SDL, Blok 1, October 2009
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Extra exercises 14-15

This is the set of exercises used for the 2 hour in-class test on October 30, 2009. Each exercise counts 50%.

It is allowed to bring written material of all kinds (books and printed papers, personal notes etc.). Electronic equipment is not allowed. Answers can be formulated in Danish or English, as preferred.

Exercise E14. Consider the autonomous differential equation for $\boldsymbol{y} \in \mathbb{R}^2$:

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \text{ where } \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} e^{y_1} - 1 - 2y_2 \\ 3y_1 - 4y_2 \end{pmatrix}. \tag{E14.1}$$

(a) Show (without finding the solution explicitly) that for any $t_0 \in \mathbb{R}$, any $\boldsymbol{\eta} \in \mathbb{R}^2$, there is a unique maximal solution $\boldsymbol{\varphi}(t)$ defined on an interval $]c^*, d^*[$ containing t_0 and satisfying

$$\boldsymbol{\varphi}(t_0) = \boldsymbol{\eta}; \tag{E14.2}$$

and that if $d^* < \infty$, then $|\boldsymbol{\varphi}(t)| \to \infty$ for $t \to d^*$.

(b) Show that $\mathbf{0}$ is a critical point.

(c) Show that the null-solution is asymptotically stable.

(*Hint:* By application of Taylor's formula to the exponential function, you can write the problem as an almost linear system, where the results of Section 4.4 in the book may be used.)

Exercise E15. Consider the differential equation for $(t, \boldsymbol{y}) \in \mathbb{R}^4$:

$$\boldsymbol{y}' = A\boldsymbol{y}, \text{ where } A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 7 \end{pmatrix}. \tag{E15.1}$$

(a) Find the eigenvalues for A, and find the corresponding eigenspaces and generalized eigenspaces.

(b) Find the fundamental matrix e^{tA}.

(c) Find the solution of the nonhomogeneous problem

$$\boldsymbol{y}' = A\boldsymbol{y} + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{E15.2}$$

Danish version of the exercises:

Exercise E14. Man betragter den autonome differentialligning for $\boldsymbol{y} \in \mathbb{R}^2$:

$$\text{(E14.1)} \qquad \boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \text{ hvor } \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} e^{y_1} - 1 - 2y_2 \\ 3y_1 - 4y_2 \end{pmatrix}.$$

(a) Vis (uden at finde løsningen explicit) at for ethvert $t_0 \in \mathbb{R}$ og ethvert $\boldsymbol{\eta} \in \mathbb{R}^2$ er der en entydigt bestemt maksimal løsning $\boldsymbol{\varphi}(t)$ som er defineret på et interval $]c^*, d^*[$ omkring t_0 og opfylder

$$\text{(E14.2)} \qquad \boldsymbol{\varphi}(t_0) = \boldsymbol{\eta};$$

og at hvis $d^* < \infty$, så vil $|\boldsymbol{\varphi}(t)| \to \infty$ for $t \to d^*$.

(b) Vis, at $\mathbf{0}$ er et kritisk punkt.

(c) Vis, at nulløsningen er asymptotisk stabil.

(*Vink:* Ved at anvende Taylors formel på eksponentialfunktionen kan man skrive problemet som et næsten-lineært system, hvor resultater fra bogens Afsnit 4.4 kan bruges.)

Exercise E15. Man betragter følgende differentialligning for $(t, \boldsymbol{y}) \in \mathbb{R}^4$:

$$\text{(E15.1)} \qquad \boldsymbol{y}' = A\boldsymbol{y}, \text{ hvor } A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 7 \end{pmatrix}.$$

(a) Find egenværdierne for A, samt de tilhørende egenrum og generaliserede egenrum.

(b) Find fundamental-matricen e^{tA}.

(c) Find løsningen til det inhomogene problem

$$\text{(E15.2)} \qquad \boldsymbol{y}' = A\boldsymbol{y} + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$