)\\ &\overset{(b)}{\simeq} (A \ltimes C)\otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C,I)\end{aligned}$$

where (a) is by lemma 3.2(i) and (b) is by [3, (A.4.24)]. Hence

$$\begin{aligned}&\mathrm{RHom}_{A\ltimes C}(\mathrm{Hom}_A(A \ltimes C, I), -)\\ &\quad\simeq \mathrm{RHom}_{A\ltimes C}((A \ltimes C)\otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C,I), -)\\ &\quad\overset{(c)}{\simeq} \mathrm{RHom}_A(\mathrm{RHom}_A(C,I), -),\\ &\quad\simeq \mathrm{RHom}_A(\mathrm{Hom}_A(C,I), -)\end{aligned}$$

where (c) is by adjunction. □

4. Bounds on the injective dimension of C

This section studies the injective dimension of a semi-dualizing module C. If this dimension is finite, then C is a dualizing module, and this forces the ring A to be Cohen-Macaulay.

The main result is proposition 4.5 which will be used in section 5 to show that finiteness of Cohen-Macaulay dimensions characterizes Cohen-Macaulay rings with a dualizing module.

LEMMA 4.1. *Let C be a semi-dualizing module for A and let M be an A-module which is Gorenstein injective over $A \ltimes C$. Then there exists a short exact sequence of A-modules,*

$$0 \to M' \to \operatorname{Hom}_A(C, I) \to M \to 0,$$

such that

(i) *The A-module I is injective.*
(ii) *The A-module M' is Gorenstein injective over $A \ltimes C$.*
(iii) *For each injective A-module J, the sequence stays exact if one applies to it the functor* $\operatorname{Hom}_A(\operatorname{Hom}_A(C, J), -)$.

PROOF. Since M is Gorenstein injective over $A \ltimes C$, it has a complete injective resolution; see [3, def. (6.1.1)]. From this can be extracted a short exact sequence of $(A \ltimes C)$-modules,

$$0 \to N \to K \to M \to 0,$$

where K is injective and N Gorenstein injective over $A \ltimes C$, which stays exact if one applies to it the functor $\operatorname{Hom}_{A \ltimes C}(L, -)$ for any injective $(A \ltimes C)$-module L.

In particular, the sequence stays exact if one applies to it the functor $\operatorname{Hom}_{A \ltimes C}(\operatorname{Hom}_A(A \ltimes C, J), -)$ for any injective A-module J, because $\operatorname{Hom}_A(A \ltimes C, J)$ is an injective $(A \ltimes C)$-module by lemma 3.1(i).

By lemma 3.1(ii), the injective $(A \ltimes C)$-module K is a direct summand in $\operatorname{Hom}_A(A \ltimes C, I)$ for some injective A-module I. If $K \oplus K' \cong$ $\cong \operatorname{Hom}_A(A \ltimes C, I)$, then by adding K' to both the first and the second module in the short exact sequence, we may assume that the sequence has the form

$$0 \to N \to \operatorname{Hom}_A(A \ltimes C, I) \xrightarrow{\eta} M \to 0.$$

The module N is still Gorenstein injective over $A \ltimes C$, and the sequence still stays exact if one applies to it the functor

$$\mathrm{Hom}_{A\ltimes C}(\mathrm{Hom}_A(A\ltimes C, J), -)$$

for any injective A-module J.

Now let us consider in detail the homomorphism η. Elements of the source $\mathrm{Hom}_A(A\ltimes C, I)$ have the form (α,γ) where $A \xrightarrow{\alpha} I$ and $C \xrightarrow{\gamma} I$ are homomorphisms of A-modules. The $(A\ltimes C)$-module structure of $\mathrm{Hom}_A(A\ltimes C, I)$ comes from the first variable, and one checks that it takes the form

$$(a,c)\cdot(\alpha,\gamma) = (a\alpha + \chi_{\gamma(c)}, a\gamma),$$

where $\chi_{\gamma(c)}$ is the homomorphism $A \to I$ given by $a \mapsto a\gamma(c)$.

The target of η is M which is an A-module. When viewed as an $(A\ltimes C)$-module, M is annihilated by the ideal $0\ltimes C$, so

(5) $$0 = (0,c)\cdot\eta(\alpha,\gamma) = \eta((0,c)\cdot(\alpha,\gamma)) = \eta(\chi_{\gamma(c)}, 0),$$

where the last $=$ follows from the previous equation.

In fact, this implies

(6) $$\eta(\alpha, 0) = 0$$

for each $A \xrightarrow{\alpha} I$. To see so, note that there is a surjection $F \to \mathrm{Hom}_A(C, I)$ with F free, and hence a surjection $C\otimes_A F \to C\otimes_A \mathrm{Hom}_A(C,I)$. The target here is isomorphic to I by [4, prop. (4.4) and obs. (4.10)], so there is a surjection $C\otimes_A F \to I$. As $C\otimes_A F$ is a direct sum of copies of C, this means that, given an element i in I, it is possible to find homomorphisms $\gamma_1,\ldots,\gamma_t : C \to I$ and elements $c_1,\ldots,c_t$ in C with $i = \gamma_1(c_1)+\cdots+\gamma_t(c_t)$. Hence the homomorphism $A \xrightarrow{\alpha} I$ given by $a \mapsto ai$ satisfies

$$\alpha = \chi_{\gamma_1(c_1)+\cdots+\gamma_t(c_t)} = \chi_{\gamma_1(c_1)} + \cdots + \chi_{\gamma_t(c_t)},$$

and so equation (5) implies equation (6).

To make use of this, observe that the canonical exact sequence of $(A\ltimes C)$-modules

(7) $$\begin{array}{ccccccccc} 0 & \to & C & \to & A\ltimes C & \to & A & \to & 0, \\ & & c & \longmapsto & (0,c), & & & & \\ & & & & (a,c) & \longmapsto & a & & \end{array}$$

induces an exact sequence

$$0 \to \mathrm{Hom}_A(A,I) \to \mathrm{Hom}_A(A\ltimes C, I) \to \mathrm{Hom}_A(C,I) \to 0$$

because I is injective. Equation (6) means that the composition of

$\mathrm{Hom}_A(A,I) \to \mathrm{Hom}_A(A \ltimes C, I)$ and $\mathrm{Hom}_A(A \ltimes C, I) \xrightarrow{\eta} M$ is zero, so $\mathrm{Hom}_A(A \ltimes C, I) \xrightarrow{\eta} M$ factors through the surjection $\mathrm{Hom}_A(A \ltimes C, I) \to$ $\to \mathrm{Hom}_A(C, I)$. This means that we can construct a commutative diagram of $(A \ltimes C)$-modules with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \mathrm{Hom}_A(A \ltimes C, I) & \xrightarrow{\eta} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & \mathrm{Hom}_A(C, I) & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

We will show that if we view the lower row as a sequence of A-modules, then it is a short exact sequence with the properties claimed in the lemma.

(i) The A-module I is injective by construction.

(ii) Applying the Snake Lemma to the above diagram embeds the vertical arrows into exact sequences. The leftmost of these gives the short exact sequence

$$0 \to \mathrm{Hom}_A(A, I) \to N \to M' \to 0.$$

Here the modules $\mathrm{Hom}_A(A,I) \cong I$ and N are Gorenstein injective over $A \ltimes C$ by lemma 3.3(ii), respectively, by construction. Hence M' is also Gorenstein injective over $A \ltimes C$ because the class of Gorenstein injective modules is injectively resolving by [9, thm. 2.6].

(iii) By construction, the upper sequence in the diagram stays exact if one applies to it the functor $\mathrm{Hom}_{A \ltimes C}(\mathrm{Hom}_A(A \ltimes C, J), -)$ for any injective A-module J. It follows that the same holds for the lower row. But taking H_0 of the isomorphism in lemma 3.4 shows

$$\mathrm{Hom}_{A \ltimes C}(\mathrm{Hom}_A(A \ltimes C, J), -) \simeq \mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -),$$

so the lower row in the diagram also stays exact if one applies to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ for any injective A-module J. □

The special case $C = A$ of the following lemma was first proved by Frankild and Holm using the octahedral axiom; the present proof is simpler. In the lemma, ${}_C\mathcal{A}(A)$ and ${}_C\mathcal{B}(A)$ denote the Auslander and Bass classes of the semi-dualizing module C, as introduced in [4, def. (4.1)]. By Σ is denoted suspension of complexes in the derived category, and by id is denoted injective dimension of complexes as defined e.g. in [3, def. (A.3.8)].

LEMMA 4.2. *Let C be a semi-dualizing module for A. Let M be a complex in ${}_C\mathcal{A}(A)$ which has non-zero homology and satisfies that* $\mathrm{Gid}_{A\ltimes C}\, M < \infty$. *Write* $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$. *Then there is a distinguished triangle in* $\mathrm{D}(A)$,

$$\Sigma^s H \to Y \to M \to,$$

where H is an A-module which is Gorenstein injective over $A \ltimes C$ and where

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \mathrm{Gid}_{A\ltimes C}\, M.$$

PROOF. By [4, prop. (4.8)] we know

$$\sup\{\, i \mid \mathrm{H}_i(C \otimes^{\mathrm{L}}_A M) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\} = s,$$

so we can pick an injective resolution of $C \otimes^{\mathrm{L}}_A M$ of the form

$$J = \cdots \to 0 \to J_s \to J_{s-1} \to \cdots.$$

Then

$$M \cong \mathrm{RHom}_A(C, C \otimes^{\mathrm{L}}_A M) \cong \mathrm{Hom}_A(C, J)$$

where the first $\cong$ is because M is in ${}_C\mathcal{A}(A)$.

Now, $\mathrm{Hom}_A(C, J)$ consists of A-modules which are Gorenstein injective over $A \ltimes C$ by lemma 3.3(ii), and if we write

$$n = \mathrm{Gid}_{A\ltimes C}\, M$$

for the finite Gorenstein injective dimension of M over $A \ltimes C$, it follows from [3, thm. (6.2.4)] or [5, thm. (2.5)] that the soft truncation

$$\cdots \to 0 \to \mathrm{Hom}_A(C, J_s) \to \cdots \to \mathrm{Hom}_A(C, J_{-n+1}) \to G \to 0 \to \cdots$$

has G Gorenstein injective over $A \ltimes C$. The truncation remains quasi-isomorphic to M.

By iterating lemma 4.1, the module G can be embedded into the complex

$$\cdots \to 0 \to \mathrm{Hom}_A(C, I_{s+1}) \to \mathrm{Hom}_A(C, I_s) \to \cdots \to \mathrm{Hom}_A(C, I_{-n+1}) \to G \to 0 \to \cdots,$$

where the I_i are injective A-modules by 4.1(i). This complex can only have non-zero homology in degree $s+1$; let us call the $(s+1)$'st homology module H so the complex is quasi-isomorphic to $\Sigma^{s+1} H$. Note that the A-module H is Gorenstein injective over $A \ltimes C$ by 4.1(ii).

By 4.1(iii), the identity on G can be lifted to a chain map

$$
\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & (C,J_s) & \longrightarrow \cdots \longrightarrow & (C,J_{-n+1}) & \longrightarrow & G & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & (C,I_{s+1}) & \longrightarrow & (C,I_s) & \longrightarrow \cdots \longrightarrow & (C,I_{-n+1}) & \longrightarrow & G & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
\tag{8}
$$

where we have omitted Hom_A everywhere for typographical reasons. (Note that this is actually just a version of Auslander's pitchfork construction.)

If we construct the mapping cone of the chain map (8), it is a standard observation that the null homotopic subcomplex

$$\cdots \longrightarrow 0 \longrightarrow G = G \longrightarrow 0 \longrightarrow \cdots$$

splits off as a direct summand. The remaining complex Q consists of modules of the form $\mathrm{Hom}_A(C, K)$ where K is an injective A-module, and it sits in homological degrees $s+1, \ldots, -n+1$.

In the derived category $\mathrm{D}(A)$, we can replace a complex with any quasi-isomorphic complex. Hence (8) can be viewed as a morphism $M \to \Sigma^{s+1}H$. Given that the mapping cone of (8) is Q up to a null-homotopic summand, this gives that there is a distinguished triangle

$$M \to \Sigma^{s+1}H \to Q \to$$

in $\mathrm{D}(A)$. Rotation gives a distinguished triangle

$$\Sigma^{s}H \to \Sigma^{-1}Q \to M \to$$

in $\mathrm{D}(A)$ where H is Gorenstein injective over $A \ltimes C$, and where $\Sigma^{-1}Q$ consists of modules of the form $\mathrm{Hom}_A(C, K)$ where K is an injective A-module, and sits in homological degrees $s, \ldots, -n$.

We claim that with $Y = \Sigma^{-1}Q$, this is the lemma's triangle. It only remains to check $\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \mathrm{Gid}_{A \ltimes C}\, M$. But Y has the form

$$Y = \cdots \to 0 \to \mathrm{Hom}_A(C, K_s) \to \cdots \to \mathrm{Hom}_A(C, K_{-n}) \to 0 \to \cdots$$

where the K_i are injective A-modules, and [4, prop. (4.4) and obs. (4.10)] give that modules of the form $\mathrm{Hom}_A(C, K)$ are acyclic for the functor $C \otimes_A -$ whence

$$C \otimes^{\mathrm{L}}_A Y \cong C \otimes_A Y.$$

Moreover, [4, prop. (4.4) and obs. (4.10)] show $C \otimes_A \mathrm{Hom}_A(C, K_i) \cong K_i$ for each i, so

$$C \otimes_A Y \cong \cdots \to 0 \to K_s \to \cdots \to K_{-n} \to 0 \to \cdots.$$

The last two equations imply

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant n = \mathrm{Gid}_{A \ltimes C} M$$

as desired. □

LEMMA 4.3. *Let C be a semi-dualizing module for A. Let M be a complex in ${}_C\mathcal{A}(A)$ with non-zero homology. Write $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$ and suppose that*

$$\mathrm{Ext}^{s+1}_A(M, H) = 0$$

for each A-module H which is Gorenstein injective over $A \ltimes C$. Then

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) = \mathrm{Gid}_{A \ltimes C} M.$$

PROOF. To prove the lemma's equation, let us first prove the inequality $\leqslant$. We may clearly suppose $\mathrm{Gid}_{A \ltimes C} M < \infty$. By lemma 4.2 there is a distinguished triangle in $\mathrm{D}(A)$,

$$\Sigma^s H \to Y \to M \to \Sigma^{s+1} H, \tag{9}$$

where H is an A-module which is Gorenstein injective over $A \ltimes C$, and where

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \mathrm{Gid}_{A \ltimes C} M. \tag{10}$$

But

$$\mathrm{Hom}_{\mathrm{D}(A)}(M, \Sigma^{s+1} H) \cong \mathrm{Ext}^{s+1}_A(M, H) = 0$$

by assumption, so the connecting morphism $M \to \Sigma^{s+1} H$ in (9) is zero, whence (9) is a split distinguished triangle with $Y \cong \Sigma^s H \oplus M$.

This implies

$$C \otimes^{\mathrm{L}}_A Y \cong (C \otimes^{\mathrm{L}}_A \Sigma^s H) \oplus (C \otimes^{\mathrm{L}}_A M)$$

from which clearly follows

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) \leqslant \mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y). \tag{11}$$

Combining the inequalities (11) and (10) shows

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) \leqslant \mathrm{Gid}_{A \ltimes C} M$$

as desired.

Let us next prove the inequality $\geqslant$. Let $t = \sup\{\, i \mid \mathrm{H}_i(C \otimes^{\mathrm{L}}_A M) \neq 0 \,\}$

and $n = \mathrm{id}_A(C \otimes^{\mathrm{L}}_A M)$. We may clearly suppose $n < \infty$. Let

$$J = \cdots \to 0 \to J_t \to \cdots \to \cdots \to J_{-n} \to 0 \to \cdots$$

be an injective resolution of $C \otimes^{\mathrm{L}}_A M$. The complex M is in ${}_C\mathcal{A}(A)$ by assumption, so we get the first $\cong$ in

$$M \cong \mathrm{RHom}_A(C, C \otimes^{\mathrm{L}}_A M) \cong \mathrm{Hom}_A(C, J).$$

Lemma 3.3(ii) implies that $\mathrm{Hom}_A(C, J)$ is a complex of Gorenstein injective modules over $A \ltimes C$. Since $\mathrm{Hom}_A(C, J)_\ell = \mathrm{Hom}_A(C, J_\ell) = 0$ for $\ell < -n$, we see

$$\mathrm{Gid}_{A \ltimes C}\, M \leqslant n = \mathrm{id}_A(C \otimes^{\mathrm{L}}_A M)$$

as desired. □

The following lemma provides some complexes to which lemma 4.3 applies. By pd is denoted projective dimension of complexes as defined e.g. in [3, def. (A.3.9)].

Lemma 4.4. *Let C be a semi-dualizing module for A. Let M be a complex of A-modules which has non-zero homology and satisfies that $\mathrm{H}_i M = 0$ for $i \ll 0$ and that $\mathrm{pd}_A\, M < \infty$. Write $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$. If H is an A-module which is Gorenstein injective over $A \ltimes C$, then*

$$\mathrm{Ext}^{s+1}_A(M, H) = 0.$$

Proof. The conditions on M imply that it has a bounded projective resolution P, and clearly

$$\mathrm{Ext}^{s+1}_A(M, H) \;\cong\; \mathrm{Ext}^1_A(Q, H)$$

when Q is the s'th cokernel of P. Since

$$\cdots \to P_{s+1} \to P_s \to Q \to 0$$

is a projective resolution of Q and since P is bounded, we have $\mathrm{pd}_A Q < \infty$. Hence it is enough to show

$$\mathrm{Ext}^1_A(Q, H) = 0$$

for each A-module Q with $\mathrm{pd}_A Q < \infty$. The case $Q = 0$ is clear, so we assume $Q \neq 0$.

To prove this, we first argue that if I is any injective A-module then

$$\mathrm{Ext}^i_A(Q, \mathrm{Hom}_A(C, I)) = 0 \tag{12}$$

for $i > 0$. For this, note that we have

$$\begin{aligned}\mathrm{RHom}_A(Q,\mathrm{Hom}_A(C,I)) &\cong \mathrm{RHom}_A(Q,\mathrm{RHom}_A(C,I))\\ &\overset{(a)}{\cong} \mathrm{RHom}_A(Q\otimes^{\mathrm{L}}_A C,I)\\ &\overset{(b)}{\cong} \mathrm{RHom}_A(C,\mathrm{RHom}_A(Q,I))\\ &\cong \mathrm{RHom}_A(C,\mathrm{Hom}_A(Q,I))\end{aligned}$$

where (a) and (b) are by adjunction, and consequently,

$$\mathrm{Ext}^i_A(Q,\mathrm{Hom}_A(C,I)) \cong \mathrm{Ext}^i_A(C,\mathrm{Hom}_A(Q,I)) \tag{13}$$

for each i. The condition $\mathrm{pd}_A Q < \infty$ implies $\mathrm{id}_A \mathrm{Hom}_A(Q,I) < \infty$, and therefore $\mathrm{Hom}_A(Q,I)$ belongs to ${}_C\mathcal{B}(A)$ by [4, prop. (4.4)]. Thus [4, obs. (4.10)] implies that the right hand side of (13) is zero for $i > 0$, proving equation (12).

Now set $n = \mathrm{pd}_A Q$. Repeated use of lemma 4.1 shows that there is an exact sequence of A-modules

$$0 \to H' \to \mathrm{Hom}_A(C,I_{n-1}) \to \cdots \to \mathrm{Hom}_A(C,I_0) \to H \to 0, \tag{14}$$

where $I_0,\ldots,I_{n-1}$ are injective A-modules. Applying $\mathrm{Hom}_A(Q,-)$ to (14) and using equation (12), we obtain

$$\mathrm{Ext}^1_A(Q,H) \;\cong\; \mathrm{Ext}^{n+1}_A(Q,H') = 0$$

as desired. Here the last equality holds because $\mathrm{pd}_A Q = n$. ☐

The following proposition uses some homological invariants for complexes of modules. In particular, injective, projective and flat dimension of complexes are used. They are denoted id, pd, and fd, and the definition can be found e.g. in [3, defs. (A.3.8), (A.3.9), and (A.3.10)].

Recall from [3, (A.5.7.4)] that when A is local with residue class field k and C is a complex of A-modules with bounded finitely generated homology, then

$$\mathrm{id}_A C = -\inf\{\, i \mid \mathrm{H}_i \mathrm{RHom}_A(k,C) \neq 0 \,\}.$$

Recall also from [3, (A.6.3)] that the width of a complex of A-modules M is defined as

$$\mathrm{width}_A M = \inf\{\, i \mid \mathrm{H}_i(M\otimes^{\mathrm{L}}_A k) \neq 0 \,\}.$$

The following is the main result of this section.

PROPOSITION 4.5. *Assume that the ring A is local and let C be a semi-dualizing module for A. Let M be a complex of A-modules which has non-zero homology and satisfies $\mathrm{H}_i M = 0$ for $i \ll 0$ and $\mathrm{fd}_A M < \infty$. Then*

$$\mathrm{id}_A C \leqslant \mathrm{Gid}_{A \ltimes C} M + \mathrm{width}_A M.$$

PROOF. Denote by k the residue class field of A. Observe that $\mathrm{H}_i M = 0$ for $i \ll 0$ implies $\mathrm{H}_i(M \otimes^{\mathrm{L}}_A k) = 0$ for $i \ll 0$, whence $\mathrm{width}_A M > -\infty$. Also, we can assume $\mathrm{width}_A M < \infty$, because the proposition is trivially true if $\mathrm{width}_A M = \infty$.

Since $\mathrm{fd}_A M < \infty$, the isomorphism [3, (A.4.23)] gives

$$\mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \cong \mathrm{RHom}_A(k, C) \otimes^{\mathrm{L}}_A M.$$

This implies (a) in

$$\begin{aligned}
&\inf\{\, i \mid \mathrm{H}_i \mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \neq 0 \,\} \\
&\overset{(a)}{=} \inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_A(k, C) \otimes^{\mathrm{L}}_A M) \neq 0 \,\} \\
&\overset{(b)}{=} \inf\{\, i \mid \mathrm{H}_i \mathrm{RHom}_A(k, C) \neq 0 \,\} + \inf\{\, i \mid \mathrm{H}_i(M \otimes^{\mathrm{L}}_A k) \neq 0 \,\} \\
&= -\mathrm{id}_A C + \mathrm{width}_A M,
\end{aligned}$$

where (b) is by [3, (A.7.9.2)]. Consequently,

$$\begin{aligned}
\mathrm{id}_A C &= -\inf\{\, i \mid \mathrm{H}_i \mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \neq 0 \,\} + \mathrm{width}_A M \\
&\leqslant \mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) + \mathrm{width}_A M \\
&= (\,*\,).
\end{aligned}$$

The condition $\mathrm{fd}_A M < \infty$ implies $\mathrm{pd}_A M < \infty$ by [11, Seconde partie, cor. (3.2.7)], and hence if we write $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$, lemma 4.4 gives

$$\mathrm{Ext}^{s+1}_A(M, H) = 0$$

when H is an A-module which is Gorenstein injective over $A \ltimes C$. But $\mathrm{fd}_A M < \infty$ also implies $M \in {}_C\mathcal{A}(A)$ by [4, prop. (4.4)], and altogether, lemma 4.3 applies to M and gives

$$(\,*\,) = \mathrm{Gid}_{A \ltimes C} M + \mathrm{width}_A M$$

as desired. □

We will also need the dual of proposition 4.5. First an easy lemma.

LEMMA 4.6. *Let C be an A-module, let I be a faithfully injective A-module, and let M be a complex of A-modules with $H_i M = 0$ for $i \ll 0$. Then*

$$\mathrm{Gid}_{A\ltimes C}\,\mathrm{Hom}_A\,(M, I) = \mathrm{Gfd}_{A\ltimes C}\,M.$$

PROOF. From lemma 3.1(i) follows that $E = \mathrm{Hom}_A(A \ltimes C, I)$ is a faithfully injective $(A \ltimes C)$-module. Hence

$$\mathrm{Gid}_{A\ltimes C}\mathrm{Hom}_{A\ltimes C}\,(M, E) = \mathrm{Gfd}_{A\ltimes C}\,M$$

follows from [3, thm. (6.4.2)].

But equation (2) in the proof of lemma 3.1 shows

$$\mathrm{Hom}_{A\ltimes C}\,(M, E) \cong \mathrm{Hom}_A\,(M, I),$$

so accordingly,

$$\mathrm{Gid}_{A\ltimes C}\,\mathrm{Hom}_A(M, I) = \mathrm{Gfd}_{A\ltimes C}\,M.$$

□

PROPOSITION 4.7. *Assume that the ring A is local and let C be a semi-dualizing module for A. Let N be a complex of A-modules which has non-zero homology and satisfies $\mathrm{H}_i N = 0$ for $i \gg 0$ and $\mathrm{id}_A N < \infty$. Then*

$$\mathrm{id}_A C \leqslant \mathrm{Gfd}_{A\ltimes C}\,N + \mathit{depth}_A N.$$

PROOF. Apply Matlis duality and lemma 4.6 to proposition 4.5. □

5. Properties of the Cohen-Macaulay dimensions

This section contains our main results on the Cohen-Macaulay dimensions, as announced in the introduction. From now on, A is assumed to be local with residue class field k.

THEOREM 5.1. *The following conditions are equivalent.*

(CM) A is a Cohen-Macaulay ring with a dualizing module.

(I1) $\mathrm{CMid}_A\,M < \infty$ *holds when M is any complex of A-modules with bounded homology.*

(I2) *There is a complex M of A-modules with bounded homology,* $\mathrm{CMid}_A\,M < \infty$, $\mathrm{fd}_A\,M < \infty$, *and* $\mathrm{width}_A\,M < \infty$.

(I3) $\mathrm{CMid}_A\,k < \infty$.

(P1) $\mathrm{CMpd}_A M < \infty$ *holds when* M *is any complex of* A*-modules with bounded homology.*

(P2) *There is a complex* M *of* A*-modules with bounded homology,* $\mathrm{CMpd}_A M < \infty$, $\mathrm{id}_A M < \infty$, *and* $\mathrm{depth}_A M < \infty$.

(P3) $\mathrm{CMpd}_A k < \infty$.

(F1) $\mathrm{CMfd}_A M < \infty$ *holds when* M *is any complex of* A*-modules with bounded homology.*

(F2) *There is a complex* M *of* A*-modules with bounded homology,* $\mathrm{CMfd}_A M < \infty$, $\mathrm{id}_A M < \infty$, *and* $\mathrm{depth}_A M < \infty$.

(F3) $\mathrm{CMfd}_A k < \infty$.

PROOF. Let us first prove that conditions (CM), (I1), (I2), and (I3) are equivalent.

(CM) $\Rightarrow$ (I1) Let A be Cohen-Macaulay with dualizing module C. Then $A \ltimes C$ is Gorenstein by [7, thm. 5.6]. If M is a complex of A-modules with bounded homology, then M is also a complex of $(A \ltimes C)$-modules with bounded homology, so

$$\mathrm{Gid}_{A \ltimes C} M < \infty$$

by [3, thm. (6.2.7)]. As C is in particular a semi-dualizing module, the definition of CMid then implies

$$\mathrm{CMid}_A M < \infty.$$

(I1) $\Rightarrow$ (I2) and (I1) $\Rightarrow$ (I3) Trivial.

(I2) $\Rightarrow$ (CM) When $\mathrm{CMid}_A M < \infty$ then the definition of CMid implies that A has a semi-dualizing module C with

$$\mathrm{Gid}_{A \ltimes C} M < \infty.$$

When

$$\mathrm{width}\, M < \infty$$

then M has non-zero homology. And finally, when

$$\mathrm{fd}\, M < \infty$$

also holds, then proposition 4.5 implies

$$\mathrm{id}_A C < \infty.$$

So C is a dualizing module for A, and hence A is Cohen-Macaulay by [3, (A.8.5.3)] and has a dualizing module.

(I3) $\Rightarrow$ (CM) When $\mathrm{CMid}_A k < \infty$, then A has a semi-dualizing module C with

$$\mathrm{Gid}_{A \ltimes C}\, k < \infty.$$

Denoting by $E_{A\ltimes C}(k)$ the injective hull of k over $A \ltimes C$, it follows from [9, thm. 2.22] that

$$\mathrm{RHom}_{A\ltimes C}(E_{A\ltimes C}(k), k)$$

has bounded homology. Denoting Matlis duality over $A \ltimes C$ by $\vee$, we have

$$\begin{aligned}\mathrm{RHom}_{A\ltimes C}(E_{A\ltimes C}(k), k) &\cong \mathrm{RHom}_{A\ltimes C}(k^{\vee}, E_{A\ltimes C}(k)^{\vee}) \\ &\cong \mathrm{RHom}_{A\ltimes C}(k, \widehat{A \ltimes C}) \\ &\overset{(a)}{\cong} \mathrm{RHom}_{A\ltimes C}(k, A \ltimes C) \otimes_{A\ltimes C} \widehat{A \ltimes C},\end{aligned}$$

where (a) is by [3, (A.4.23)]. Since the completion $\widehat{A \ltimes C}$ is faithfully flat over $A \ltimes C$, it follows that also $\mathrm{RHom}_{A\ltimes C}(k, A \ltimes C)$ has bounded homology, whence $A \ltimes C$ is a Gorenstein ring.

But then C is a dualizing module for A by [7, thm. 5.6] and so again, A is Cohen-Macaulay with a dualizing module.

Similar proofs give that also (CM), (P1), (P2), and (P3) as well as (CM), (F1), (F2), and (F3) are equivalent. For this, proposition 4.7 should be used instead of proposition 4.5. □

Remark 5.2. In condition (I2) of theorem 5.1, one could consider for M either the ring A itself, or the Koszul complex $K(x_1, \ldots, x_r)$ on any sequence of elements $x_1, \ldots, x_r$ in the maximal ideal. These complexes satisfy $\mathrm{fd}_A\, M < \infty$ and $\mathrm{width}_A\, M < \infty$, and so either of the conditions

$$\mathrm{CMid}_A\, A < \infty \quad \text{and} \quad \mathrm{CMid}_A\, K(x_1, \ldots, x_r) < \infty$$

is equivalent to A being a Cohen-Macaulay ring with a dualizing module.

Similarly, in conditions (P2) and (F2), one could consider for M either the injective hull of the residue class field, $E_A(k)$, or a dualizing complex D (if one is known to exist). These complexes satisfy $\mathrm{id}_A\, M < \infty$ and $\mathrm{depth}_A\, M < \infty$, and so either of the conditions

$$\mathrm{CMpd}_A\, E_A(k) < \infty \quad \text{and} \quad \mathrm{CMpd}_A\, D < \infty$$

and

$$\mathrm{CMfd}_A\, E_A(k) < \infty \quad \text{and} \quad \mathrm{CMfd}_A\, D < \infty$$

is equivalent to A being a Cohen-Macaulay ring with a dualizing module.

REMARK 5.3. Our viewpoint of emphasizing the Cohen-Macaulay homological dimensions makes us think of theorem 5.1 as a main result. However, proposition 4.5 actually implies a more precise result, namely, if there is a complex M with non-zero homology, $\mathrm{H}_i M = 0$ for $i \ll 0$, and

$$\mathrm{Gid}_{A \ltimes C} M < \infty, \quad \mathrm{fd}_A M < \infty, \quad \text{and} \quad \mathrm{width}_A M < \infty,$$

then A is Cohen-Macaulay with dualizing module C.

Similarly, proposition 4.7 implies that if there is a complex N with non-zero homology and $\mathrm{H}_i N = 0$ for $i \gg 0$, then either of

$$\mathrm{Gpd}_{A \ltimes C} N < \infty, \quad \mathrm{id}_A N < \infty, \quad \text{and} \quad \mathrm{depth}_A N < \infty$$

and

$$\mathrm{Gfd}_{A \ltimes C} N < \infty, \quad \mathrm{id}_A N < \infty, \quad \text{and} \quad \mathrm{depth}_A N < \infty$$

implies that A is Cohen-Macaulay with dualizing module C.

THEOREM 5.4. *Let M be an A-module. Then*

$$\mathrm{CMid}_A M < \infty \Leftrightarrow \mathrm{CMid}_A M \leq \dim A,$$

$$\mathrm{CMpd}_A M < \infty \Leftrightarrow \mathrm{CMpd}_A M \leq \dim A,$$

$$\mathrm{CMfd}_A M < \infty \Leftrightarrow \mathrm{CMfd}_A M \leq \dim A.$$

PROOF. The implications $\Leftarrow$ are trivial.

To see the implication $\Rightarrow$ for CMpd, observe that when M is given there exists a semi-dualizing module C with

$$\mathrm{CMpd}_A M = \mathrm{Gpd}_{A \ltimes C} M.$$

When this is finite, we have

$$\mathrm{Gpd}_{A \ltimes C} M \leq \mathrm{FPD}(A \ltimes C)$$

by [9, thm. 2.28], where FPD denotes the (big) finitistic projective dimension. But

$$\mathrm{FPD}(A \ltimes C) = \dim A \ltimes C$$

by [11, Seconde partie, thm. (3.2.6)], and A and $A \ltimes C$ are finitely generated as modules over each other, so

$$\dim A \ltimes C = \dim A.$$

Together, this establishes the implication $\Rightarrow$ for CMpd.

The implication $\Rightarrow$ for CMfd follows by a similar argument, using that the finitistic flat dimension satisfies $\mathrm{FFD}(A \ltimes C) \leq \mathrm{FPD}(A \ltimes C)$ because of [11, Seconde partie, cor. (3.2.7)].

Finally, the implication $\Rightarrow$ for CMid follows from the one for CMfd using Matlis duality and lemma 4.6. □

The following results use CMdim, the Cohen-Macaulay dimension introduced by Gerko in [8], and Gdim, the G-dimension originally introduced by Auslander and Bridger in [1].

LEMMA 5.5. *Let C be a semi-dualizing module for A and let M be a finitely generated A-module. If*

$$\mathrm{Gpd}_{A \ltimes C} M < \infty$$

then

$$\mathrm{CMdim}_A M = \mathrm{Gpd}_{A \ltimes C} M.$$

PROOF. Combining [8, proof of thm. 3.7] with [8, def. 3.2'] shows

$$\mathrm{CMdim}_A M \leqslant \mathrm{Gpd}_{A \ltimes C} M.$$

So $\mathrm{Gpd}_{A \ltimes C} M < \infty$ implies $\mathrm{CMdim}_A M < \infty$ and hence

$$\mathrm{CMdim}_A M = \mathrm{depth}_A A - \mathrm{depth}_A M$$

by [8, thm. 3.8].

On the other hand,

$$\mathrm{Gpd}_{A \ltimes C} M = G\text{-}\mathrm{dim}_C M$$

by [10, prop. 3.1], where $G\text{-}\mathrm{dim}_C M$ is the homological dimension introduced in [4, def. (3.11)]. So $G\text{-}\mathrm{dim}_C M$ is finite and hence

$$G\text{-}\mathrm{dim}_C M = \mathrm{depth}_A A - \mathrm{depth}_A M$$

by [4, thm. (3.14)].

Combining the last three equations shows

$$\mathrm{CMdim}_A M = \mathrm{Gpd}_{A \ltimes C} M$$

as desired. □

THEOREM 5.6. *Let M be a finitely generated A-module. Then*

$$\mathrm{CMdim}_A M \leqslant \mathrm{CMpd}_A M \leqslant \mathrm{Gdim}_A M,$$

and if one of these numbers is finite then the inequalities to its left are equalities.

Proof. The first inequality is clear from lemma 5.5, since $\mathrm{CMpd}_A\, M$ is defined as the infimum of all $\mathrm{Gpd}_{A\ltimes C}\, M$.

For the second inequality, note that the ring A is itself a semi-dualizing module, so the definition of CMpd gives $\leqslant$ in

$$\mathrm{CMpd}_A\, M \leqslant \mathrm{Gpd}_{A\ltimes A} M = \mathrm{Gpd}_A\, M = \mathrm{Gdim}_A\, M,$$

where the first $=$ is by [10, cor. 2.17], and the second $=$ holds by [3, cor. (4.4.6)] or [5, prop. (2.11)(b)] because M is finitely generated.

Equalities: If $\mathrm{Gdim}_A\, M < \infty$ then $\mathrm{CMdim}_A\, M < \infty$ by [8, thm. 3.7]. But $\mathrm{Gdim}_A\, M < \infty$ implies

$$\mathrm{Gdim}_A\, M = \mathrm{depth}_A\, A - \mathrm{depth}_A\, M$$

by [3, thm. (2.3.13)], and similarly, $\mathrm{CMdim}_A\, M < \infty$ implies

$$\mathrm{CMdim}_A\, M = \mathrm{depth}_A A - \mathrm{depth}_A\, M$$

by [8, thm. 3.8]. So it follows that $\mathrm{CMdim}_A\, M = \mathrm{Gdim}_A\, M$, and hence both inequalities in the theorem must be equalities.

If $\mathrm{CMpd}_A\, M < \infty$ then by the definition of CMpd there exists a semi-dualizing module C over A with $\mathrm{Gpd}_{A\ltimes C}\, M < \infty$. But by lemma 5.5, any such C has

$$\mathrm{CMdim}_A\, M = \mathrm{Gpd}_{A\ltimes C}\, M,$$

so the first inequality in the theorem is an equality. □

We finish with two clasically flavoured results.

Proposition 5.7 [Auslander-Buchsbaum formula]. *Let M be a finitely generated A-module. If* $\mathrm{CMpd}_A\, M < \infty$, *then*

$$\mathrm{CMpd}_A\, M = \mathrm{depth}_A A - \mathrm{depth}_A\, M.$$

Proof. By the definition of CMpd, we can pick a semi-dualizing A-module C such that

$$\mathrm{CMpd}_A\, M = \mathrm{Gpd}_{A\ltimes C}\, M.$$

But M is finitely generated, so

$$\mathrm{Gpd}_{A\ltimes C}\, M = \mathrm{Gdim}_{A\ltimes C} M$$

by [3, cor. (4.4.6)] or [5, prop. (2.11)(b)]. The Auslander-Bridger formula gives

$$\mathrm{Gdim}_{A\ltimes C}\, M = \mathrm{depth}_{A\ltimes C}\, A \ltimes C - \mathrm{depth}_{A\ltimes C}\, M.$$

Finally, it remains to note that

$$\operatorname{depth}_{A\ltimes C} A \ltimes C = \operatorname{depth}_A A$$

and

$$\operatorname{depth}_{A\ltimes C} M = \operatorname{depth}_A M$$

because $A \ltimes C$ and A are finitely generated over each other. □

PROPOSITION 5.8 [Bass formula]. *Assume that A has a dualizing complex and let $N \neq 0$ be a finitely generated A-module. If* $\operatorname{CMid}_A N < \infty$, *then*

$$\operatorname{CMid}_A N = \operatorname{depth} A.$$

PROOF. By the definition of CMid, we can pick a semi-dualizing A-module C such that $\operatorname{CMid}_A N = \operatorname{Gid}_{A\ltimes C} N$. By [5, thm. (6.4)], finiteness of $\operatorname{Gid}_{A\ltimes C} N$ implies that

$$\operatorname{Gid}_{A\ltimes C} N = \operatorname{depth}_{A\ltimes C} A \ltimes C,$$

since $N \neq 0$ is finitely generated over $A \ltimes C$, and since $A \ltimes C$ is finitely generated over A and so has a dualizing complex because A does. Finally, we have

$$\operatorname{depth}_{A\ltimes C} A \ltimes C = \operatorname{depth}_A A$$

again. □

Acknowledgments. We wish to thank the referee for a number of excellent suggestions.

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Manoscritto pervenuto in redazione il 30 settembre 2005