Spectral results for mixed problems and fractional order elliptic operators

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January 4–10, 2015

1. Krein formula for the mixed problem

Let Ω be smooth open bounded $\subset \mathbb{R}^n$, with boundary $\partial\Omega = \Sigma$. Denote $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$. Denote by $H^s(\mathbb{R}^n)$ the L_2-Sobolev space of order $s \in \mathbb{R}$, $H^s(\Omega) = r_\Omega H^s(\mathbb{R}^n)$, $\dot{H}^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$.

Consider a symmetric strongly elliptic second-order differential operator on Ω with real C^∞-coefficients,

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j(a_{jk}(x)\partial_k u) + a_0(x)u.$$

The associated sesquilinear form $a(u,v) = \sum_{j,k=1}^{n}(a_{jk}\partial_k u, \partial_j v) + (a_0 u, v)$ is coercive on $H^1(\Omega)$, and we add a constant to a_0 to make it positive. Set $\nu u = \sum n_j \gamma_0(a_{jk}\partial_k u)$ ($= \gamma_1 u$ when $A = -\Delta$), the conormal derivative. Realizations of A:

The maximal realization $A_{\max}$, $D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$.

The Dirichlet realization A_γ with $D(A_\gamma) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$.

The Neumann realization A_ν with $D(A_\nu) = \{u \in H^2(\Omega) \mid \nu u = 0\}$.

A mixed realization $A_{\nu,U}$. Here U is a smooth open subset of Σ, and $D(A_{\nu,U}) = \{u \in H^1(\Omega) \cap D(A_{\max}) \mid \nu u = 0 \text{ on } U,\ \gamma_0 u = 0 \text{ on } \Sigma \setminus U\}$.

The latter three are defined variationally from the form $a(u,v)$ considered on $\dot{H}^1(\overline{\Omega})$, $H^1(\Omega)$, resp. $H^1_U(\Omega) = \{u \in H^1(\Omega) \mid \operatorname{supp} \gamma_0 u \subset \overline{U}\}$.
They are selfadjoint positive, and whereas $D(A_\gamma)$ and $D(A_\nu) \subset H^2(\Omega)$, it is known that $D(A_{\nu,U}) \subset H^{\frac32 - \varepsilon}(\Omega)$ only.

Let $Z = \ker A_{\max}$, and let K_γ denote the Poisson operator $K_\gamma \colon \varphi \mapsto u$ solving the semihomogeneous Dirichlet problem

$$Au = 0 \text{ on } \Omega,\ \gamma_0 u = \varphi \text{ on } \Sigma,$$

it maps e.g. $K_\gamma \colon H^{-\frac12}(\Sigma) \xrightarrow{\sim} Z$, closed subset of $L_2(\Omega)$.

Let $P = \nu K_\gamma$, the **Dirichlet-to-Neumann operator**; it is known to be a pseudodifferential operator on Σ of order 1.

Proposition 1. *Let $x' \in \Sigma$ and choose coordinates such that the interior normal is $(0,\dots,0,1)$. Write the principal symbol of A at x' as $a_{nn}(x')\xi_n^2 + 2b(x',\xi')\xi_n + c(x',\xi')$, and let*

$$m(x',\xi') = a_{nn}(x')c(x',\xi') - b(x',\xi')^2,$$

it is positive for $\xi' \neq 0$ by the ellipticity of A.
Then P has principal symbol $p^0(x',\xi') = -m(x',\xi')^{\frac12}$ at x'.
Hence if M is a selfadjoint differential operator on Σ with principal symbol $m(x',\xi')$, $P = -M^{\frac12} +$ order 0.

Define for Σ the restriction operator $r^+\colon \varphi \mapsto \varphi|_U$, and the extension operator $e^+\colon \psi \mapsto \{\psi \text{ on } U, 0 \text{ on } \Sigma \setminus U\}$.

When Q is an operator over Σ we denote $r^+ Q e^+ = Q_+$ (truncation).

Let $X = \dot{H}^{-\frac{1}{2}}(\overline{U})$ (the subspace of distributions in $H^{-\frac{1}{2}}(\Sigma)$ supported in $\overline{U}$). Its dual space is $X^* = H^{\frac{1}{2}}(U) = r^+ H^{\frac{1}{2}}(\Sigma)$.

Define $V = K_\gamma(X) \subset Z$ and denote the restriction $K_\gamma|_X$ by

$$K_{\gamma,X}\colon X \overset{\sim}{\to} V, \text{ with adjoint } K^*_{\gamma,X}\colon V \overset{\sim}{\to} X^*.$$

In J. Math. An. Appl. '11 we showed:

Theorem 2. *For the mixed problem there is an operator L mapping $D(L) \subset X$ onto X^* such that the Kreĭn resolvent formula holds:*

$$A^{-1}_{\nu,U} - A^{-1}_\gamma = \mathrm{i}_V K_{\gamma,X} L^{-1} K^*_{\gamma,X} \, \mathrm{pr}_V \equiv T. \qquad (1)$$

Here L acts like $-P_+$ and has

$$D(L) = \{\varphi \in X \mid P_+\varphi \in X^*\} \subset \dot{H}^{1-\varepsilon}(\overline{U}).$$

We want to find the spectral behavior of the Krein term T.

Question: What is L^{-1}? (It does NOT act like $-(P^{-1})_+$). L^{-1} was studied in '11 using tools from Eskin '81, Birman-Solomiak '77, Laptev '81. This lead to a spectral asymptotic formula for T when $A = -\Delta +$ lower order terms near Σ, so that $P = -(-\Delta_\Sigma)^{\frac{1}{2}} +$ l.o.t. on Σ.

2. Boundary problems for fractional order operators

Now a better tool is available: Boundary value theories for fractional powers of elliptic operators. This will allow general A and P.

A basic example of a pseudodifferential operator (ps.d.o.) of noninteger order is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$:

$$(-\Delta)^a u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)), \quad \hat{u}(\xi) = \mathcal{F}u = \int_{\mathbb{R}^{n'}} e^{-ix\cdot\xi} u(x)\,dx.$$

Currently of interest both in probability, finance, mathematical physics and geometry. More general example: M^a, where M is a 2'order strongly elliptic differential operator with smooth coefficients on $\mathbb{R}^{n'}$. M^a is a ps.d.o. of order $2a$ by Seeley '66.

Let U be bounded smooth open $\subset \mathbb{R}^{n'}$. Dirichlet problem for M^a on U?

Let $m_a(u,v) = (M^a u, v)$ for $u, v \in C_0^\infty(U)$. It satisfies

$$\operatorname{Re} m_a(u,u) \ge c\|u\|_a^2 - k\|u\|_0^2,\ c > 0, k \in \mathbb{R},$$

and its closure with domain $\dot{H}^a(\overline{U})$ defines a convenient operator M^a_{Dir} in $L_2(U)$ by variational theory. It acts like M^a_+, with $D(M^a_{\mathrm{Dir}}) \subset \dot{H}^a(\overline{U})$. It represents the problem

$$M^a_+ u = f, \quad u \text{ sought in } \dot{H}^a(\overline{U}). \qquad (2)$$

What is $D(M^a_{\mathrm{Dir}})$? What are the regularity properties of solutions of (2)? Here the results are quite recent.
Ros-Oton and Serra (J.Math.Pur.Appl.'14) showed by potential theory and integral operator methods, when $M = -\Delta$ and U is $C^{1,1}$, that

$$f \in L_\infty(U) \implies u \in d^a C^\alpha(\overline{U}) \cap C^a(\overline{U}), \qquad (3)$$

for some $\alpha > 0$. Here $d(x) = \mathrm{dist}(x, \partial U)$. They stated that they did not know of other regularity results for $(-\Delta)^a$ in the literature.

Ps.d.o. methods? The Boutet de Monvel calculus, initiated in '71, requires integer order plus a so-called 0-transmission property at ∂U. M^a is not covered.

But we have recently developed another calculus. It is based on a more general μ-transmission property, introduced by Hörmander in his 1985 book (in fact in an unpublished lecture note from IAS Princeton 1965). Here M^a has the a-transmission property, since the symbol has even parity and is of order $2a$.
It allows to improve the information in (3) to $u \in d^a C^a(\overline{U})$ and to get higher regularity: $f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t}(\overline{\Omega})$ for $t > 0$ (except for t or $a + t$ integer, slightly weaker result).
(G Adv.Math.'15, Anal&PDE'14.)

The results rely on constructing an approximate inverse of M^a_{Dir} (a parametrix).
Consider a localized situation where U and $\complement\overline{U}$ are replaced by, resp. $\mathbb{R}^{n'}_\pm = \{x \mid x_{n'} \gtrless 0\}$. There exist **order-reducing operators**:

Theorem 3. *There exist two families of ps.d.o.s $\Lambda^{(t)}_\pm$ of order $t \in \mathbb{R}$, preserving support in $\overline{\mathbb{R}}^{n'}_\pm$, respectively, such that for all $s \in \mathbb{R}$,*

$$\Lambda^{(t)}_+ \colon \dot H^s(\overline{\mathbb{R}}^{n'}_+) \overset{\sim}{\to} \dot H^{s-t}(\overline{\mathbb{R}}^{n'}_+), \quad (\Lambda^{(t)}_-)_+ \colon H^s(\mathbb{R}^{n'}_+) \overset{\sim}{\to} H^{s-t}(\mathbb{R}^{n'}_+).$$

Then M^a_+ can be linked to an operator in the BdM calculus:

Theorem 4. *On $\dot H^a(\overline{\mathbb{R}}^{n'}_+)$, the operator M^a_+ can be written in the form*

$$M^a_+ = (\Lambda^{(a)}_-)_+ r^+ Q \Lambda^{(a)}_+, \qquad (4)$$

where Q is a ps.d.o. of order 0 in the Boutet de Monvel calculus, such that the problem

$$Q_+ v = g, \quad \operatorname{supp} v \subset \overline{\mathbb{R}}^{n'}_+, \qquad (5)$$

is well-posed. Here the solution to (2) is found as $\Lambda^{(-a)}_+ e^+ v$, when $g = (\Lambda^{(-a)}_-)_+ f$.

Theorem 5. *Let* $\widetilde{Q}_+ + G_0$ *be a parametrix for* (5) *(*G_0 *being a sing. Green op. of class and order* 0 *in the Boutet de Monvel calculus). Then the operator* M^a_{Dir} *has the parametrix*

$$R = (\Lambda_+^{(-a)})_+(\widetilde{Q}_+ + G_0)(\Lambda_-^{(-a)})_+. \qquad (6)$$

Similar results can be obtained in the situation of the manifold $\Sigma = \partial\Omega$ and its subset U (of dimension $n' = n - 1$).

Formula (6) can be used to get a spectral asymptotic estimate for R.

Theorem 6. *Let*

$$\mathcal{P} = P_{1,+} \dots P_{l_0,+}(P_{0,+} + G)P_{l_0+1,+} \dots P_{l,+},$$

where P_0 *is of order 0,* G *is a singular Green on* U *of order and class* 0*, and the* P_j *are of order* $-t_j < 0$*. Let* $t = t_1 + \dots + t_l$*. Then the singular values* $s_k(\mathcal{P})$ *satisfy:*

$$s_k(\mathcal{P})k^{t/(n-1)} \to C(\mathcal{P}) \text{ for } k \to \infty,$$

where $C(\mathcal{P})$ *is defined from the principal symbols on* U*.*

Corollary 7. *For* R *in* (6),

$$s_k(R)k^{2a/(n-1)} \to C(R).$$

3. Application to the mixed problem

For the mixed problem, we were aiming to find the spectral asymptotics of the Krein term

$$T = \mathrm{i}_V K_{\gamma,X} L^{-1} K^*_{\gamma,X} \,\mathrm{pr}_V \,.$$

Recall that we are here in a selfadjoint case. We know that L acts like $-P_+$, where P is the Dirichlet-to-Neumann operator. It was shown in Proposition 1 that P is of the form

$$P = -M^{\frac{1}{2}} - S,$$

where M is a selfadjoint 2' order differential operator on Σ and S is a ps.d.o. of order 0.

Then the operator L appearing in the Krein term acts like

$$L = -P_+ = M_+^{\frac{1}{2}} + S_+, \text{ with } D(L) \subset \dot{H}^{1-\varepsilon}(\overline{U}).$$

Here $M_+^{\frac{1}{2}}$ acts like $M_{\mathrm{Dir}}^{\frac{1}{2}}$.

Now we can use that we have found a parametrix R of $M_{\mathrm{Dir}}^{\frac{1}{2}}$. Since L is invertible, we can deduce that

$$L^{-1} = R + S_1, \qquad (7)$$

where S_1 is of order -2.

For the eigenvalues of the Krein term we show by commutation:

$$\mu_k(T) = \mu_k(\mathrm{i}_V K_{\gamma,X} L^{-1} K^*_{\gamma,X}\,\mathrm{pr}_V) = \mu_k(L^{-1} K^*_{\gamma,X} K_{\gamma,X}) = \mu_k(L^{-1} P_{1,+})$$

where $P_1 = K^*_\gamma K_\gamma$ is a ps.d.o. of order -1.

With $P_2 = P_1^{\frac{1}{2}}$ we deduce moreover, using also (7):

$$\mu_k(T) = \mu_k(P_{2,+} L^{-1} P_{2,+} + S_2) = \mu_k(P_{2,+} R P_{2,+} + S_3)$$

where S_2 and S_3 by various perturbation arguments will not enter in the principal asymptotics.

Now Theorem 6 can be applied to $P_{2,+} R P_{2,+}$. This leads to

Theorem 8. *The eigenvalues of T satisfy*

$$\mu_k(T) k^{2/(n-1)} \to C(T) \text{ for } k \to \infty,$$

where $C(T)$ is an integral over U of a function defined from the principal symbols:

$$C(T) = \tfrac{1}{(n-1)(2\pi)^{n-1}} \int_U \int_{|\xi'|=1} \Big(\frac{a_{nn}(x')}{2m(x',\xi')} \Big)^{\frac{n-1}{2}} d\omega(\xi') dx'.$$

(G JMAA'15)