Rend. Sem. Mat. Univ. Pol. Torino - Vol. 66, 4 (2008)
Second Conf. on Pseudo-Diff. Operators: Invited Lectures

**G. Grubb**

# KREIN RESOLVENT FORMULAS FOR ELLIPTIC BOUNDARY PROBLEMS IN NONSMOOTH DOMAINS*

**Abstract.** The paper reports on a recent construction of $M$-functions and Kreĭn resolvent formulas for general closed extensions of an adjoint pair, and their implementation to boundary value problems for second-order strongly elliptic operators on smooth domains. The results are then extended to domains with $C^{1,1}$ Hölder smoothness, by use of a recently developed calculus of pseudodifferential boundary operators with nonsmooth symbols.

## 1. Introduction

In the study of boundary value problems for ordinary differential equations, the Weyl-Titchmarsh $m$-function has played an important role for many years; it allows a reduction of questions concerning the resolvent $(\widetilde{A}-\lambda)^{-1}$ of a realisation $\widetilde{A}$ to questions concerning an associated family $M(\lambda)$ of matrices, holomorphic in $\lambda \in \rho(\widetilde{A})$. Moreover, there is a formula describing the difference between $(\widetilde{A}-\lambda)^{-1}$ and the resolvent of a well-known reference problem in terms of $M(\lambda)$, a so-called Kreĭn resolvent formula. The concepts have also been introduced in connection with the abstract theories of extensions of symmetric operators or adjoint pairs in Hilbert spaces, initiated by Kreĭn [22] and Vishik [32]. The literature on this is abundant, and we refer to e.g. Brown, Marletta, Naboko and Wood [10] and Brown, Grubb and Wood [9] for accounts of the development, and references. For elliptic partial differential equations in higher dimensions, concrete interpretations of $M(\lambda)$ have been taken up in recent years, e.g. in Amrein and Pearson [5], Behrndt and Langer [6], and in [10]; here $M(\lambda)$ is a family of operators defined over the boundary. In the present paper we report on the latest development in nonsymmetric cases worked out in [9]; it uses the early work of Grubb [14] as an important ingredient.

The interest of this in a context of pseudodifferential operators is that $M(\lambda)$ in elliptic cases, and also in some nonelliptic cases, is a pseudodifferential operator ($\psi$do), to which $\psi$do methods can be applied. The new results in the present paper are concerned with situations with a nonsmooth boundary. Our strategy here is to apply the nonsmooth pseudodifferential boundary operator ($\psi$dbo) calculus introduced by Abels [3]. We show that when the domain is $C^{1,1}$ and the given strongly elliptic second-order operator $A$ has smooth coefficients, then indeed the $M$-function can be defined as a generalized $\psi$do over the boundary, and a Kreĭn formula holds. Selfadjoint cases have been treated under various nonsmoothness hypotheses in Gesztesy and Mitrea [12], Posilicano and Raimondi [29], but the present study allows nonselfadjoint operators, and includes a discussion of Neumann-type boundary conditions. Besides bounded domains, we also treat exterior domains and perturbed halfspaces.

*It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

The author thanks Helmut Abels for useful conversations.

## 2. Abstract results

We begin by recalling the theory of extensions and $M$-functions established in works of Brown, Wood and the author [9] and [14].

There is given an adjoint pair of closed, densely defined linear operators $A_{\min}$, $A'_{\min}$ in a Hilbert space $H$:

$$A_{\min} \subset (A'_{\min})^* = A_{\max}, \quad A'_{\min} \subset (A_{\min})^* = A'_{\max}.$$

Let $\mathcal{M}$ denote the set of linear operators lying between the minimal and maximal operator:

$$\mathcal{M} = \{\widetilde{A} \mid A_{\min} \subset \widetilde{A} \subset A_{\max}\}, \quad \mathcal{M}' = \{\widetilde{A}' \mid A'_{\min} \subset \widetilde{A}' \subset A'_{\max}\}.$$

Write $\widetilde{A}u$ as $Au$ for any $\widetilde{A}$, and $\widetilde{A}'u$ as $A'u$ for any $\widetilde{A}'$. Assume that there exists an $A_\gamma \in \mathcal{M}$ with $0 \in \rho(A_\gamma)$; then $A_\gamma^* \in \mathcal{M}'$ with $0 \in \rho(A_\gamma^*)$. We shall define $M$-functions for *any* closed $\widetilde{A} \in \mathcal{M}$.

First recall some details from the treatment of extensions in [14]: Denote

$$Z = \ker A_{\max}, \quad Z' = \ker A'_{\max}.$$

Define the basic non-orthogonal decompositions

$$\begin{aligned} D(A_{\max}) &= D(A_\gamma) \dot{+} Z, \text{ denoted } u = u_\gamma + u_\zeta = \mathrm{pr}_\gamma u + \mathrm{pr}_\zeta u, \\ D(A'_{\max}) &= D(A_\gamma^*) \dot{+} Z', \text{ denoted } v = v_{\gamma'} + v_{\zeta'} = \mathrm{pr}_{\gamma'} v + \mathrm{pr}_{\zeta'} v; \end{aligned}$$

here $\mathrm{pr}_\gamma = A_\gamma^{-1} A_{\max}$, $\mathrm{pr}_\zeta = I - \mathrm{pr}_\gamma$, and $\mathrm{pr}_{\gamma'} = (A_\gamma^*)^{-1} A'_{\max}$, $\mathrm{pr}_{\zeta'} = I - \mathrm{pr}_{\gamma'}$. By $\mathrm{pr}_V u = u_V$ we denote the *orthogonal projection* of $u$ onto $V$.

The following "abstract Green's formula" holds:

$$(Au, v) - (u, A'v) = ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z). \tag{1}$$

It can be used to show that when $\widetilde{A} \in \mathcal{M}$ and we set $W = \overline{\mathrm{pr}_{\zeta'} D(\widetilde{A}^*)}$, then

$$\{\{u_\zeta, (Au)_W\} \mid u \in D(\widetilde{A})\} \text{ is a graph.}$$

Denoting the operator with this graph by $T$, we have:

THEOREM 1. [14] *For the closed* $\widetilde{A} \in \mathcal{M}$, *there is a* 1–1 *correspondence*

$$\widetilde{A} \textit{ closed} \longleftrightarrow \begin{cases} T : V \to W, \textit{ closed, densely defined} \\ \textit{with } V \subset Z,\ W \subset Z', \textit{ closed subspaces.} \end{cases}$$

*Here* $D(T)=\mathrm{pr}_\zeta D(\widetilde{A})$, $V=\overline{D(T)}$, $W=\overline{\mathrm{pr}_{\zeta'} D(\widetilde{A}^*)}$, *and*

$$Tu_\zeta=(Au)_W \text{ for all } u\in D(\widetilde{A}), \text{ (the \textbf{defining equation})}.$$

*In this correspondence,*

(i) $\widetilde{A}^*$ *corresponds similarly to* $T^*:W\to V$.
(ii) $\ker\widetilde{A}=\ker T$; $\quad \mathrm{ran}\,\widetilde{A}=\mathrm{ran}\,T+(H\ominus W)$.
(iii) *When* $\widetilde{A}$ *is invertible,*

$$\widetilde{A}^{-1}=A_\gamma^{-1}+\mathrm{i}_{V\to H}\,T^{-1}\,\mathrm{pr}_W\,.$$

Here $\mathrm{i}_{V\to H}$ indicates the injection of $V$ into $H$ (it is often left out).
Now provide the operators with a spectral parameter $\lambda$, then this implies, with

$$Z_\lambda=\ker(A_{\max}-\lambda),\quad Z'_{\bar\lambda}=\ker(A'_{\max}-\bar\lambda),$$
$$D(A_{\max})=D(A_\gamma)\dot{+}Z_\lambda,\quad u=u_\gamma^\lambda+u_\zeta^\lambda=\mathrm{pr}_\gamma^\lambda u+\mathrm{pr}_\zeta^\lambda u, \text{ etc.:}$$

COROLLARY 1. *Let* $\lambda\in\rho(A_\gamma)$. *For the closed* $\widetilde{A}\in\mathcal{M}$, *there is a* 1–1 *correspondence*

$$\widetilde{A}-\lambda\longleftrightarrow\begin{cases}T^\lambda:V_\lambda\to W_{\bar\lambda}, \text{ closed, densely defined}\\ \text{with } V_\lambda\subset Z_\lambda,\ W_{\bar\lambda}\subset Z'_{\bar\lambda}, \text{ closed subspaces.}\end{cases}$$

*Here* $D(T^\lambda)=\mathrm{pr}_\zeta^\lambda D(\widetilde{A})$, $V_\lambda=\overline{D(T^\lambda)}$, $W_{\bar\lambda}=\overline{\mathrm{pr}_{\zeta'}^{\bar\lambda} D(\widetilde{A}^*)}$, *and*

$$T^\lambda u_\zeta^\lambda=((A-\lambda)u)_{W_{\bar\lambda}} \text{ for all } u\in D(\widetilde{A}).$$

*Moreover,*
(i) $\ker(\widetilde{A}-\lambda)=\ker T^\lambda$; $\quad \mathrm{ran}(\widetilde{A}-\lambda)=\mathrm{ran}\,T^\lambda+(H\ominus W_{\bar\lambda})$.
(ii) *When* $\lambda\in\rho(\widetilde{A})\cap\rho(A_\gamma)$,

$$(\widetilde{A}-\lambda)^{-1}=(A_\gamma-\lambda)^{-1}+\mathrm{i}_{V_\lambda\to H}(T^\lambda)^{-1}\,\mathrm{pr}_{W_{\bar\lambda}}\,.$$

This gives a Kreĭn resolvent formula for any closed $\widetilde{A}\in\mathcal{M}$.
The operators $T$ and $T^\lambda$ are related in the following way: Define

$$E^\lambda=I+\lambda(A_\gamma-\lambda)^{-1},\quad F^\lambda=I-\lambda A_\gamma^{-1},$$
$$E'^{\bar\lambda}=I+\bar\lambda(A_\gamma^*-\bar\lambda)^{-1},\quad F'^{\bar\lambda}=I-\bar\lambda(A_\gamma^*)^{-1},$$

then $E^\lambda F^\lambda=F^\lambda E^\lambda=I$, $E'^{\bar\lambda}F'^{\bar\lambda}=F'^{\bar\lambda}E'^{\bar\lambda}=I$ on $H$. Moreover, $E^\lambda$ and $E'^{\bar\lambda}$ restrict to homeomorphisms

$$E_V^\lambda:V\xrightarrow{\sim}V_\lambda,\quad E_W'^{\bar\lambda}:W\xrightarrow{\sim}W_{\bar\lambda},$$

with inverses denoted $F_V^\lambda$ resp. $F_W'^{\bar\lambda}$. In particular, $D(T^\lambda)=E_V^\lambda D(T)$.

THEOREM 2. *Let $G^{\lambda}_{V,W} = -\operatorname{pr}_W \lambda E^{\lambda}\, \mathrm{i}_{V\to H}$; then*

$$(E'^{\bar\lambda}_W)^* T^{\lambda} E^{\lambda}_V = T + G^{\lambda}_{V,W}. \tag{2}$$

*In other words, $T$ and $T^{\lambda}$ are related by the commutative diagram (where the horizontal maps are homeomorphisms)*

$$\begin{tikzcd} V_{\lambda} \arrow[d, "T^{\lambda}"'] & V \arrow[l, "E^{\lambda}_V"'] \arrow[d, "T+G^{\lambda}_{V,W}"] \\ W_{\bar\lambda} \arrow[r, "(E'^{\bar\lambda}_W)^*"'] & W \end{tikzcd} \qquad D(T^{\lambda}) = E^{\lambda}_V D(T).$$

This is a straightforward elaboration of [16], Prop. 2.6.

Now let us introduce boundary triplets and $M$-functions. The general setting is the following: There is given a pair of Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and two pairs of "boundary operators"

$$\begin{pmatrix}\Gamma_1\\ \\ \Gamma_0\end{pmatrix} : D(A_{\max}) \to \begin{matrix}\mathcal{H}\\ \times\\ \mathcal{K}\end{matrix}, \quad \begin{pmatrix}\Gamma'_1\\ \\ \Gamma'_0\end{pmatrix} : D(A'_{\max}) \to \begin{matrix}\mathcal{K}\\ \times\\ \mathcal{H}\end{matrix},$$

bounded with respect to the graph norm and surjective, such that

$$D(A_{\min}) = D(A_{\max}) \cap \ker\Gamma_1 \cap \ker\Gamma_0, \quad D(A'_{\min}) = D(A'_{\max}) \cap \ker\Gamma'_1 \cap \ker\Gamma'_0,$$

and for all $u \in D(A_{\max})$, $v \in D(A'_{\max})$,

$$(Au, v) - (u, A'v) = (\Gamma_1 u, \Gamma'_0 v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma'_1 v)_{\mathcal{K}}.$$

Then the three pairs $\{\mathcal{H}, \mathcal{K}\}$, $\{\Gamma_1, \Gamma_0\}$ and $\{\Gamma'_1, \Gamma'_0\}$ are said to form a *boundary triplet*. (See [10] and [9] for references to the literature on this.)

Note that under our assumptions, the choice

$$\mathcal{H} = Z', \quad \mathcal{K} = Z, \quad \begin{pmatrix}\Gamma_1 u\\ \Gamma_0 u\end{pmatrix} = \begin{pmatrix}(Au)_{Z'}\\ u_{\zeta}\end{pmatrix}, \quad \begin{pmatrix}\Gamma'_1 v\\ \Gamma'_0 v\end{pmatrix} = \begin{pmatrix}(A'v)_Z\\ v_{\zeta'}\end{pmatrix}, \tag{3}$$

defines a boundary triplet, cf. (1).

Following [10], the boundary triplet is used to define operators $A_T \in \mathcal{M}$ and $A'_{T'} \in \mathcal{M}'$ for any pair of operators $T \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $T' \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ by

$$D(A_T) = \ker(\Gamma_1 - T\Gamma_0), \quad D(A'_{T'}) = \ker(\Gamma'_1 - T'\Gamma'_0). \tag{4}$$

Then they show:

PROPOSITION 1. *For $\lambda \in \rho(A_T)$, there is a well-defined M-function $M_T(\lambda)$ determined by*

$$M_T(\lambda) : \operatorname{ran}(\Gamma_1 - T\Gamma_0) \to \mathcal{K}, \quad M_T(\Gamma_1 - T\Gamma_0)u = \Gamma_0 u \text{ for all } u \in Z_\lambda.$$

*Likewise, for $\lambda \in \rho(A'_{T'})$, the function $M'_{T'}(\lambda)$ is determined similarly by*

$$M'_{T'}(\lambda) : \operatorname{ran}(\Gamma'_1 - T'\Gamma'_0) \to \mathcal{H}, \quad M'_{T'}(\Gamma'_1 - T'\Gamma'_0)v = \Gamma'_0 v \text{ for all } v \in Z'_\lambda.$$

*Here, when $\rho(A_T) \neq \emptyset$,*

$$(A_T)^* = A'_{T^*}.$$

This was set in relation to Theorem 1 in [9]: Take the boundary triplet defined in (3). Then the formula for $D(A_T)$ in (4) is the same as the defining equation (2) for $D(\widetilde{A})$. For the sake of generality, allow also unbounded, densely defined, closed operators $T : Z \to Z'$; then in fact the formulas in Proposition 1 still lead to a well-defined $M$-function $M_T(\lambda)$. We denote $A_T$ by $\widetilde{A}$ and $M_T(\lambda)$ by $M_{\widetilde{A}}(\lambda)$, when they come from the special choice (3) of boundary triplet. Then we have:

THEOREM 3. *Let $\widetilde{A}$ correspond to $T : Z \to Z'$ by Theorem 1. For any $\lambda \in \rho(\widetilde{A})$, $M_{\widetilde{A}}(\lambda)$ is in $\mathcal{L}(Z', Z)$ and satisfies*

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1} \operatorname{i}_{Z' \to H}.$$

*Moreover, $M_{\widetilde{A}}(\lambda)$ relates to $T$ and $T^\lambda$ by:*

$$M_{\widetilde{A}}(\lambda) = -(T + G^\lambda_{Z,Z'})^{-1} = -F^\lambda_Z (T^\lambda)^{-1} (F'^{\bar{\lambda}}_{Z'})^*, \text{ for } \lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma).$$

This takes care of those operators $\widetilde{A}$ for which $\operatorname{pr}_\zeta D(\widetilde{A})$ is dense in $Z$ and $\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)$ is dense in $Z'$. But the construction extends in a natural way to all the closed $\widetilde{A} \in \mathcal{M}$, giving the following result:

THEOREM 4. *Let $\widetilde{A}$ correspond to $T : V \to W$ by Theorem 1. For any $\lambda \in \rho(\widetilde{A})$, there is a well-defined $M_{\widetilde{A}}(\lambda) \in \mathcal{L}(W, V)$, holomorphic in $\lambda$ and satisfying*

(i) $M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1} \operatorname{i}_{W \to H}$.

(ii) *When $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$,*

$$M_{\widetilde{A}}(\lambda) = -(T + G^\lambda_{V,W})^{-1}.$$

(iii) *For $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$, it enters in a Kreĭn resolvent formula*

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \operatorname{i}_{V_\lambda \to H} E^\lambda_V M_{\widetilde{A}}(\lambda) (E'^{\bar{\lambda}}_W)^* \operatorname{pr}_{W_{\bar{\lambda}}}.$$

Other Kreĭn-type resolvent formulas in a general framework of *relations* can be found in Malamud and Mogilevskiĭ [26, Section 5.2].

### 3. Neumann-type conditions for second-order operators

The abstract theory can be applied to elliptic realisations by use of suitable mappings going to and from the boundary, allowing an interpretation in terms of boundary conditions. We shall demonstrate this in the strongly elliptic second-order case.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ of one of the following three types: 1) $\Omega$ is bounded, 2) $\Omega$ is the complement of a bounded set (i.e., is an exterior domain), or 3) there is a ball $B(0,R)$ with center 0 and radius $R$ such that $\Omega \setminus B(0,R) = \mathbb{R}^n_+ \setminus B(0,R)$ (we then call $\Omega$ a perturbed halfspace). More general sets or manifolds could be considered in a similar way, namely the so-called admissible manifolds as defined in the book [19].

The sets will in the present section be assumed to be $C^\infty$; later from Section 5 on they will be taken to be $C^{k,\sigma}$, where $k$ is an integer $\geq 0$ and $\sigma \in ]0,1]$. (Recall that the norm on the Hölder space $C^{k,\sigma}(V)$ is

$$\|u\|_{C^{k,\sigma}(V)} = \sup_{|\alpha|\leq k, x\in V} |D^\alpha u(x)| + \sup_{|\alpha|=k, x\neq y} |D^\alpha u(x) - D^\alpha u(y)|\,|x-y|^{-\sigma}.)$$

We then denote $k+\sigma = \tau$.

That a bounded domain $\Omega$ is $C^{k,\sigma}$ means that there is an open cover $\{U_j\}_{j=1,\dots,J}$ of $\partial\Omega$ such that by an affine coordinate change for each $j$, $U_j$ is a box $\{\max_{k\leq n} |y_k| < a_j\}$, and

$$\Omega \cap U_j = \{(y',y_n) \mid \max_{k<n} |y_k| < a_j,\ f_j(y') < y_n < a_j\},$$
$$\partial\Omega \cap U_j = \{(y',y_n) \mid \max_{k<n} |y_k| < a_j, y_n = f_j(y')\},$$

with $C^{k,\sigma}$-functions $f_j$ such that $|f_j(y')| < a_j$ for $\max_{k<n} |y_k| < a_j$. The diffeomorphism (coordinate change)

$$F_j : (y',y_n) \mapsto (y', y_n - f_j(y')) \tag{5}$$

is then also $C^{k,\sigma}$. The sets $U_j$ must be supplied with a suitable bounded open set $U_0$ with closure contained in $\Omega$, to get a full cover of $\overline{\Omega}$.

For exterior domains, we cover $\partial\Omega$ similarly, then this must be supplied with a suitable open set $U_0$ with closure contained in $\Omega$ to get a full cover of $\overline{\Omega}$; here $U_0$ contains the complement of a ball, $U_0 \supset \mathbb{R}^n \setminus B(0,R')$.

For a perturbed halfspace, we cover $\partial\Omega \cap B(0,R+1)$ as above, and supply this with $U_0 = \{x \mid x_n > -\varepsilon, |x| > R\}$ to get a full cover of $\overline{\Omega}$.

The boundary $\partial\Omega$ will be denoted $\Sigma$. We assume in the present section that $\Omega$ is $C^\infty$; then $\Sigma$ is an $(n-1)$-dimensional $C^\infty$ manifold without boundary.

Let $A = \sum_{|\alpha|\leq 2} a_\alpha D^\alpha$ with $C^\infty$ coefficients $a_\alpha$ given on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ (containing $U_0$ in the perturbed halfspace case), and uniformly strongly elliptic:

$$\operatorname{Re} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq c_0|\xi|^2, \text{ all } x \in \widetilde{\Omega}, \xi \in \mathbb{R}^n,$$

$c_0 > 0$. The formal adjoint $A' = \sum_{|\alpha|\le 2} D^\alpha \bar{a}_\alpha = \sum_{|\alpha|\le 2} a'_\alpha D^\alpha$ likewise has $C^\infty$ coefficients $a'_\alpha$ and is strongly elliptic on $\widetilde{\Omega}$. We asume that the coefficients and all their derivatives are bounded.

We denote by $A_{\max}$ resp. $A_{\min}$ the maximal resp. minimal realisations of $A$ in $L_2(\Omega) = H$; they act like $A$ in the distribution sense and have the domains

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}, \quad D(A_{\min}) = H_0^2(\Omega)$$

(using $L_2$ Sobolev spaces). Similarly, $A'_{\max}$ and $A'_{\min}$ denote the maximal and minimal realisations in $L_2(\Omega)$ of the formal adjoint $A'$; here $A_{\max} = {A'_{\min}}^*$, $A'_{\max} = {A_{\min}}^*$.

Denote $\gamma_j u = (\partial_n^j u)|_\Sigma$, where $\partial_n$ is the derivative along the interior normal $\vec{n}$ at $\Sigma$. Let $s_0(x')$ be the coefficient of $-\partial_n^2$ when $A$ is written in terms of normal and tangential derivatives at $x' \in \Sigma$; it is bounded with bounded inverse. Denoting

$$s_0\gamma_1 = \nu_1, \quad \bar{s}_0\gamma_1 = \nu'_1,$$

we have the Green's formula for $A$ valid for $u, v \in H^2(\Omega)$,

$$(6) \qquad (Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\nu_1 u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \nu'_1 v + \mathcal{A}'_0 \gamma_0 v)_{L_2(\Sigma)},$$

where $\mathcal{A}'_0$ is a certain first-order differential operator over $\Sigma$. The formula extends e.g. to $u \in H^2(\Omega)$, $v \in D(A'_{\max})$, as

$$(7) \qquad (Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\nu_1 u, \gamma_0 v)_{\frac12, -\frac12} - (\gamma_0 u, \nu'_1 v + \mathcal{A}'_0 \gamma_0 v)_{\frac32, -\frac32},$$

where $(\cdot,\cdot)_{s,-s}$ denotes the duality pairing between $H^s(\Sigma)$ and $H^{-s}(\Sigma)$. (Cf. Lions and Magenes [24] for this and the next results.)

The Dirichlet realisation $A_\gamma$ is defined as usual by variational theory (the Lax-Milgram lemma); it is the restriction of $A_{\max}$ with domain

$$D(A_\gamma) = D(A_{\max}) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega),$$

where the last equality follows by elliptic regularity theory. By addition of a constant to $A$ if necessary, we can assume that the spectrum of $A_\gamma$ is contained in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$. For $\lambda \in \rho(A_\gamma)$, $s \in \mathbb{R}$, let

$$Z_\lambda^s(A) = \{u \in H^s(\Omega) \mid (A - \lambda)u = 0\};$$

it is a closed subspace of $H^s(\Omega)$. The trace operators $\gamma_0$, $\gamma_1$ and $\nu_1$ extend by continuity to continuous maps

$$\gamma_0 : Z_\lambda^s(A) \to H^{s-\frac12}(\Sigma), \quad \gamma_1, \nu_1 : Z_\lambda^s(A) \to H^{s-\frac32}(\Sigma),$$

for all $s \in \mathbb{R}$. When $\lambda \in \rho(A_\gamma)$, let $K_\gamma^\lambda : \varphi \mapsto u$ denote the Poisson operator from $H^{s-\frac12}(\Sigma)$ to $H^s(\Omega)$ solving the semi-homogeneous Dirichlet problem

$$(8) \qquad (A - \lambda)u = 0 \text{ in } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Sigma.$$

It is well-known that $K^\lambda_\gamma$ maps homeomorphically

$$K^\lambda_\gamma : H^{s-\frac12}(\Sigma) \xrightarrow{\sim} Z^s_\lambda(A),$$

for all $s \in \mathbb{R}$, with $\gamma_0$ acting as an inverse there. The analogous operator for $A' - \bar\lambda$ is denoted $K'^{\bar\lambda}_\gamma$.

We shall now recall from [9, 14] how the statements in Section 2 are interpreted in terms of boundary conditions. In the rest of this section, we abbreviate $H^s(\Sigma)$ to $H^s$. With the notation from Section 1,

$$Z^0_0(A) = Z, \quad Z^0_0(A') = Z', \quad Z^0_\lambda(A) = Z_\lambda, \quad Z^0_\lambda(A') = Z'_\lambda.$$

We denote by $\gamma_{Z_\lambda}$ the restriction of $\gamma_0$ to a mapping from $Z_\lambda$ (closed subspace of $L_2(\Omega)$) to $H^{-\frac12}$; its adjoint $\gamma^*_{Z_\lambda}$ goes from $H^{\frac12}$ to $Z_\lambda$:

$$\gamma_{Z_\lambda} : Z_\lambda \xrightarrow{\sim} H^{-\frac12}, \text{ with adjoint } \gamma^*_{Z_\lambda} : H^{\frac12} \xrightarrow{\sim} Z_\lambda.$$

There is a similar notation for the primed operators. When $\lambda = 0$, this index is left out.

These homeomorphisms allow "translating" an operator $T : Z \to Z'$ to an operator $L : H^{-\frac12} \to H^{\frac12}$, as in the diagram

$$\begin{tikzcd} Z \arrow[r, "\gamma_Z"] \arrow[d, "T"'] & H^{-\frac12} \arrow[d, "L"] \\ Z' \arrow[r, "(\gamma^*_{Z'})^{-1}"'] & H^{\frac12} \end{tikzcd} \qquad D(L) = \gamma_0 D(T), \tag{9}$$

whereby $(Tz, z') = (L\gamma_0 z, \gamma_0 z')_{\frac12,-\frac12}$.

We moreover define the Dirichlet-to-Neumann operators for each $\lambda \in \rho(A_\gamma)$,

$$P^\lambda_{\gamma_0,\nu_1} = \nu_1 K^\lambda_\gamma; \quad P'^{\bar\lambda}_{\gamma_0,\nu'_1} = \nu'_1 K'^{\bar\lambda}_\gamma; \tag{10}$$

they are first-order elliptic pseudodifferential operators over $\Sigma$, continuous from $H^{s-\frac12}$ to $H^{s-\frac32}$ for all $s \in \mathbb{R}$, and Fredholm in case $\Sigma$ is bounded. (Their pseudodifferential nature and ellipticity was explained e.g. in [15]).

For general trace maps $\beta$ and $\eta$ we write

$$P^\lambda_{\beta,\eta} : \beta u \mapsto \eta u, \quad u \in Z^s_\lambda(A), \tag{11}$$

when this operator is well-defined.

Introduce the trace operators $\Gamma$ and $\Gamma'$ (from [14], where they were called $M$ and $M'$) by

$$\Gamma = \nu_1 - P^0_{\gamma_0,\nu_1}\gamma_0 = \nu_1 A^{-1}_\gamma A_{\max}, \quad \Gamma' = \nu'_1 - P'^0_{\gamma_0,\nu'_1}\gamma_0 = \nu'_1 (A^*_\gamma)^{-1} A'_{\max}. \tag{12}$$

Here $\Gamma$ and $\Gamma'$ map $D(A_{\max})$ resp. $D(A'_{\max})$ continuously onto $H^{\frac12}$. With these pseudodifferential boundary operators there is a generalized Green's formula valid *for all* $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

(13) $$(Au,v)_{L_2(\Omega)} - (u,A'v)_{L_2(\Omega)} = (\Gamma u, \gamma_0 v)_{\frac12,-\frac12} - (\gamma_0 u, \Gamma' v)_{-\frac12,\frac12}.$$

In particular,

(14) $$(Au,w) = (\Gamma u, \gamma_0 w)_{\frac12,-\frac12} \text{ for all } w \in Z_0^0(A') = Z'.$$

(Cf. [14], Th. III 1.2.) By composition with suitable isometries $\Lambda_t : H^s(\Sigma) \to H^{s-t}(\Sigma)$, (13) can be turned into a standard boundary triplet formula

(15) $$(Au,v)_{L_2(\Omega)} - (u,A'v)_{L_2(\Omega)} = (\Gamma_1 u, \Gamma'_0 v)_{L_2(\Sigma)} - (\Gamma_0 u, \Gamma'_1 v)_{L_2(\Sigma)},$$

with $\Gamma_1 = \Lambda_{\frac12}\Gamma$, $\Gamma'_1 = \Lambda_{\frac12}\Gamma'$, $\Gamma_0 = \Gamma'_0 = \Lambda_{-\frac12}\gamma_0$ and $\mathcal{H} = \mathcal{K} = L_2(\Sigma)$.

There is a general "translation" of the abstract results in Section 1 to statements on closed realisations $\widetilde{A}$ of $A$. First let $\widetilde{A}$ correspond to $T : Z \to Z'$ (i.e., assume $V = Z$, $W = Z'$). Then in view of (9) and (14), the defining equation in Theorem 1 is turned into

$$(\Gamma u, \gamma_0 z')_{\frac12,-\frac12} = (L\gamma_0 u, \gamma_0 z')_{\frac12,-\frac12}, \text{ all } z' \in Z'.$$

Since $\gamma_0 z'$ runs through $H^{-\frac12}$, this means that $\Gamma u = L\gamma_0 u$, also written

$$\nu_1 u = (L + P^0_{\gamma_0,\nu_1})\gamma_0 u.*$$

Thus $\widetilde{A}$ represents a *Neumann-type condition*

(16) $$\nu_1 u = C\gamma_0 u, \text{ with } C = L + P^0_{\gamma_0,\nu_1}.$$

This allows all first-order $\psi$do's $C$ to enter, namely by letting $L$ act as $C - P^0_{\gamma_0,\nu_1}$.

*The elliptic case:* Consider a Neumann-type boundary condition

(17) $$\nu_1 u = C\gamma_0 u,$$

where $C$ is a first-order classical $\psi$do on $\Sigma$. Let $\widetilde{A}$ be the restriction of $A_{\max}$ with domain

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \nu_1 u = C\gamma_0 u\}.$$

Now the boundary condition satifies the Shapiro-Lopatinskiĭ condition (is *elliptic*) if and only if $L$ is elliptic; then in fact

(18) $$D(\widetilde{A}) = \{u \in H^2(\Omega) \mid \nu_1 u = C\gamma_0 u\}.$$

Then the adjoint $\widetilde{A}^*$ equals the operator that is defined similarly from $A'$ by the boundary condition

$$\nu'_1 v = (C^* - \mathcal{A}'_0)\gamma_0 v,$$

likewise elliptic.

When we do the above considerations for $\widetilde{A} - \lambda$, we get $L^\lambda$ satisfying the diagram

$$\begin{tikzcd} Z \arrow[r, "E_Z^\lambda"] \arrow[d, "T+G^\lambda_{Z,Z'}"'] & Z_\lambda \arrow[r, "\gamma_{Z_\lambda}"] \arrow[d, "T^\lambda"'] & H^{-\frac{1}{2}} \arrow[d, "L^\lambda"] \\ Z' \arrow[r, "(F'^{\bar\lambda}_{Z'})^*"'] & Z'_{\bar\lambda} \arrow[r, "(\gamma^*_{Z'_{\bar\lambda}})^{-1}"'] & H^{\frac{1}{2}} \end{tikzcd} \qquad D(L^\lambda) = D(L).$$

Here the horizontal maps are homeomorphisms, and they compose as $\gamma_{Z_\lambda} E_Z^\lambda = \gamma_Z$, $(\gamma^*_{Z'_{\bar\lambda}})^{-1}(F'^{\bar\lambda}_{Z'})^* = (\gamma^*_{Z'})^{-1}$, so

$$L^\lambda = \gamma_Z^{-1}(T + G^\lambda_{Z,Z'})\gamma^*_{Z'}.$$

In terms of $L^\lambda$, the boundary condition reads:

$$\nu_1 u = (L^\lambda + P^\lambda_{\gamma_0,\nu_1})\gamma_0 u.$$

Note that $L^\lambda + P^\lambda_{\gamma_0,\nu_1} = C = L + P^0_{\gamma_0,\nu_1}$, so

$$L^\lambda = L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}.$$

As shown in [9], this leads to:

THEOREM 5. *Assumptions as in the start of Section* 3*, with $C^\infty$ domain and operator. Let $\widetilde{A}$ correspond to $T : Z \to Z'$, carried over to $L : H^{-\frac{1}{2}} \to H^{\frac{1}{2}}$. Then $\widetilde{A}$ represents the boundary condition* (16)*. Moreover:*

(i) *For $\lambda \in \rho(A_\gamma)$, $P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1} \in \mathcal{L}(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$ and*

$$L^\lambda = L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}.$$

(ii) *For $\lambda \in \rho(\widetilde{A})$, there is a related M-function $\in \mathcal{L}(H^{\frac{1}{2}}, H^{-\frac{1}{2}})$*

$$M_L(\lambda) = \gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big)A_\gamma^{-1}\,\mathrm{i}_{Z' \to H}\,\gamma^*_{Z'}.$$

(iii) *For $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$,*

$$M_L(\lambda) = -(L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1})^{-1} = -(L^\lambda)^{-1}.$$

(iv) *For $\lambda \in \rho(A_\gamma)$,*

$$\begin{aligned} \ker(\widetilde{A} - \lambda) &= K^\lambda_\gamma \ker L^\lambda, \\ \operatorname{ran}(\widetilde{A} - \lambda) &= \gamma^*_{Z'_{\bar\lambda}} \operatorname{ran} L^\lambda + \operatorname{ran}(A_{\min} - \lambda), \end{aligned}$$

*so that* $H\setminus(\operatorname{ran}(\widetilde{A}-\lambda))=Z'_{\bar\lambda}\setminus(\gamma^*_{Z'_{\bar\lambda}}\operatorname{ran}L^\lambda)$.

(v) *For* $\lambda\in\rho(\widetilde{A})\cap\rho(A_\gamma)$ *there is a Kreĭn resolvent formula:*

$$
\begin{aligned}
(\widetilde{A}-\lambda)^{-1}&=(A_\gamma-\lambda)^{-1}-\mathrm{i}_{Z_\lambda\to H}\,\gamma^{-1}_{Z_\lambda}M_L(\lambda)(\gamma^*_{Z'_{\bar\lambda}})^{-1}\,\mathrm{pr}_{Z'_{\bar\lambda}}\\
&=(A_\gamma-\lambda)^{-1}-K^\lambda_\gamma M_L(\lambda)(K'^{\bar\lambda}_\gamma)^*.
\end{aligned}
\tag{19}
$$

(vi) *In particular, if* $C$ *is a* $\psi$*do of order* $1$ *such that* $C-P^0_{\gamma_0,\nu_1}$ *is elliptic, and* $\rho(\widetilde{A})\cap\rho(A_\gamma)\neq\emptyset$*, then* $D(L)=H^{\frac32}$*, and*

$$
M_L(\lambda)=-(C-P^\lambda_{\gamma_0,\nu_1})^{-1}
\tag{20}
$$

*is elliptic of order* $-1$ *for all* $\lambda\in\rho(\widetilde{A})$*. Here* $\widetilde{A}$ *satisfies* (18) *with* (16)*.*

Note that with the notation (11), $C-P^\lambda_{\gamma_0,\nu_1}=-P^\lambda_{\gamma_0,\nu_1-C\gamma_0}$, and $M_L(\lambda)=P^\lambda_{\nu_1-C\gamma_0,\gamma_0}$.

Observe the simple last formula in (19), where $K^\lambda_\gamma$ is the Poisson operator for $A-\lambda$, the adjoint being a trace operator of class zero.

The Kreĭn formula is consistent with formulas found for selfadjoint cases with Robin-type conditions in other works, such as Posilicano [28], Posilicano and Raimondi [29], Gesztesy and Mitrea [12], when one observes that

$$
(K'^{\bar\lambda}_\gamma)^*=\nu_1(A_\gamma-\lambda)^{-1};
\tag{21}
$$

this follows from the fact that for $\varphi\in H^{-\frac12}(\Sigma)$ and $v=K'^{\bar\lambda}_\gamma\varphi$, $f\in L_2(\Omega)$ and $u=(A_\gamma-\lambda)^{-1}f$, one has using Green's formula (7):

$$
\begin{aligned}
(f,K'^{\bar\lambda}_\gamma\varphi)_{L_2(\Omega)}&=((A-\lambda)u,v)_{L_2(\Omega)}-(u,(A'-\bar\lambda)v)_{L_2(\Omega)}\\
&=(\nu_1u,\gamma_0v)_{\frac12,-\frac12}-(\gamma_0u,\nu'_1v+\mathcal{A}'_0\gamma_0v)_{\frac32,-\frac32}=(\nu_1(A_\gamma-\lambda)^{-1}f,\varphi)_{\frac12,-\frac12}.
\end{aligned}
$$

For the general case of $\widetilde{A}$ corresponding to $T:V\to W$ with subspaces $V\subset Z$, $W\subset Z'$, there is a related "translation" to boundary conditions. Details are given in [9], let us here just mention some ingredients:

We use the notation in (15) ff. Set

$$
X_1=\overline{\Gamma_0D(\widetilde{A})}=\Lambda_{-\frac12}\gamma_0V\subset L_2(\Sigma),\quad Y_1=\overline{\Gamma_0D(\widetilde{A}^*)}=\Lambda_{-\frac12}\gamma_0W\subset L_2(\Sigma),
$$

where $\Gamma_0$ restricts to homeomorphisms

$$
\Gamma_{0,V}:V\xrightarrow{\sim}X_1,\quad \Gamma_{0,W}:W\xrightarrow{\sim}Y_1.
$$

Then $T : V \to W$ is carried over to $L_1 : X_1 \to Y_1$ by

$$\begin{array}{ccc} V & \xrightarrow{\Gamma_{0,V}} & X_1 \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle L_1} \\ W & \xrightarrow[(\Gamma^*_{0,W})^{-1}]{} & Y_1 \end{array} \qquad D(L_1) = \Gamma_0 D(T),$$

The boundary condition is:

$$\Gamma_0 u \in D(L_1), \quad L_1 \Gamma_0 u = \mathrm{pr}_{Y_1} \Gamma_1 u.$$

There is a similar reduction for $\widetilde{A} - \lambda$ when $\lambda \in \rho(A_\gamma)$, and we find that

$$L_1^\lambda = L_1 + \mathrm{pr}_{Y_1} \Lambda_{\frac{1}{2}} (P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}) \Lambda_{\frac{1}{2}} \mathrm{i}_{X_1 \to L_2(\Sigma)} .$$

There is an $M$-function $M_{L_1}(\lambda) : Y_1 \to X_1$ defined for $\lambda \in \rho(\widetilde{A})$. It equals $-(L_1^\lambda)^{-1}$ when $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$, and there is then a Kreĭn resolvent formula

$$\begin{aligned} (\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda \to H} \Gamma^{-1}_{0,V_\lambda} M_{L_1}(\lambda) (\Gamma^*_{0,W_{\bar\lambda}})^{-1} \mathrm{pr}_{W_{\bar\lambda}} \\ &= (A_\gamma - \lambda)^{-1} - K^\lambda_{\gamma,X_1} M_{L_1}(\lambda) (K'^{\bar\lambda}_{\gamma,Y_1})^*; \end{aligned}$$

here $K^\lambda_{\gamma,X_1} : X_1 \subset L_2(\Sigma) \xrightarrow{\Lambda_{\frac{1}{2}}} H^{-\frac{1}{2}}(\Sigma) \xrightarrow{K^\lambda_\gamma} L_2(\Omega)$.

For higher order elliptic operators, and systems, there are similar results on $M$-functions and Kreĭn resolvent formulas, see [9]. In such cases there occur interesting subspace situations where $X$ and $Y$ are (homeomorphic to) full products of Sobolev spaces over $\Sigma$.

## 4. The nonsmooth ψdbo calculus

The study of the smooth case was formulated in [9] in terms of the pseudodifferential boundary operator (ψdbo) calculus, which was initiated by Boutet de Monvel [8] and further developed e.g. in Grubb [17], [19] (we refer to these works or to [20] for details on the calculus). The ψdbo theory has been adapted to nonsmooth situations by Abels in [3], by use of ideas from the adaptation of ψdo's to nonsmooth cases by Kumano-go and Nagase [23], Taylor [30]. The operators considered by Abels have symbols that satisfy the usual estimates in the conormal variables $\xi', \xi, \eta_n$, pointwise in the space variable $x$, but are only of class $C^{k,\sigma}$ in $x$ (so that the symbol estimates hold with respect to $C^{k,\sigma}$-norm in $x$). (For $\tau = k + \sigma$ integer, one could replace $C^{k,\sigma}$ by the so-called Zygmund space $C^\tau = B^\tau_{\infty,\infty}$, which is slightly larger, and gives the scale of spaces slightly better interpolation properties, cf. Abels [1, 2], but we shall let that aspect lie.) We call $(k, \sigma)$ the Hölder smoothness of the operator and its symbol.

The theory allows the operators to act beween $L_p$-based Besov and Bessel-potential spaces ($1 < p < \infty$), but we shall here just use it in the case $p = 2$ (although an extension to $p \neq 2$ would also be interesting). Some important results of [3] are:

THEOREM 6. $1^\circ$ *One has that the continuous mapping property*

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} H^{s+m}(\mathbb{R}^n_+)^N \\ \times \\ H^{s+m-\frac{1}{2}}(\mathbb{R}^{n-1})^M \end{array} \to \begin{array}{c} H^{s}(\mathbb{R}^n_+)^{N'} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M'} \end{array}$$

*holds when $\mathcal{A}$ is a Green operator on $\mathbb{R}^n_+$ of order $m \in \mathbb{Z}$ and class $r$, with Hölder smoothness $(k,\sigma)$, provided that (with $\tau = k + \sigma$)*

1. $|s| < \tau$ *if* $N' \neq 0$,
2. $|s - \frac{1}{2}| < \tau$ *if* $M' \neq 0$,
3. $s + m > r - \frac{1}{2}$ *if* $N \neq 0$ *(class restriction).*

$2^\circ$ *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be as in $1^\circ$, with symbols $a_1$ resp. $a_2$ and constants $k_1, \sigma_1, \tau_1, m_1, N_1, \dots$ resp. $k_2, \sigma_2, \tau_2, m_2, N_2, \dots$. Assume that $N_2' = N_1, M_2' = M_1$, so that the operators can be composed. Let $k_3 = \min\{k_1, k_2\}$, $\sigma_3 = \min\{\sigma_1, \sigma_2\}$, $\tau_3 = \min\{\tau_1, \tau_2\}$, $0 < \theta < \min\{1, \tau_2\}$. The boundary symbol composition $a_1 \circ_n a_2$ is a Green symbol $a_3$ of order $m_3 = m_1 + m_2$, class $r_3 = \max\{r_1 + m_2, r_2\}$ and Hölder smoothness $(k_3, \sigma_3)$, defining a Green operator $\mathcal{A}_3$. The remainder is continuous:*

$$\mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_3 : \begin{array}{c} H^{s+m_3-\theta}(\mathbb{R}^n_+)^{N_2} \\ \times \\ H^{s+m_3-\frac{1}{2}-\theta}(\mathbb{R}^{n-1})^{M_2} \end{array} \to \begin{array}{c} H^{s}(\mathbb{R}^n_+)^{N_1'} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M_1'} \end{array},$$

*if the following conditions are satisfied:*

1. $|s| < \tau_3$ *and* $s - \theta > -\tau_2$ *if* $N_1' > 0$, $|s - \frac{1}{2}| < \tau_3$ *and* $s - \frac{1}{2} - \theta > -\tau_2$ *if* $M_1' > 0$;
2. $-\tau_2 + \theta < s + m_1 < \tau_2$ *if* $N_1 > 0$, $-\tau_2 + \theta < s + m_1 - \frac{1}{2} < \tau_2$ *if* $M_1 > 0$;
3. $s + m_1 > r_1 - \frac{1}{2}$ *if* $N_1 > 0$, $s + m_3 - \theta > r_2 - \frac{1}{2}$ *if* $N_2 > 0$ *(class restrictions).*

$3^\circ$ *Let $\mathcal{A}$ be as in $1^\circ$, and polyhomogeneous and uniformly elliptic with principal symbol $a^0$ (here $N = N' > 0$). Then there is a Green operator $\mathcal{B}^0$ (the operator with symbol $(a^0)^{-1}$ if $m = 0$) of order $-m$, class $r - m$ and Hölder smoothness $(k, \sigma)$, continuous in the opposite direction of $\mathcal{A}$, such that $\mathcal{R} = \mathcal{A}\mathcal{B}^0 - I$ is continuous:*

$$\mathcal{R} : \begin{array}{c} H^{s-\theta}(\mathbb{R}^n_+)^{N} \\ \times \\ H^{s-\theta-\frac{1}{2}}(\mathbb{R}^{n-1})^{M'} \end{array} \to \begin{array}{c} H^{s}(\mathbb{R}^n_+)^{N} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M} \end{array},$$

*if, with $\tau = k + \sigma$,*

1. $-\tau+\theta<s<\tau$;

2. $s-\frac{1}{2}>-\tau+\theta$ *if* $M$ *or* $M'>0$;

3. $s-\theta>r-m-\frac{1}{2}$ *(class restriction).*

See [3] (Theorems 1.1, 1.2 and 6.4). For integer $\tau$, the results are worked out there for symbols in Zygmund spaces, but they imply the results with Hölder spaces, see also [1, 2]. The class restrictions are imposed even when the operators have $C^\infty$ coefficients. $\mathcal{B}^0$ is called a parametrix of $\mathcal{A}$.

Abels has also generalized the calculus of [19] for symbols depending on a parameter $\mu$ to nonsmooth coefficients; again the estimates in the cotangent variables $\xi',\xi,\eta_n,\mu$ are the usual ones, but valid in $x$ w.r.t. Hölder norms.

We recall from the theory of ψdo's that $P$ is said to be "in $x$-form" resp. "in $y$-form", when it is defined from a symbol $p$ by

$$Pu=c\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi,\ \text{resp. } Pu=c\int e^{i(x-y)\cdot\xi}p(y,\xi)u(y)\,dyd\xi,$$

$c=(2\pi)^{-n}$; the concept extends to ψdbo's. In Theorem 6, all the operators labeled with $\mathcal{A}$ are in $x$-form. So is $\mathcal{B}^0$ when $m=0$; otherwise it is a composition of an operator in $x$-form with an order-reducing operator system to the left, see Remark 1 below. The adjoints of operators in $x$-form are operators in $y$-form. [3] does not discuss the reduction from $y$-form to $x$-form; some indications may be inferred from Taylor [31], Ch. 1 §9. For operators in $y$-form one has at least the results that can be derived from the above results by transposition.

REMARK 1. An important tool in the calculus is "order-reducing operators". There are two types, one acting over the domain and one acting over the boundary:

$$\begin{aligned}&\Lambda^r_{-,+}=\mathrm{OP}(\lambda^r_-(\xi))_+ : H^t(\mathbb{R}^n_+)\xrightarrow{\sim}H^{t-r}(\mathbb{R}^n_+),\\ &\Lambda^r_0=\mathrm{OP}'(\langle\xi'\rangle^r) : H^t(\mathbb{R}^{n-1})\xrightarrow{\sim}H^{t-r}(\mathbb{R}^{n-1}),\ \text{all } t\in\mathbb{R},\end{aligned}$$

with inverses $\Lambda^{-r}_{-,+}$ resp. $\Lambda^{-r}_0$. Here $\lambda^r_-$ is the "minus-symbol" defined in [18] Prop. 4.2 as a refinement of $(\langle\xi'\rangle-i\xi_n)^r$. In Theorem 6 3°, whereas $\mathcal{B}^0$ is the operator with symbol $(a^0)^{-1}$ when $m=0$, one applies the zero-order construction to $\mathcal{A}_1=\mathcal{A}\begin{pmatrix}\Lambda^{-m}_{-,+}&0\\0&\Lambda^{-m}_0\end{pmatrix}$ to define $\mathcal{B}^0=\begin{pmatrix}\Lambda^{-m}_{-,+}&0\\0&\Lambda^{-m}_0\end{pmatrix}\mathcal{B}^0_1$ when $m\neq0$.

It should be noted that when e.g. $P_+$ is as in Theorem 6 1°, then

$$\Lambda^r_{-,+}P_+ : H^{s+m}(\mathbb{R}^n_+)\to H^{s-r}(\mathbb{R}^n_+)\ \text{for } -\tau<s<\tau,\tag{22}$$

whereas the composition rule Theorem 6 2° shows that $\Lambda^r_{-,+}P_+$ can be written as the sum of an operator in the calculus $\mathrm{OP}'(\lambda^r_{-,+}\circ_n p(x,\xi)_+)$ in $x$-form and a remainder, such that the sum maps $H^{s'+m+r}(\mathbb{R}^n_+)\to H^{s'}(\mathbb{R}^n_+)$ for $-\tau<s'<\tau$; this gives a mapping property like in (22) but with $-\tau+r<s<\tau+r$. This apparently extends the range, but the decompositions into a primary part and a remainder are not the same; $\Lambda^r_{-,+}P_+$ is not in $x$-form but is an operator in $x$-form composed to the left with $\Lambda^r_{-,+}$, not equal to $\mathrm{OP}'(\lambda^r_{-,+} \circ_n p(x,\xi)_+)$. Compositions to the right with $\Lambda^r_{-,+}$ are simpler and preserve $x$-form directly. We shall say that operators formed by composing an operator in $x$-form with an order-reducing operator to the left are "in order-reduced $x$-form".

Coordinate changes give some inconveniences in the nonsmooth calculus because, in a $C^{k,\sigma}$-setting, the action of $D_j$ after a $C^{k,\sigma}$-coordinate change gets Jacobian factors that are $C^{k-1,\sigma}$, and higher powers $D^\alpha$ get coefficients in $C^{k-|\alpha|,\sigma}$ (when $k-|\alpha|\ge 0$).

We say that an operator is a generalized Green operator (of one of the respective types) if it is the sum of an operator defined from symbols in the calculus and a remainder of lower order (for $s$ in an interval, specified in each case or understood from the context).

## 5. Resolvent formulas in the case of non-smooth domains

To treat one difficulty at a time, we consider in the following the case where the domain is nonsmooth, but the operator $A$ is given with smooth coefficients (this includes of course constant coefficients).

Let $\Omega$ be an open set in $\mathbb{R}^n$ of one of the three types described in Section 3, of class $C^{k,\sigma}$. We still take $A$ with $C^\infty$-coefficients on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, as described in Section 2.

Recall from Grisvard [13] (Th. 1.3.3.1, 1.5.1.2, 1.4.1.1, 1.5.3.4):

THEOREM 7. *Let $\Omega$ be bounded and $C^{k,\sigma}$, let $\tau = k+\sigma$.*

$1^\circ$ *When $\Phi$ is a $C^{k,\sigma}$-diffeomorphism, $\tau$ integer, then $u \in H^s_{\mathrm{loc}} \implies u\circ\Phi \in H^s_{\mathrm{loc}}$ for $|s|\le\tau$.*

$2^\circ$ *One can for $|s|\le\tau$, integer, define $H^s(\Sigma)$ to be the space of distributions $u$ on $\Sigma$ such that for each $j$, $u\circ F_j^{-1}$ is in $H^s$ on $\{y' \mid \max|y_k| \le a_j\}$. The trace map $\gamma_0 : H^s(\Omega)\to H^{s-\frac12}(\Sigma)$ is well-defined for $\frac12 < s \le \tau$, and the trace map $\gamma_1 : H^s(\Omega)\to H^{s-\frac32}(\Sigma)$ is well-defined for $\frac32 < s\le\tau$. There is a continous right inverse of each map, and of the two maps jointly for $\frac32 < s \le \tau$.*

$3^\circ$ *Let $\varphi$ be $C^{k_1,\sigma_1}$, $\tau_1 = k_1+\sigma_1$, then $u\mapsto\varphi u$ is continuous in $H^s(\mathbb{R}^n)$ for $|s|\le\tau_1$ if $\tau_1$ is integer, $|s|<\tau_1$ if $\tau_1$ is non-integer.*

$4^\circ$ *When $\tau\ge 2$ and $A$ is a second-order differential operator on $\Omega$ in a divergence form ($A = -\sum_{j,k}\partial_j a_{jk}\partial_k + \sum_k a_k\partial_k + a_0$) with $C^{0,1}$-coefficients, and we define the associated oblique Neumann trace operators by*

$$\nu_A = \sum_{j,k} n_j a_{jk}\gamma_0\partial_k, \quad \nu_{A'} = \sum_{j,k} n_k \bar a_{jk}\gamma_0\partial_j, \tag{23}$$

*there holds a Green's formula*

$$(Au,v)_{L_2(\Omega)} - (u,A'v)_{L_2(\Omega)} = (\nu_A u,\gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u,\nu_{A'} v - \textstyle\sum_k n_k \bar{a}_k \gamma_0 v)_{L_2(\Sigma)}, \tag{24}$$

*for* $u,v \in H^2(\Omega)$.

The Green's formula (24) can be reorganized as (6); for our $A$ with smooth coefficients, $\nu_1$, $\nu_1'$ and $\mathcal{A}_0'$ get $C^{k-1,\sigma}$-coefficients when $\Omega$ is $C^{k,\sigma}$.

We define the Dirichlet realisation $A_\gamma$ of $A$, with domain $D(A_\gamma) = D(A_{\max}) \cap H_0^1(\Omega)$ by the usual variational construction, and we shall assume that $A_\gamma$ is invertible. Its adjoint is the analogous operator for $A'$.

By the difference quotient method of Nirenberg [27] one has that $D(A_\gamma) = H^2(\Omega) \cap H_0^1(\Omega)$ when $\tau \geq 2$ (this fact is also derived below); detailed proofs are e.g. found in the textbooks of Evans [11] (for $C^2$-domains) or McLean [25] (for $C^{1,1}$-domains).

Also the extended Green's formula (7) is valid when $\tau \geq 2$; this follows by an extension of the proof in Lions and Magenes [24], as mentioned in [13] Remark 1.5.3.5. It follows that the generalized Green's formula (13) holds, when $\Gamma$ and $\Gamma'$ are defined by

$$\Gamma = \nu_1 A_\gamma^{-1} A_{\max}, \quad \Gamma' = \nu_1'(A_\gamma^*)^{-1} A'_{\max}. \tag{25}$$

The local coordinates (cf. (5)) are used to reduce the curved situation to the flat situation; then the boundary becomes straight but nonsmoothness is imposed on the symbols.

In the following we work out what the nonsmooth ψdbo method can give for the Dirichlet problem; this can be regarded as a basic exercise in the calculus (some other cases appear in works of Abels and coauthors).

First we consider the case of a uniformly strongly elliptic second-order operator on $\mathbb{R}^n_+$ — which we for simplicity of notation also call $A$ — with Hölder smoothness $(k_1,\sigma_1)$ and $\tau_1 = k_1 + \sigma_1$, together with a Dirichlet trace operator,

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^{s+2}(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix};$$

it is continuous for

$$-\tau_1 < s < \tau_1, \quad s > -\tfrac{3}{2}, \tag{26}$$

extended to $|s| \leq \tau_1$ if integer (cf. Theorem 7 3°). To prepare for an application of Theorem 6, we apply order-reducing operators (cf. Remark 1) to reduce to order 0, introducing

$$\mathcal{A}_1 = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} \mathcal{A}\Lambda_{-,+}^{-2} = \begin{pmatrix} A\Lambda_{-,+}^{-2} \\ \\ \Lambda_0^2 \gamma_0 \Lambda_{-,+}^{-2} \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}, \tag{27}$$

for $s$ as in (26) ff. By Theorem 6 3° it has a parametrix $\mathcal{B}_1^0$ of order 0 and class $-1$ defined from the principal symbols,

$$\mathcal{B}_1^0 = \begin{pmatrix} R_1^0 & K_1^0 \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac12}(\mathbb{R}^{n-1}) \end{matrix} \to H^s(\mathbb{R}^n_+), \tag{28}$$

for $s$ satisfying

$$-\tau_1 + \tfrac12 < s < \tau_1, \quad s > -\tfrac32; \tag{29}$$

here the remainder $\mathcal{R}_1 = \mathcal{A}_1 \mathcal{B}_1^0 - I$ satisfies

$$\mathcal{R}_1 : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta-\frac12}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac12}(\mathbb{R}^{n-1}) \end{matrix}, \tag{30}$$

when $0 < \theta < \min\{1, \tau_1\}$,

$$-\tau_1 + \tfrac12 + \theta < s < \tau_1, \quad s > -\tfrac32 + \theta. \tag{31}$$

Then the equation $\mathcal{A}_1 \mathcal{B}_1^0 = I + \mathcal{R}_1$, also written

$$\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} \mathcal{A} \Lambda_{-,+}^{-2} \mathcal{B}_1^0 = I + \mathcal{R}_1,$$

implies by composition to the left with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix}$ and to the right with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}$:

$$\mathcal{A} \Lambda_{-,+}^{-2} \mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = I + \mathcal{R}, \text{ with } \mathcal{R} = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix} \mathcal{R}_1 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}.$$

Hence

$$\mathcal{B}^0 = \Lambda_{-,+}^{-2} \mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = \begin{pmatrix} R^0 & K^0 \end{pmatrix}$$

is a parametrix of $\mathcal{A}$, with

$$\mathcal{A} \mathcal{B}^0 = I + \mathcal{R}, \tag{32}$$

$$\mathcal{B}^0 : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac32}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+), \quad \mathcal{R} : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta+\frac32}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac32}(\mathbb{R}^{n-1}) \end{matrix}, \tag{33}$$

for $s$ as in (29) resp. (31). With the notation from Remark 1, $\mathcal{B}^0$ is in order-reduced $x$-form.

Now consider the situation where $A$ has smooth coefficients and the domain is nonsmooth. We shall go through the parametrix and inverse construction in the case where the Hölder smoothness of the domain is $(1,1)$ so that $\tau = 2$. We have the direct operator

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix}, \tag{34}$$

it is continuous for $-\frac{3}{2} < s \le 0$ (recall the restriction $s+2 \le 2$ coming from Theorem 7 2°).

For each $i = 1, \dots, J$, the diffeomorhism (5) carries $\Omega \cap U_j$ over to $V_j = \{(y', y_n) \mid \max_{k<n} |y_k| < a_j, 0 < y_n < a_j - f_j(y')\}$, such that $\partial\Omega \cap U_j$ is mapped to $\{(y', y_n) \mid \max_{k<n} |y_k| < a_j, y_n = 0\}$. When the smooth differential operator $A$ is transformed to local coordinates in this way, the principal part of the resulting operator $\underline{A}$ has Hölder smoothness $(0,1)$, so here $\tau_1 = 1$. In each of these charts one constructs a parametrix $\underline{\mathcal{B}}^0$ for $\begin{pmatrix} \underline{A} \\ \gamma_0 \end{pmatrix}$ as above (the coefficients of $\underline{A}$ can be assumed to be extended to $\overline{\mathbb{R}}^n_+$). When $\Omega$ is bounded or is an exterior domain, one uses for the set $U_0$ a parametrix of $A$ without changing coordinates. In the perturbed halfspace case, for the set $U_0$ one extends $A$ smoothly to $\overline{\mathbb{R}}^n_+$ and uses a smooth version of the above construction. These parametrices are carried back to the curved situation and pieced together using a partition of unity subordinate to the cover $\{U_0, U_1, \dots, U_J\}$, as indicated in [19], p. 228 (the first factor $\varphi_i$ in each term in (2.4.77) should be replaced by a function $\eta_i \in C_0^\infty(U_i)$ such that $\eta_i \varphi_i = \varphi_i$, to get preservation of the principal symbol after summation). Here the coordinate changes allow the smoothness to remain at $(0,1)$; cf. [2], in particular Section 5.3 there. The sum over $i$ is then a parametrix of (34); its composition with $\mathcal{A}$ gives the identity plus a remainder of lower order, for values $s$ as indicated above.

In the subsequent compositions below, it will always be understood that they take place in local coordinates (after decomposing the operators in pieces supported in the $U_i$ by use of suitable partitions of unity) and are taken back to the curved situation afterwards.

In the present construction, we shall actually carry a spectral parameter along that will be useful for discussions of invertibility. So we now replace the originally given $A$ by $A - \lambda$, to be studied for large negative $\lambda$.

The parametrix will be of the form

$$\mathcal{B}^0(\lambda) = \begin{pmatrix} R^0(\lambda) & K^0(\lambda) \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega); \tag{35}$$

with $(k_1, \sigma_1) = (0,1)$ the condition (29) means that $-\frac{1}{2} < s < 1$, so that, along with the restriction coming from Theorem 7, we have altogether that

$$-\tfrac{1}{2} < s \le 0 \tag{36}$$

is allowed. The remainder maps as follows:

$$\mathcal{R}(\lambda) = \mathcal{A}(\lambda)\mathcal{B}^0(\lambda) - I : \begin{matrix} H^{s-\theta}(\Omega) \\ \times \\ H^{s-\theta+\frac{3}{2}}(\Sigma) \end{matrix} \to \begin{matrix} H^{s}(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \tag{37}$$

for

$$-\tfrac{1}{2} + \theta < s \leq 0. \tag{38}$$

In order to get hold of the exact inverse, we shall use an old trick of Agmon [4], which implies a useful $\lambda$-dependent estimate of the remainder: Write $-\lambda = \mu^2$ ($\mu > 0$), introduce an extra variable $t \in S^1$, and replace $\mu$ by $D_t = -i\partial_t$; let

$$\widehat{A} = A + D_t^2 \text{ on } \Omega \times S^1. \tag{39}$$

Then $\widehat{A}$ is strongly elliptic on $\Omega \times S^1$, and by the preceding construction (carried out with local coordinates respecting the product structure),

$$\widehat{\mathcal{A}} = \begin{pmatrix} \widehat{A} \\ \gamma_0 \end{pmatrix} \text{ has a parametrix } \widehat{\mathcal{B}}^0,$$

with mapping properties of $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}} = \widehat{\mathcal{A}}\widehat{\mathcal{B}}^0 - I$ as in (35) and (37) with $\Omega, \Sigma$ replaced by $\widehat{\Omega} = \Omega \times S^1$, $\widehat{\Sigma} = \Sigma \times S^1$.

For functions $w$ of the form $w(x,t) = u(x)e^{i\mu t}$,

$$\widehat{\mathcal{A}} w = \begin{pmatrix} (A + \mu^2)w \\ \gamma_0 w \end{pmatrix},$$

and similarly, the parametrix $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}}$ act on such functions like $\mathcal{B}^0(\lambda)$ and $\mathcal{R}(\lambda)$ applied in the $x$-coordinate.

Moreover, for $w(x,t) = u(x)e^{i\mu t}$, $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|w\|_{H^s(\mathbb{R}^n \times S^1)} \simeq \|(1 - \Delta + \mu^2)^s u(x)\|_{L_2(\mathbb{R}^n)} \simeq \|(1 + |\xi|^2 + \mu^2)^{s/2}\hat{u}(\xi)\|_{L_2},$$

with similar relations for Sobolev spaces over other sets. Norms as in the right-hand side are called $H^{s,\mu}$-norms; they were extensively used [19], see the Appendix there for the definition on subsets. The important observation is now that when $s' < s$ and $w(x,t) = u(x)e^{i\mu t}$, then

$$\begin{aligned} \|w\|_{H^{s'}(\mathbb{R}^n \times S^1)} &\simeq \|(1 + |\xi|^2 + \mu^2)^{s'/2}\hat{u}(\xi)\|_{L_2} \\ &\leq \langle\mu\rangle^{s'-s}\|(1 + |\xi|^2 + \mu^2)^{s/2}\hat{u}(\xi)\|_{L_2} \simeq \langle\mu\rangle^{s-s'}\|w\|_{H^s(\mathbb{R}^n \times S^1)}, \end{aligned}$$

with constants independent of $u$ and $\mu$. Analogous estimates hold with $\mathbb{R}^n$ replaced by $\Omega$ or $\Sigma$.

Applying this principle to the estimates of the remainder $\widehat{\mathcal{R}}$, we find that

$$\|\mathcal{R}(\lambda)u\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)} \le c_s\|u\|_{H^{s-\theta,\mu}(\Omega)\times H^{s-\theta+\frac{3}{2},\mu}(\Sigma)} \\ \le c_s'\langle\mu\rangle^{-\theta}\|u\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)}$$

for $s$ as in (38).

For each $s$, take a fixed $\lambda$ with $|\lambda|$ so large that $c_s'\langle\mu\rangle^{-\theta} \le \frac{1}{2}$. Then $I+\mathcal{R}(\lambda)$ has the inverse $I+\mathcal{R}'(\lambda) = I+\sum_{k\ge 1}(-\mathcal{R}(\lambda))^k$ (converging in the operator norm for operators on $H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)$), and

$$\mathcal{A}(\lambda)\mathcal{B}^0(\lambda)(I+\mathcal{R}'(\lambda)) = I.$$

This gives a right inverse

$$\mathcal{B}(\lambda) = \mathcal{B}^0(\lambda)+\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda) = \begin{pmatrix} R(\lambda) & K(\lambda)\end{pmatrix},$$

with the same Sobolev space continuity as $\mathcal{B}^0(\lambda)$, and $\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda)$ of lower order. Since

$$\mathcal{A}(\lambda)\mathcal{B}(\lambda) = \begin{pmatrix}(A-\lambda)R(\lambda) & (A-\lambda)K(\lambda)\\ \gamma_0R(\lambda) & \gamma_0K(\lambda)\end{pmatrix} = \begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}, \tag{40}$$

$R(\lambda)$ solves

$$(A-\lambda)u = f,\quad \gamma_0u = 0, \tag{41}$$

and $K(\lambda)$ solves

$$(A-\lambda)u = 0,\quad \gamma_0u = \psi. \tag{42}$$

For such large $\lambda$, $R(\lambda)$ coincides with the resolvent of $A_\gamma$ defined by variational theory, and $K(\lambda)$ is the Poisson-type operator we called $K_\gamma^\lambda$ in Section 3;

$$(A_\gamma-\lambda)^{-1}: H^s(\Omega)\to H^{s+2}(\Omega),\quad K_\gamma^\lambda: H^{s+\frac{3}{2}}(\Sigma)\to H^{s+2}(\Omega), \tag{43}$$

for $s$ satisfying (36).

The mapping properties extend to all the $\lambda$ for which the operators are well-defined, especially to $\lambda=0$. For $A_\gamma^{-1}$, this goes as follows: When $u\in H^1(\Omega)$ and $f\in H^s(\Omega)$ with $s<1$, $f+\lambda u$ is likewise in $H^s(\Omega)$. Then $A_\gamma u = f+\lambda u$ allows the conclusion $u\in H^{s+2}(\Omega)$. The argument works for all $s$ satisfying (36) (for each such $s$, there is room to take $\theta>0$ so small that (38) is satified. Moreover, since $A_\gamma^{-1}-(A_\gamma-\lambda)^{-1} = -\lambda A_\gamma^{-1}(A_\gamma-\lambda)^{-1}$ is of lower order than $A_\gamma^{-1}$, $A_\gamma^{-1}$ equals a nonsmooth $\psi$dbo plus a lower-order remainder.

The Poisson operator solving (42) can be further described as follows (for all $\lambda\in\rho(A_\gamma)$): There is a right inverse $\mathcal{K}: H^{s+\frac{3}{2}}(\Sigma)\to H^{s+2}(\Omega)$ of $\gamma_0$ for $-\frac{3}{2}<s\le 0$ (cf. Theorem 7 2°). When we set $v = u-\mathcal{K}\varphi$, we find that $v$ should solve

$$(A-\lambda)v = -(A-\lambda)\mathcal{K}\varphi,\quad \gamma_0v = 0,$$

to which we apply the preceding results; then when $\lambda \in \rho(A_\gamma)$,

$$K^\lambda_\gamma = \mathcal{K} - (A_\gamma - \lambda)^{-1}(A-\lambda)\mathcal{K}; \tag{44}$$

solves (42) uniquely. It maps $H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega)$ for $s$ satisfying (36).

Since our original operator had $C^\infty$ coefficients, the same construction works for the adjoint Dirichlet problem, so we also here get the mapping properties

$$(A'_\gamma - \bar\lambda)^{-1} : H^s(\Omega) \to H^{s+2}(\Omega), \quad K'^{\bar\lambda}_\gamma : H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega), \tag{45}$$

for $s$ satisfying (36).

The condition $s > -\frac{1}{2}$ prevents the Poisson operator from starting from $H^{-\frac{1}{2}}(\Sigma)$, which would be needed for an analysis as in Section 3. Fortunately, it is possible to get supplementing information in other ways.

By (7) we have, analogously to (21), that $K^\lambda_\gamma$ is the adjoint of a trace operator of class 0 as follows:

$$K^\lambda_\gamma = (\nu'_1(A'_\gamma - \bar\lambda)^{-1})^*; \tag{46}$$

(it is used here that $\mathcal{A}'_0\gamma_0(A'_\gamma - \bar\lambda)^{-1} = 0$).

Now use the mapping property in (45). The resolvent can be composed with $\nu'_1$ for $s > -\frac{1}{2}$, so

$$\nu'_1(A'_\gamma - \lambda)^{-1} = (K^\lambda_\gamma)^* : H^s(\Omega) \to H^{s+\frac{1}{2}}(\Sigma) \text{ for } -\tfrac{1}{2} < s \le 0.$$

It follows that

$$K^\lambda_\gamma : H^{s'-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega), \tag{47}$$

when $0 \le s' < \frac{1}{2}$. In particular, $s' = 0$ is allowed.

Taking this together with the larger values that were covered by (43), we find that (47) holds for

$$0 \le s' \le 2; \tag{48}$$

the intermediate values are included by interpolation. We denote $s'$ by $s$ from here on.

One can analyze the structure of $K^\lambda_\gamma$ for the low values of $s$ further, decomposing it into terms belonging to the calculus and lower-order remainders. There is a difficulty here in the fact that order-reducing operators as well as operators in $y$-form enter, and both types affect the $s$-values for which the decompositions and mapping properties are valid (cf. Remark 1). We refrain from including a deeper analysis.

There is a similar result for $K'^{\bar\lambda}_\gamma$. The adjoints also extend, e.g.

$$(K'^{\bar\lambda}_\gamma)^* : H^s_0(\overline{\Omega}) \to H^{s+\frac{1}{2}}(\Sigma), \text{ for } -2 \le s \le 0; \tag{49}$$

recall that $H^s_0(\overline{\Omega}) = H^s(\Omega)$ when $|s| < \frac{1}{2}$. To sum up, we have shown:

THEOREM 8. *When $\Omega$ is $C^{1,1}$ and $A$ has $C^\infty$-coefficients, the solution operators $K^\lambda_\gamma$ and $K'^{\bar\lambda}_\gamma$ for* (8) *and its primed version map $H^{s-\frac12}(\Sigma)$ to $H^s(\Omega)$ for $0\le s\le 2$. They are generalized Poisson operators in the sense that for $s\in]\frac32,2]$, they can be written as the sum of a Poisson operator of Hölder smoothness $(0,1)$, in order-reduced $x$-form, and a lower order operator.*

The next step is to study $P^\lambda_{\gamma_0,\nu_1}=\nu_1K^\lambda_\gamma$ and $P'^{\bar\lambda}_{\gamma_0,\nu'_1}=\nu'_1K'^{\bar\lambda}_\gamma$, cf. (10) ff.

We have immediately from the mapping properties established above, that

$$P^\lambda_{\gamma_0,\nu_1},P'^{\bar\lambda}_{\gamma_0,\nu'_1}:H^{s-\frac12}(\Sigma)\to H^{s-\frac32}(\Sigma), \tag{50}$$

when $\frac32<s\le 2$. Let us also introduce the operator $\nu''_1=\nu'_1+\mathcal{A}'_0\gamma_0$, then Green's formula (7) takes the form

$$(Au,v)_{L_2(\Omega)}-(u,A'v)_{L_2(\Omega)}=(\nu_1u,\gamma_0v)_{\frac12,-\frac12}-(\gamma_0u,\nu''_1v)_{\frac32,-\frac32}, \tag{51}$$

for $u\in H^2(\Omega)$, $v\in D(A'_{\max})$, and $P'^{\bar\lambda}_{\gamma_0,\nu''_1}$ (cf. (11)) likewise maps as in (50) ff. Applying (51) to functions $u,v$ with $Au=0$, $A'v=0$, we see that $P^\lambda_{\gamma_0,\nu_1}$ and $P'^{\bar\lambda}_{\gamma_0,\nu''_1}$ are contained in each other's adjoints. Therefore $P^\lambda_{\gamma_0,\nu_1}$ considered in (50) has the extension $(P'^{\bar\lambda}_{\gamma_0,\nu''_1})^*$, which is continuous from $H^{s'+\frac32}(\Sigma)$ to $H^{s'+\frac12}(\Sigma)$ for $-2\le s'<-\frac32$. This extends the statement in (50) to the values $0\le s<\frac12$, and by interpolation we obtain the validity of (50) for $0\le s\le 2$.

$P^\lambda_{\gamma_0,\nu_1}$ can in the localizations to $\mathbb{R}^n_+$ be described as the composition of the operator $\nu_1=s_0\gamma_1$ (with $s_0\in C^{0,1}$) and a generalized Poisson operator consisting of an operator in order-reduced $x$-form having $C^{0,1}$-smoothness plus a remainder of lower order. For $s\in]\frac32,2]$ we can apply Theorem 6 2° to the compositions, using that $K^\lambda_\gamma$ is locally the sum of a composition $\Lambda^{-2}_{-,+}K^0_1(\lambda)\Lambda^2_0$ (multiplied with smooth cut-off functions) where $K^0_1(\lambda)$ is in $x$-form, and a remainder of lower order. This implies that $P^\lambda_{\gamma_0,\nu_1}$, apart from the remainder term coming from $K^\lambda_\gamma$, is the sum of a first-order ψdo in $x$-form with $C^{0,1}$-smoothness and a remainder term, mapping $H^{t+1}(\Sigma)$ to $H^t(\Sigma)$ for $|t|<1$, resp. $H^{t+1-\theta}(\Sigma)$ to $H^t(\Sigma)$ for $-1+\theta<t<1$. With $s-\frac12=t+1$, $s$ runs in $]\frac12,\frac52[$ resp. $]\frac12+\theta,\frac52[$ here, which covers the interval $s\in]\frac32,2]$ allowed by the other remainder.

For low values of $s$ there is again the difficulty that we are dealing with a composition with ingredients of order-reducing operators and $x$- or $y$-form operators, which each have different rules for the spaces in which the decompositions and mapping properties are valid, and we refrain from a further discussion here.

Observe moreover that $P^\lambda_{\gamma_0,\nu_1}$ is elliptic (the principal symbol is invertible) — since this is known for $P^0_{\gamma_0,\gamma_1}$ ( [4], [15]).

This shows:

THEOREM 9. *Assumptions as in Theorem* 8. $P^{\lambda}_{\gamma_0,\nu_1}$ *and* $P'^{\bar\lambda}_{\gamma_0,\nu'_1}$ *map* $H^{s-\frac{1}{2}}(\Sigma)$ *to* $H^{s-\frac{3}{2}}(\Sigma)$ *for* $s \in [0,2]$. *They are generalized elliptic* $\psi$*do's of order* 1*, in the sense that for* $s \in ]\frac{3}{2},2]$*, they have the form of an elliptic principal part in x-form of Hölder smoothness* $(0,1)$ *plus a lower order part.*

With these mapping properties it is straightforward to verify that $\Gamma$ and $\Gamma'$ defined in (25) satisfy the full statement in (12).

When more smoothness of $\Omega$ is assumed, the representation of $P^{\lambda}_{\gamma_0,\nu_1}$ as the sum of a principal part in $x$-form and a lower-order term can of course be extended to larger intervals than found above.

## 6. Interpretation of realisations

We now have all the ingredients to interpret the abstract characterisation of closed realisations $\widetilde{A}$ in terms of operators $T : V \to W$ recalled in Section 2, to boundary conditions. In fact, we have the mappings defined from the trace operator $\gamma_0$

$$\gamma_{Z_\lambda} : Z_\lambda \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma), \quad \gamma^*_{Z_\lambda} : H^{\frac{1}{2}}(\Sigma) \xrightarrow{\sim} Z_\lambda,$$

and the mappings defined from Poisson-type operators

$$K^{\lambda}_{\gamma} : H^{-\frac{1}{2}}(\Sigma) \to H^0(\Omega), \quad (K^{\lambda}_{\gamma})^* : H^0(\Omega) \to H^{\frac{1}{2}}(\Sigma),$$

as well as the versions with primes. Then the various definitions recalled in Section 3 for the smooth case, carrying $T^{\lambda} : V_\lambda \to W_{\bar\lambda}$ over to $L^{\lambda} : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ if $V = Z$, $W = Z'$, resp. to $L^{\lambda}_1 : X_1 \to Y_1$ in general, are effective in exactly the same way, and all the diagrams are valid in this situation.

In this way, $\widetilde{A}$ is determined by a Neumann-type boundary condition

$$\nu_1 u = (L + P^0_{\gamma_0,\nu_1})\gamma_0 u$$

in the case $V = Z$, $W = Z'$, and by a condition involving projections in the general case.

The adjoint $\widetilde{A}$ is determined by the boundary condition

$$\nu'_1 u = (L^* + P'^0_{\gamma_0,\nu'_1})\gamma_0 u$$

in the case $V = Z$, $W = Z'$ (resp. by a condition involving projections in the general case), where $L^*$ is the adjoint of $L$, considered as a generally unbounded operator from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$.

There is a well-defined $M$-function $M_L(\lambda)$, which coincides with $-(L^{\lambda})^{-1}$ for $\lambda \in \rho(A_\gamma) \cap \rho(\widetilde{A})$; here (20) and (19) hold. Suitably modified results hold in cases of general $V, W$.

For the case $V = Z$, $W = Z'$, we have obtained:

THEOREM 10. *When* $\Omega$ *is* $C^{1,1}$ *and* $A$ *has* $C^\infty$ *coefficients, bounded with bounded derivatives on a neighborhood of* $\Omega$*, and is uniformly strongly elliptic, then Theorem* 5 (i)–(v) *and* (20) *are valid.*

Gesztesy and Mitrea have in [12] established Kreĭn resolvent formulas for the Laplacian under a weaker smoothness hypothesis, namely that $\Omega$ is $C^{1,\sigma}$ with $\sigma > \frac{1}{2}$. Here they treat *selfadjoint* realisations determined by Robin-type boundary conditions

$$\gamma_1 u = B\gamma_0 u, \tag{52}$$

with $B$ compact from $H^1$ to $H^0$ (assured if $B$ is of order $< 1$). Posilicano and Raimondi [29] describe results for *selfadjoint* realisations in case $\Omega$ is $C^{1,1}$ and the coefficients of $A$, when it is written in symmetric divergence form, are $C^{0,1}$ satisfying various hypotheses. They remark that their treatment works for boundary conditions (52) with $\gamma_1$ replaced by the oblique Neumann trace operator $\nu_A$ (23) connected with the divergence form. Here $B$ is taken of order $< 1$, so it is a Robin-type perturbation of the natural Neumann condition.

It is an important point in the present treatment, besides that it deals with non-selfadjoint situations, that Neumann-type conditions (17) with general ψdo's $C$ of order 1 are included in the detailed discussion.

Furthermore, our pseudodifferential strategy allows the application of ellipticity concepts:

When $C$ is a generalized pseudodifferential operator of order 1 and Hölder smoothness $(0,1)$, $L = C - P^0_{\gamma_0,\nu_1}$ is a generalized pseudodifferential operator of order 1 and Hölder smoothness $(0,1)$, and vice versa. $L$ is elliptic precisely when the model boundary value problem for $A$ with the boundary condition (17) is uniquely solvable at all $(x',\xi')$ with $\xi' \neq 0$ in the boundary cotangent space (this is the Shapiro-Lopatinskiĭ condition). $L^\lambda$ is then also elliptic at each $\lambda \in \rho(A_\gamma)$ (since $P^\lambda_{\gamma_0,\nu_1} - P^0_{\gamma_0,\nu_1}$ is of order $< 1$).

Moreover, there is then a parametrix of $L$, and this can be used to investigate the regularity of the domain of $L$. Likewise, each $L^\lambda$ has a parametrix then. However, we want to set the true inverse $-M_L(\lambda)$ in relation to such a parametrix.

Restrict the attention to the case where $C$ is a first-order *differential* operator on $\Sigma$ with $C^{0,1}$-coefficients; then we can say more about $M_L(\lambda)$ with the present methods.

Assume a little more, namely that there is a ray $\lambda = -\mu^2 e^{i\theta}$, $\mu \in \mathbb{R}$, such that when we include $\lambda$ in the principal symbol of $P^\lambda_{\gamma_0,\nu_1}$, then the principal symbol of $L^\lambda = C - P^\lambda_{\gamma_0,\nu_1}$ is invertible for $|\xi'|^2 + |\mu|^2 \geq 1$ ("parameter-ellipticity"). Let $s \in ]\frac{3}{2}, 2]$. As in Section 5, we can invoke the system for $\widehat{A}$ on $\widehat{\Omega} = \Omega \times S^1$ (39) coupled with the same boundary operator (constant in the $t$-direction)

$$\widehat{\mathcal{A}} = \begin{pmatrix} \widehat{A} \\ \nu_1 - C\gamma_0 \end{pmatrix} : H^s(\widehat{\Omega}) \to \begin{matrix} H^{s-2}(\widehat{\Omega}) \\ \times \\ H^{s-\frac{3}{2}}(\widehat{\Sigma}) \end{matrix};$$

it is elliptic and has a parametrix $\widehat{\mathcal{B}}^0$. For the functions $u(x,t) = w(x)e^{i\mu t}$, this gives a

$\lambda$-dependent parametrix family for $\mathcal{A}(\lambda) = \big(A - \lambda\nu_1 - C\gamma_0\big)$ (when $|\lambda| \geq 1$) such that the remainder in the composition with $\mathcal{A}(\lambda)$ is $O(\langle\mu\rangle^{-\theta})$ for $\lambda \to \infty$ on the ray. Then there is a true inverse of $\mathcal{A}(\lambda)$, hence of $L^\lambda$, for sufficiently large $\lambda$ on the ray. We can follow this up for the operator $\widehat{L} = C - \widehat{P}_{\gamma_0,\nu_1}$ over $\widehat{\Sigma}$, which gives $L^\lambda$ when applied to functions $\varphi(x')e^{i\mu t}$. Here $\widehat{L}$ has a parametrix $\widehat{\widehat{L}}$ such that $\widehat{L}\widehat{\widehat{L}} - I$ is of negative order; this gives a parametrix $\widetilde{L}^\lambda$ of $L^\lambda$ such that $L^\lambda\widetilde{L}^\lambda - I$ has an $O(\langle\mu\rangle^{-\theta})$ estimate. For sufficiently large $\lambda$ on the ray this allows us to write $M_L(\lambda) = -(L^\lambda)^{-1}$ as $-\widetilde{L}^\lambda + \mathcal{R}$ with $\mathcal{R}$ of lower order. More precisely, $\widetilde{L}^\lambda$ is obtained as a composition of an operator in $x$-form with an order-reducing operator to the left; it maps from $H^{s-\frac{3}{2}}$ to $H^{s-\frac{1}{2}}$, and the remainder maps from $H^{s-\frac{3}{2}-\theta}$ to $H^{s-\frac{1}{2}}$. (The $s \in ]\frac{3}{2}, 2]$ run inside the interval where the parametrix construction for elliptic first-order ψdo's of Hölder smoothness $(0,1)$ works, as in Theorem 6 3° and Remark 1.) In this sense, $M_L(\lambda)$ is a generalized ψdo of order $-1$.

Using this information for $s = 2$, we see that $M_L(\lambda)$ map $H^{\frac{1}{2}}$ not just to $H^{-\frac{1}{2}}$, but to $H^{\frac{3}{2}}$. Then $D(L) = D(L^\lambda) = H^{\frac{3}{2}}$ and $D(\widetilde{A})$ is in $H^2(\Omega)$.

If, moreover, $C^*$ has Hölder smoothness $C^{0,1}$, the adjoint $\widetilde{A}^*$ is of the same type. In particular, there is selfadjointness if $A$ and $L$ are formally selfadjoint. This gives a very satisfactory version of the Kreĭn formula.

THEOREM 11. *If, in addition to the hypotheses of Theorem* 10, *$C$ is a first-order differential operator with Hölder smoothness* $(0,1)$ *and the principal symbol of* $L^\lambda = C - P^\lambda_{\gamma_0,\nu_1}$ *is parameter-elliptic on a ray* $\lambda = -\mu^2 e^{i\theta}$, $\mu \in \mathbb{R}$, *then* $D(L) = H^{\frac{3}{2}}(\Sigma)$, *and* $M_L(\lambda)$ *is for large* $\lambda$ *on the ray the sum of an elliptic ψdo of order* $-1$ *and Hölder smoothness* $(0,1)$, *in order-reduced x-form, and a lower-order term. Then* $D(\widetilde{A}) \subset H^2(\Omega)$.

*If, moreover,* $C^*$ *has Hölder smoothness* $(0,1)$, *the adjoint* $\widetilde{A}^*$ *is defined similarly from of* $L^*$ *with* $D(L^*) = H^{\frac{3}{2}}$, $D(\widetilde{A}^*) \subset H^2(\Omega)$. *In particular,* $\widetilde{A}$ *is selfadjoint if $A$ and $L$ are formally selfadjoint.*

From the point of view of the systematic parameter-dependent calculus of [19], the symbols of $C$ and $P^\lambda_{\gamma_0,\nu_1}$ have "regularity $\nu = +\infty$" when $C$ is a differential operator, so there is a parametrix with the same "regularity $+\infty$".

*Pseudodifferential* operators $C$ can be included in the discussion if the symbol classes in [19] are used in a more definitive way (here when $C$ is of order 1, it has "regularity 1", and the same will hold for the resulting principal symbols of $L^\lambda$ and $M_L(\lambda)$). Considerations with finite positive "regularity" play an important role in [1,2]. We hope to return to such cases in future works, but here just wanted to show what can be done using Agmon's principle.

**References**

[1] ABELS H., *Stokes equations in asymptotically flat domains and the motion of a free surface*, Thesis at TU Darmstadt, Shaker Verlag, Aachen 2003.

[2] ABELS H., *Reduced and generalized Stokes resolvent equations in asymptotically flat layers, Part II: $H_\infty$-calculus*, J. Math. Fluid Mech. **7** (2005), 223–260.

[3] ABELS H., *Pseudodifferential boundary value problems with nonsmooth coefficients*, Communications Part. Diff. Equ. **30** (2005), 1463–1503.

[4] AGMON S., *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

[5] AMREIN W.O. AND PEARSON D.B., *M operators: a generalisation of Weyl-Titchmarsh theory*, J. Comp. Appl. Math.**171** (2004), 1–26.

[6] BEHRNDT J. AND LANGER M., *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Funct. Anal. **243** (2007), 536–565.

[7] BOUTET DE MONVEL L., *Comportement d'un opérateur pseudo-différentiel sur une variété à bord, I-II*, J. d'Analyse Fonct. **17** (1966), 241–304.

[8] BOUTET DE MONVEL L, *Boundary problems for pseudodifferential operators*, Acta Math. **126** (1971), 11–51.

[9] BROWN B.M., GRUBB G., WOOD I.G., *M-functions for closed extensions of adjoint pairs of operators, with applications to elliptic boundary problems*, arXiv:0803.3630, to appear in Math. Nachr.

[10] BROWN B.M., MARLETTA M., NABOKO S. AND WOOD I.G., *Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices*, J. Lond. Math. Soc. **77** (2) (2008), 700–718.

[11] EVANS L.C., *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island 1998.

[12] GESZTESY F. AND MITREA M., *Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein type resolvent formulas for Schrödinger operators on bounded domains*, arXiv:0803.3179.

[13] GRISVARD P., *Elliptic problems in nonsmooth domains*, Pitman, Boston MA 1985.

[14] GRUBB G., *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (3) (1968), 425–513. Available from www.numdam.org.

[15] GRUBB G., *On coerciveness and semiboundedness of general boundary problems*, Israel J. Math. **10** (1971), 32–95.

[16] GRUBB G., *Properties of normal boundary problems for elliptic even-order systems*, Ann. Sc. Norm. Sup. Pisa, Ser. IV **1** (1974), 1–61. Available from www.numdam.org.

[17] GRUBB G., *Singular Green operators and their spectral asymptotics*, Duke Math. J. **51** (1984), 477–528.

[18] GRUBB G., *Pseudodifferential boundary problems in $L_p$ spaces*, Comm. Part. Diff. Eq. **15** (1990), 289–340.

[19] GRUBB G., *Functional calculus of pseudodifferential boundary problems*, Progress in Math. **65**, Birkhäuser, Boston 1996.

[20] GRUBB G., *Distributions and operators*, Graduate Texts in Mathematics **252**, Springer-Verlag, New York 2009.

[21] KATO T., *Perturbation theory for linear operators*, Grundlehren Math. Wiss. **132**, Springer-Verlag, Berlin 1966.

[22] KREĬN M.G., *Theory of self-adjoint extensions of symmetric semi-bounded operators and applications I*, Mat. Sb. **20** (62) (1947), 431–495 (Russian).

[23] KUMANO-GO H. AND NAGASE M., *Pseudo-differential operators with non-regular symbols and applications*, Funkcial. Ekvac. **21** (1978), 151–192.

[24] LIONS J.-L. AND MAGENES E., *Problèmes aux limites non homogènes et applications*, **1**. Éditions Dunod, Paris 1968.

[25] MCLEAN W., *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge 2000.

[26] MALAMUD M.M. AND MOGILEVSKIĬ V.I., *Kreĭn type formula for canonical resolvents of dual pairs of linear relations*, Methods Funct. Anal. Topology **8** (4) (2002), 72–100.

[27] NIRENBERG, L. *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 649–675.

[28] POSILICANO A., *Self-adjoint extensions of restrictions*, Operators and Matrices **2** (2008), 483–506.

[29] POSILICANO A. AND RAIMONDI L., *Krein's resolvent formula for self-adjoint extensions of symmetric second order elliptic differential operators*, arXiv:0804.3312.

[30] TAYLOR M.E., *Pseudodifferential operators and nonlinear PDE. Progress in Mathematics, 100*, Birkhäuser Boston, Inc., Boston MA 1991.

[31] TAYLOR M.E., *Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials, Mathematical Surveys and Monographs, 81*, American Mathematical Society, Providence RI 2000.

[32] VISHIK V.I., *On general boundary value problems for elliptic differential operators*, Trudy Mosc. Mat. Obsv **1** (1952), 187–246, Amer. Math. Soc. Transl. **24** (2) (1963), 107–172.

**AMS Subject Classification: 35J25, 47A10, 58J40.**

Gerd GRUBB, Department of Mathematical Sciences, Copenhagen University,
Universitetsparken 5, 2100 Copenhagen, DENMARK
`grubb@math.ku.dk`