Fractional order operators on bounded domains

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1. Fractional-order pseudodifferential operators

A prominent example of a fractional-order pseudodifferential operator (ps.d.o.) is the fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$; it is currently of great interest in probability, finance, mathematical physics and differential geometry.
Recall how the Fourier transform $\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx$ is used to describe differential operators:

$$Pu = \sum\nolimits_{|\alpha|\le m} a_\alpha(x) D^\alpha u = \mathcal{F}^{-1}\big(p(x,\xi)\hat{u}(\xi)\big) = \operatorname{Op}(p)u,$$

$$\text{where } p(x,\xi) = \sum\nolimits_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \text{ the symbol.}$$

Extending this to more general functions $p(x,\xi)$ as symbols, we get ps.d.o's. For example,

$$(-\Delta)^a u = \operatorname{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)),$$

a ps.d.o. of order $2a$.
When A is a strongly elliptic second-order variable-coefficient differential operator on $\mathbb{R}^n$, one can define the powers A^a as ps.d.o.s of order $2a$.

In the classical ps.d.o. theory, symbols are taken *polyhomogeneous*:

$$p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi), \text{ where } p_j(x,t\xi) = t^{m-j} p_j(x,\xi)$$

(modified suitably near $\xi = 0$). The order is m (in $\mathbb{R}$ or in $\mathbb{C}$).

The powers A^a have such symbol expansions, with $m = 2a$.

The elliptic case is when the principal symbol p_0 is invertible; then $Q = \mathrm{Op}(p_0^{-1})$ is a good approximation to an inverse of P.

The theory extends to manifolds by use of local coordinates.

The fractional Laplacian $(-\Delta)^a$ can also be described as a convolution operator:

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2a}}\, dy;$$

the kernel function $c_{n,a}|y|^{-n-2a}$ equals $\mathcal{F}^{-1}|\xi|^{2a}$.
This is generalized in the current literature to other convolution operators where $c_{n,a}|y|^{-n-2a}$ is replaced by a homogeneous, even positive function $K(y)$, possibly with less smoothness.
Also more general $K(y)$ occur; the operators no longer being classical ps.d.o.s, but still translation-invariant (i.e., x-independent).

One of the difficulties with these operators is that they are *nonlocal*.

To circumvent the nonlocalness of $(-\Delta)^a$, recent studies have often been based on an observation by Caffarelli and Silvestre (CPDE '07), who showed that $(-\Delta)^a$ is the Dirichlet-to-Neumann operator for the degenerate elliptic equation in dimension $n+1$:

$$\nabla_{x,y} \cdot (y^{1-2a} \nabla_{x,y} v(x,y)) = 0, \text{ when } (x,y) \in \mathbb{R}^n \times \mathbb{R}_+,$$

with Dirichlet data $v(x,0)$ and (adapted) Neumann data $y^{1-2a}\partial_y v(x,y)|_{y=0}$. This allows an analysis of some questions via local (differential) operators.

But for restrictions to a bounded domain Ω this point of view introduces the difficulty of how to deal with the cylinder $\partial\Omega \times \mathbb{R}_+$.

2. Action on a bounded domain

Let $\Omega \subset \mathbb{R}^n$, smooth bounded. Because of the nonlocal character of $(-\Delta)^a$, it is not obvious how to study it on Ω. Boutet de Monvel '71 initiated a ps.d. boundary operator theory, but it requires integer order and a certain transmission property at $\partial\Omega$, which excludes $(-\Delta)^a$.

We shall need some **function spaces:**
1) The *Sobolev spaces* defined for $1 < p < \infty$ (omit p when $p = 2$):

$$\begin{aligned} H_p^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid (1-\Delta)^{s/2} u \in L_p(\mathbb{R}^n)\} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}, \\ H_p^s(\Omega) &= r^+ H_p^s(\mathbb{R}^n), \\ \dot{H}_p^s(\overline{\Omega}) &= \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}. \end{aligned}$$

Here $\langle\xi\rangle = (|\xi|^2+1)^{\frac{1}{2}}$, r^+ restricts to Ω, e^+ extends by zero on $\complement\Omega$. (The notation with $\dot{H}$ stems from Hörmander's books '63 and '85.)
2) The *Hölder spaces* $C^{k,\sigma}(\overline{\Omega})$ where $k \in \mathbb{N}_0$, $0 < \sigma \leq 1$, also denoted $C^s(\overline{\Omega})$ with $s = k + \sigma$ when $\sigma < 1$. For $s \in \mathbb{N}_0$, $C^s(\overline{\Omega})$ is the usual space of continuously differentiable functions. Hölder-Zygmund-spaces $C_*^s(\overline{\Omega})$ generalize $C^s(\overline{\Omega})$ ($s \in \mathbb{R}_+ \setminus \mathbb{N}$) to all $s \in \mathbb{R}$ with good interpolation properties. (The spaces C_*^s are also known as the Besov spaces $B^s_{\infty,\infty}$.)

Let $P = (-\Delta)^a$. An operator representing a homogeneous Dirichlet problem for P can be defined as the Friedrichs extension P_{Dir} of $r^+ P|_{C_0^\infty(\Omega)}$ (which is ≥ 0); it has domain

$$D(P_{\mathrm{Dir}}) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+ P u \in L_2(\Omega)\}.$$

It represents the following problem for P:

$$\begin{aligned} r^+ P u &= f \text{ on } \Omega, \\ \operatorname{supp} u &\subset \overline{\Omega}. \end{aligned} \tag{1}$$

There is existence and uniqueness of solution when $f \in L_2(\Omega)$, and one can then ask for regularity properties of u in terms of f.

For example, if f is in $L_\infty(\Omega)$ or a Hölder space $C^t(\overline{\Omega})$, where is u?

An important achievement was when Ros-Oton and Serra in arXiv '12 (J. Math. Pures Appl. '14) showed that when Ω is $C^{1,1}$,

$$f \in L_\infty(\Omega) \implies u \in d(x)^a C^\alpha(\overline{\Omega}) \cap C^a(\overline{\Omega}), \text{ for some } \alpha > 0,$$

$d(x) = \operatorname{dist}(x, \partial\Omega)$ near Ω. (With a slight improvement on α if f is more smooth.)

They stated that they did not know of other results on boundary regularity for the problem (1) in the literature.

I got interested in the problem in 2013, when I in connection with a memorial talk on Lars Hörmander found that he had some basic ingredients of a pseudodifferential theory for such problems in an old IAS Princeton lecture note from 1965.
This is now further developed (G arXiv '13, Adv. Math. '15), (G arXiv '14, Anal. PDE '14), and implies best possible results when Ω is C^∞:

$$f \in L_\infty(\Omega) \implies u \in e^+ d^a C^a(\overline{\Omega}), \quad \text{when } a \neq \tfrac{1}{2}, \tag{2}$$

$$f \in C^t(\overline{\Omega}) \implies u \in e^+ d^a C^{a+t}(\overline{\Omega}), \text{ for } t > 0,\ a+t, 2a+t \notin \mathbb{N}, \tag{3}$$

$$f \in C^\infty(\overline{\Omega}) \implies u \in e^+ d^a C^\infty(\overline{\Omega}). \tag{4}$$

In the exceptional cases, the conclusion holds with ε subtracted from the Hölder exponent.

Note the important factor d^a. Even when f is in $C^\infty(\overline{\Omega})$, u is not so; it has the singularity d^a; it is $d^{-a}u$ that is in $e^+C^\infty(\overline{\Omega})$.

The ps.d.o. theory behind this will now be described.

3. Pseudodifferential boundary problems

Let P be a classical ps.d.o. The Boutet de Monvel theory imposes a *transmission condition* (for the symbol at $\partial\Omega$) that assures that r^+Pe^+ preserves $C^\infty(\overline{\Omega})$. It is not satisfied by $(-\Delta)^a$ (or powers A^a of elliptic differential operators A).

Hörmander presented in the lecture note '65 and in his book '85 (with a different notation):

For $\operatorname{Re}\mu > -1$, define

$$\mathcal{E}_\mu(\overline{\Omega}) = e^+(d^\mu C^\infty(\overline{\Omega})),$$

where $d(x)$ is a smooth positive extension into Ω of $\operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. Generalized to $\operatorname{Re}\mu \le -1$ by taking distribution derivatives.

Definition 1. *A classical ps.d.o. of order $m \in \mathbb{C}$ is said to have the* μ-transmission property *at $\partial\Omega$ (for short: to be* of type μ*), when*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu),$$

for all indices; here $x \in \partial\Omega$ and ν denotes the interior normal at x.

It is a kind of twisted symmetry condition on the normal ν to Ω. Boutet de Monvel's transmission condition is the case $\mu = 0$, $m \in \mathbb{Z}$.

$(-\Delta)^a$ satisfies Def. 1 with $m = 2a$ and $\mu = a$, since $|\xi|^{2a}$ is *even* in ξ.

Then (Th. 18.2.18 in Hörmander's book '85):

Theorem 2. *r^+P maps $\mathcal{E}_\mu(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$* **if and only if** *the symbol has the μ-transmission property at $\partial\Omega$.*

The unpublished '65 lecture notes moreover described a *solvability theory* in L_2 Sobolev spaces for operators of type μ, when they in addition have a certain factorization property of the principal symbol, introduced by Vishik and Eskin '64:

Definition 3. *P (of order m) has the factorization index μ_0 when, in local coordinates where Ω is replaced by $\mathbb{R}^n_+ = \{x = (x', x_n) \mid x' \in \mathbb{R}^{n-1}, x_n > 0\}$,*

$$p_0(x', 0, \xi', \xi_n) = p_0^-(x', \xi', \xi_n) p_0^+(x', \xi', \xi_n),$$

with $p_0^\pm$ homogeneous in ξ of degrees μ_0 resp. $m - \mu_0$, and $p_0^\pm$ extending to $\{\operatorname{Im} \xi_n \lessgtr 0\}$ analytically in ξ_n.

Here $\operatorname{Op}(p_0^\pm(x', \xi))$ on $\mathbb{R}^n$ preserve support in $\overline{\mathbb{R}}^n_+$ resp. $\overline{\mathbb{R}}^n_-$.

A factorization always exists with $\mu_0(x')$; here μ_0 is constant in x'.

We have developed the theory further, in H^s_p and Hölder spaces (G'14, G'15). Consider a simple example:

Example. The symbol $\langle\xi\rangle^{2a}$ of $(1-\Delta)^a$ on $\mathbb{R}^n$ is factorized, relative to $\Omega=\mathbb{R}^n_+$ as

$$\langle\xi\rangle^{2a}=(\langle\xi'\rangle-i\xi_n)^a(\langle\xi'\rangle+i\xi_n)^a.$$

Then, when we define $\Xi^t_\pm=\operatorname{Op}((\langle\xi'\rangle\pm i\xi_n)^t)$ (generalized ps.d.o.s),

$$(1-\Delta)^a=\Xi^a_-\,\Xi^a_+.$$

Here $\Xi^t_\pm$ preserves support in $\overline{\mathbb{R}}^n_\pm$ since the symbol is analytic for $\operatorname{Im}\xi_n \lessgtr 0$. Hence the $\Xi^t_\pm$ have mapping properties

$$\Xi^t_+\colon \dot H^s_p(\overline{\mathbb{R}}^n_+)\xrightarrow{\sim}\dot H^{s-t}_p(\overline{\mathbb{R}}^n_+),\quad r^+\Xi^t_-e^+\colon H^s_p(\mathbb{R}^n_+)\xrightarrow{\sim}H^{s-t}_p(\mathbb{R}^n_+),$$

with inverses Ξ^{-t}_+ resp. $r^+\Xi^{-t}_-e^+$.

Now the Dirichlet problem

$$r^+(1-\Delta)^a u=f,\quad \operatorname{supp}u\subset\overline{\mathbb{R}}^n_+,$$

can be solved by inverting Ξ^a_+ and $r^+\Xi^a_-e^+$:

$$u=\Xi^{-a}_+e^+r^+\Xi^{-a}_-e^+f.$$

For data in $H^s_p(\mathbb{R}^n_+)$, $s\geq 0$, the solution space will be

$$\Xi^{-a}_+(e^+H^{s+a}_p(\mathbb{R}^n_+))\equiv H^{a(s+2a)}_p(\overline{\mathbb{R}}^n_+).$$

These functions are H^{s+2a}_p inside $\mathbb{R}^n_+$, but behave like $x_n^aH^{s+a}_p$ at $\partial\mathbb{R}^n_+$.

For the study in spaces over Ω, we need a replacement of the $\Xi^a_\pm$. Here we are greatly helped by developments of the Boutet de Monvel theory produced in more recent years, such as the following construction of order-reducing operators as in G '90 (CPDE):

Theorem 4. *There exist classical ps.d.o.s $\Lambda^{(t)}_\pm$ of order $t \in \mathbb{C}$ defining homeomorphisms*

$$\Lambda^{(t)}_+ : \dot{H}^s_p(\overline{\Omega}) \overset{\sim}{\to} \dot{H}^{s-\operatorname{Re} t}_p(\overline{\Omega}),$$
$$r^+\Lambda^{(t)}_- e^+ : H^s_p(\Omega) \overset{\sim}{\to} H^{s-\operatorname{Re} t}_p(\Omega),$$

with inverses $\Lambda^{(-t)}_+$ resp. $r^+\Lambda^{(-t)}_- e^+$; here $(r^+\Lambda^{(t)}_- e^+)^* = \Lambda^{(t)}_+$ for $t \in \mathbb{R}$.

With these operators, we can define the spaces

$$H^{\mu(s)}_p(\overline{\Omega}) = \Lambda^{(-\mu)}_+ e^+ H^{s-\operatorname{Re}\mu}_p(\Omega), \quad s > \operatorname{Re}\mu - 1/p'$$

(they have a cumbersome definition for $p = 2$ in the old lecture note). The spaces satisfy

$$H^{\mu(s)}_p(\overline{\Omega}) \begin{cases} = \dot{H}^s(\overline{\Omega}) \text{ for } s - \operatorname{Re}\mu \in]-1/p', 1/p[, \\ \subset d^\mu H^{s-\operatorname{Re}\mu}(\Omega) + \dot{H}^s(\overline{\Omega}) \text{ for } s - \operatorname{Re}\mu > 1/p. \end{cases}$$

There is then the result:

Theorem 5. *Let P be elliptic of order m, with factorization index μ_0, and of type μ_0 (mod 1). Let $s > \operatorname{Re}\mu_0 - 1/p'$. The homogeneous Dirichlet problem*

$$r^+ Pu = f, \quad \operatorname{supp} u \subset \overline{\Omega}, \quad (5)$$

considered for $u \in \dot{H}_p^{\operatorname{Re}\mu_0 - 1/p' + \varepsilon}(\overline{\Omega})$, satifies:

$$f \in H_p^{s - \operatorname{Re} m}(\Omega) \implies u \in H_p^{\mu_0(s)}(\overline{\Omega}). \quad (6)$$

Moreover, the mapping

$$r^+ P \colon H_p^{\mu_0(s)}(\overline{\Omega}) \to H_p^{s - \operatorname{Re} m}(\Omega) \quad (7)$$

is Fredholm.

Proof ingredients: The following ps.d.o. is of order 0 and type 0 with factorization index 0:

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)}.$$

Here (5) can be transformed to the equation

$$r^+ Qv = g, \text{ where } v = \Lambda_+^{(\mu_0)} u,\ g = r^+ \Lambda_-^{(\mu_0 - m)} e^+ f.$$

A closer analysis shows that $Q_+ = r^+ Q e^+$ is elliptic in the Boutet de Monvel calculus *without extra trace or Poisson operators*, so using a parametrix of it, we can construct a parametrix for (5).

The regularity result (6) implies Hölder estimates by Sobolev embedding theorems. Optimal Hölder estimates are obtained by showing a version of (6)–(7) directly in Hölder-Zygmund spaces $C^s_*(\overline{\Omega})$.

Returning to $P = (-\Delta)^a$ or A^a: It is of order $2a$, type a, and has factorization index a. Then we get the sharp regularity results mentioned in the beginning, e.g.,

$$f \in C^t(\overline{\Omega}) \implies u \in e^+ d^a C^{a+t}(\overline{\Omega}), \text{ for } t > 0,\ a+t, 2a+t \notin \mathbb{N},$$

with a loss of ε in the excepted cases.

The most important observation is the presence of the factor d^a. It is not u, but $d^{-a}u$, that is smooth up to the boundary.

One can also ask for equations with prescribed nonzero boundary values. There is a meaningful problem for $P = (-\Delta)^a$ or A^a defined in terms of the boundary value

$$\gamma_{a-1,0} u = (d^{1-a} u)|_{\partial\Omega}.$$

For this, one can show that the nonhomogeneous problem

$$r^+ Pu = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \quad \gamma_{a-1,0} u = \varphi,$$

is well-posed when u is sought in $H_p^{(a-1)(s)}(\overline{\Omega})$, a larger space than $H_p^{a(s)}(\overline{\Omega})$. There are also other well-posed boundary value problems.

4. Integration-by-parts formulas

The regularity result of Ros-Oton and Serra was actually a step in the proof of a certain "integration by parts" formula (R&S arXiv '12, Arch. Rat. Mech. '14). Namely for u with $(-\Delta)^a u \in C^{0,1}$, $\operatorname{supp} u \subset \overline{\Omega}$,

$$2\int_\Omega (x\cdot\nabla u)\,(-\Delta)^a u\,dx = (2a-n)\int_\Omega u(-\Delta)^a u\,dx \\ + \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,\gamma_0(u/d^a)^2\,d\sigma, \quad (8)$$

$$\int_\Omega ((-\Delta)^a u\,\partial_j u' + \partial_j u\,(-\Delta)^a u')\,dx \\ = \Gamma(1+a)^2\int_{\partial\Omega}\nu_j\gamma_0(u/d^a)\gamma_0(u'/d^a)\,d\sigma, \quad (9)$$

here ν is the interior normal to $\partial\Omega$, and $\gamma_0(u/d^a)$ denotes the boundary value of u/d^a from Ω. The two formulas are essentially equivalent.
The first formula leads to a (so-called Pohozaev) formula for solutions of nonlinear equations

$$r^+(-\Delta)^a u = f(u), \quad \operatorname{supp} u \subset \overline{\Omega},$$

that can be used to show results on (non-)solvability.

The formulas are remarkable, treating constant-coefficient ps.d.o.s on a *curved* domain, giving a *local* boundary contribution.

R&S have recently with Valdinoci extended the result to other translation invariant singular integral operators.
Under some regularity hypotheses, these operators coincide with constant-coefficient ps.d.o.s P of order $2a$ with even, positive, homogeneous symbol. E.g. (8) extends to:

$$
\begin{aligned}
2\int_\Omega (x\cdot\nabla u)\,Pu\,dx &= (2a-n)\int_\Omega uPu\,dx \\
&\quad + \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)s_0(x)\gamma_0(u/d^a)^2\,d\sigma, \qquad (10)
\end{aligned}
$$

where $s_0(x)$ is the value of the symbol of P on $\nu(x)$ at $x\in\partial\Omega$.

Their method: A delicate localization where the formula is shown first on rays transversal to the boundary, using a factorization $P=P^{\frac12}P^{\frac12}$. The method is entirely real, with $P^{\frac12}$ selfadjoint but acting in a complicated way on the solutions of $Pu=f\in C^{0,1}$.

Here $P^{\frac12}$ is of a type much different from that of P, and it seems natural to try to use the factorization $p_0=p_0^-p_0^+$ instead. Moreover, ps.d.o. methods could allow variable coefficients and lower-order terms, and nonselfadjointness.

Aiming for this, I have recently worked out a full asymptotic factorization $P \sim P^- P^+$ in x-dependent cases. This has allowed me to show, for bounded smooth sets $\Omega \subset \mathbb{R}^n$:

Theorem 6. *Let P be a classical elliptic ps.d.o. on $\mathbb{R}^n$ of order $2a$ with even symbol. Then for u, u' with $r^+ Pu, r^+ P^* u'$ in $H_p^1(\Omega)$ (some $p > n/a$), $\operatorname{supp} u, \operatorname{supp} u' \subset \overline{\Omega}$, there holds for $j = 1, \dots, n$:*

$$\begin{aligned}&\int_\Omega (Pu\, \partial_j \bar{u}' + \partial_j u\, \overline{P^* u'})\, dx \\ &= \Gamma(a+1)^2 \int_{\partial\Omega} \nu_j s_0 \gamma_0(u/d^a)\, \gamma_0(\bar{u}'/d^a)\, d\sigma + \int_\Omega [P, \partial_j] u\, \bar{u}'\, dx,\end{aligned}$$

$$\int_\Omega (Pu\, (x \cdot \nabla \bar{u}') + (x \cdot \nabla u)\, \overline{P^* u'})\, dx = -n \int_\Omega Pu\, \bar{u}'\, dx$$

$$\Gamma(a+1)^2 \int_{\partial\Omega} (x \cdot \nu) s_0 \gamma_0(u/d^a)\, \gamma_0(\bar{u}'/d^a)\, d\sigma + \int_\Omega [P, x \cdot \nabla] u\, \bar{u}'\, dx.$$

Distribution extensions with $H_p^1(\Omega)$ replaced by $H_2^{\frac12 - a + \varepsilon}(\Omega)$.

The first formula shows that the price of having P depend on x_j is an extra integral over Ω involving the commutator with ∂_j.

For translation invariant operators with lower-order terms we have in particular:

Corollary 7. *When P is selfadjoint and x-independent, one has for real u:*

$$
\begin{aligned}
2\int_\Omega (x\cdot\nabla u)\,Pu\,dx &= -n\int_\Omega u\,Pu\,dx \\
&+\Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)s_0\gamma_0(u/d^a)^2\,d\sigma+\int_\Omega u\,\mathrm{Op}(\xi\cdot\nabla p(\xi))u\,dx.
\end{aligned}
$$

If $p(\xi)$ is homogeneous of degree $2a$, $\xi\cdot\nabla p(\xi)=2a\,p(\xi)$ by Euler's formula, and we find the formula (10) of R&S&V '15. But the corollary also applies to nonhomogeneous cases:

Example. Let $p(\xi)=(|\xi|^2+m^2)^a$ with $0<a<1$. Then $\xi\cdot\nabla p(\xi)=2a|\xi|^2(|\xi|^2+m^2)^{a-1}>0$ for $\xi\neq 0$. Now if u is a real function solving

$$
r^+Pu=\lambda u \text{ in } \Omega,\quad \operatorname{supp} u\subset\overline{\Omega},
$$

for some $\lambda\in\mathbb{R}$, we can use the above formula to conclude that

$$
\gamma_0(u/d^a)=0 \implies \int_\Omega u\,\mathrm{Op}(|\xi|^2(|\xi|^2+m^2)^{a-1})u\,dx=0 \implies u\equiv 0,
$$

a kind of unique continuation principle.
Nonlinear applications will also be pursued.

Extra info:

The Pohozaev-type formula resulting from Corollary 7 is:

$$
\begin{aligned}
-2n\int_\Omega F(u)\,dx + n\int_\Omega u\,f(u)\,dx \\
&= \int_\Omega uP_1u\,dx + \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(d^{-a}u)^2\,d\sigma,
\end{aligned}
$$

where $P_1 = \mathrm{Op}(\xi\cdot\nabla_\xi p(\xi))$; here $F(t) = \int_0^t f(s)\,ds$.