Lectures in Noncommutative Geometry Seminar 2005

TRACE FUNCTIONALS AND TRACE DEFECT FORMULAS . . .

I. Traces on classical ψdo's.

We consider:

X — compact boundaryless n-dimensional manifold (closed).

E — hermitian vector bundle over X.

$\mathcal{A}$ — the 'algebra' of classical ψdo's A acting in E.

On pseudodifferential operators:

Recall that a differential operator of order $m \geq 0$ on $\mathbb{R}^n$ can be written:

$$Au = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = \mathcal{F}^{-1}_{\xi \to x}\Big(\sum_\alpha a_\alpha(x)\xi^\alpha \hat{u}(\xi)\Big)$$

$$= \mathrm{OP}(a(x,\xi))u, \text{ with } a(x,\xi) = \sum_\alpha a_\alpha(x)\xi^\alpha.$$

A classical pseudodifferential symbol of order $\nu \in \mathbb{R}$:

$$a(x,\xi) \sim a_\nu(x,\xi) + a_{\nu-1}(x,\xi) + \dots + a_{\nu-j}(x,\xi) + \dots$$

$a_{\nu-j}(x, t\xi) = t^{\nu-j} a(x,\xi)$ for $|\xi| \geq 1$, $t \geq 1$.

Elliptic, when the principal symbol $a_\nu(x,\xi) \neq 0$ for $|\xi| \geq 1$.

Defines a pseudodifferential operator (ψdo):

$$Au = \mathrm{Op}(a)u = \mathcal{F}^{-1}_{\xi \to x}(a(x,\xi)\hat{u}(\xi))$$

Continuous from $H^s(\mathbb{R}^n)$ to $H^{s-\nu}(\mathbb{R}^n)$. Composition:

$$\mathrm{Op}(a(x,\xi))\,\mathrm{Op}(b(x,\xi)) = \mathrm{Op}(a\#b), \text{ where}$$

$$a\#b \sim a \cdot b + \textstyle\sum_{\alpha\neq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a\, \partial_x^\alpha b.$$

Elliptic operators have (approximate) inverses.

Ψdo's are defined on manifolds *by use of local coordinates.*

Consider A on a closed manifold X. For $\nu < -n$, A is trace-class, $\mathrm{Tr}\, A = \int_X \mathrm{tr}\, K_A(x,x)dx$. Here $\mathrm{Tr}([A, A']) = 0$, where $[A, A'] = AA' - A'A$.

A *trace functional* $\ell(A)$ is a linear functional that vanishes on commutators: $\ell([A, A']) = 0$. Search for nontrivial trace functionals on higher-order ψdo's!

(I) Wodzicki, Guillemin ca. '84: *The noncommutative residue*

$$\operatorname{res}(A) = \int_X \int_{|\xi|=1} \operatorname{tr} a_{-n}(x,\xi)\, đS(\xi) dx;$$

it has a coordinate invariant meaning. ($đ = (2\pi)^{-n} d$.)

- Local (depends only on certain homogeneous terms in a).
- Defined for all $A \in \mathcal{A}$, unique up to a factor.
- Vanishes for $\nu \notin \mathbb{Z}$.
- Vanishes for $\nu < -n$; does **not** extend $\operatorname{Tr} A$!

(II) Kontsevich and Vishik ca. '94: *The canonical trace* $\mathrm{TR}(A)$

- Global (depends on the full structure).
- Defined only for some A, namely in the cases:

  (1) $\nu < -n$, then $\mathrm{TR}\, A = \operatorname{Tr} A$;

  (2) $\nu \notin \mathbb{Z}$;

  (3) $\nu \in \mathbb{Z}$, n odd, A has even-even parity;

  (4) $\nu \in \mathbb{Z}$, n even, A has even-odd parity. (Added by GG.)

(Will give formula later.)

Parity properties:

even-even alternating parity: Even order terms are even in ξ,

$$a_{\nu-j}(x,-\xi) = (-1)^{\nu-j} a_{\nu-j}(x,\xi) \text{ for } |\xi| \geq 1.$$

Example: Differential operators and their parametrices.

even-odd alternating parity: Even order terms are odd in ξ,

$$a_{\nu-j}(x,-\xi) = (-1)^{\nu-j-1} a_{\nu-j}(x,\xi) \text{ for } |\xi| \geq 1.$$

Example: $D|D|^{-1}$, D a selfadj. first-order elliptic diff. op.

The trace property holds in the following sense:

$$\mathrm{TR}([A, A']) = 0 \text{ in the cases}$$

(1′) $\nu + \nu' < -n$,

(2′) $\nu + \nu' \in \mathbb{R} \setminus \mathbb{Z}$.

(3′) ν and $\nu' \in \mathbb{Z}$, n is odd, A and A' are both even-even or both even-odd.

(4′) ν and $\nu' \in \mathbb{Z}$, n is even, A is even-odd and A' is even-even.

Both $\operatorname{res} A$ and $\operatorname{TR} A$ were originally defined by use of

complex powers:

Let P be elliptic of even order $m > 0$, say $P > 0$.

Define $\zeta(A, P, s) = \operatorname{Tr}(AP^{-s})$, the *generalized zeta function*, holomorphic for $\operatorname{Re} s > (n+\nu)/m$, extends meromorphically to $\mathbb{C}$ with simple poles in

$$\{(n+\nu-j)/m \mid j \in \mathbb{N}\} \cup \{-k \mid k \in \mathbb{N}\};$$

here $\mathbb{N} = \{0, 1, 2, \dots\}$.

In particular, ζ has a Laurent expansion at $s = 0$:

$$\zeta(A, P, s) \sim \tfrac{1}{s} C_{-1}(A, P) + C_0(A, P) + \textstyle\sum_{l \geq 1} C_l(A, P) s^l.$$

Then

(I) $\operatorname{res} A = m \cdot C_{-1}(A, P)$, the residue at $s = 0$.

(II) In the cases (1)–(4) (with P even-even for (3)–(4)), $C_{-1}(A, P) = 0$ and

$$\operatorname{TR} A = C_0(A, P).$$

NB! Independent of P!

Three operator families:

P: strongly elliptic ps.d.o. on X of even order $m > 0$.

Resolvent $(P-\lambda)^{-1}$,

Heat operator e^{-tP},

Power operator P^{-s} (defined as 0 on $\ker P$).

Can be obtained from one another:

$$\text{Resolvent } (P-\lambda)^{-1} \quad \underset{\text{Laplace transf.}}{\overset{\text{Cauchy int.}}{\rightleftarrows}} \quad e^{-tP} \text{ Heat operator}$$

$$\text{Cauchy int. } \searrow\!\sim \qquad\qquad \sim\!\swarrow \text{ Mellin transf.}$$

$$\Gamma(s)P^{-s}$$

Power operator

Example of Cauchy integral:

$$P^{-s} = \frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}(P-\lambda)^{-1}\, d\lambda.$$

Three equivalent asymptotic trace expansions:

The resolvent trace expansion:

$$\operatorname{Tr}(A(P-\lambda)^{-N}) \sim \sum_{j\ge 0} \tilde c_j (-\lambda)^{-\frac{\nu+n-j}{m}-N}$$
$$+\sum_{k\ge 0}\big(\tilde c'_k \log(-\lambda)+\tilde c''_k\big)(-\lambda)^{-k-N},$$

for $\lambda\to\infty$ in $\mathbb{C}\setminus\overline{\mathbb{R}}_+$. ($N>(\nu+n)/m$.)

The heat trace expansion:

$$\operatorname{Tr}(Ae^{-tP}) \sim \sum_{j\ge 0} c_j t^{\frac{j-\nu-n}{m}} + \sum_{k\ge 0}(-c'_k\log t + c''_k)t^k$$

for $t\to 0+$.

The complex power trace expansion:

$$\Gamma(s)\operatorname{Tr}(AP^{-s}) \sim \sum_{j\ge 0}\frac{c_j}{s+\frac{j-\nu-n}{m}} + \sum_{k\ge 0}\Big(\frac{c'_k}{(s+k)^2}+\frac{c''_k}{s+k}\Big);$$

where the right-hand side gives the pole structure of the meromorphic extension. Division by $\Gamma(s)$ gives simple poles, and

$$C_{-1}(A,P)=\tilde c'_0 = c'_0,\quad C_0(A,P)=\tilde c_{n+\nu}+\tilde c''_0 = c_{n+\nu}+c''_0;$$

where we set $\tilde c_{n+\nu}=c_{n+\nu}=0$ if $n+\nu\notin\mathbb{N}$. In cases (1)–(4),

$$C_0(A,P)=c''_0=\operatorname{TR} A.$$

Moreover, in cases (1)–(4),

$$c_0'' = \mathrm{TR}(A) = \int_X ⨍ \mathrm{tr}\, a(x,\xi)\, đ\xi dx;$$

it has a coordinate invariant meaning. Here $⨍ f(x,\xi)\, đ\xi$ is a *partie finie* integral, defined as follows: When $f(x,\xi)$ is a classical symbol of order ν, then

$$\int_{|\xi|\le R} f(x,\xi)\, đ\xi \sim \sum_{j\in\mathbb{N}, j\ne n+\nu} a_j(x) R^{n+\nu-j} + a_0'(x)\log R + a_0''(x)$$

for $R\to\infty$, and one sets $⨍ f(x,\xi)\, đ\xi = a_0''(x)$.

Instead of considering powers AP^{-s}, one can deduce these results directly from trace expansions of resolvents $A(P-\lambda)^{-1}$, using the calculus of G-Seeley '95. Details in vol. 366 of AMS Comtemp. Math. Proc., 2005.

II. Trace defect formulas.

Consider $C_0(A,P)$ in general. When (1)–(4) do not hold, $C_0(A,P)$ will depend on P and need not vanish on $[A,A']$. However, there are formulas for the **trace defects**:

(a) $C_0(A,P) - C_0(A,P') = -\frac{1}{m}\operatorname{res}(A(\log P - \log P'))$,

(b) $C_0([A,A'],P) = -\frac{1}{m}\operatorname{res}(A[A',\log P])$,

showing in particular that they are **local**. ((a) by Okikiolu '95, Konts.-V. '95, (a)+(b) by Melrose-Nistor '96 unpublished.) Their proofs go via the holomorphic family P^{-s}, with

$$\tfrac{d}{ds}P^{-s}|_{s=0} = -\log P.$$

$\log P$ has symbol $m\log|\xi| + b(x,\xi)$, where b is classical of order 0. Thus

$$A(\log P - \log P') \text{ is classical of order } \nu,$$

$$A(A'\log P - \log P A') \text{ is classical of order } \nu,$$

so res is defined.

Question: Do similar formulas hold for manifolds with boundary?

A reasonable ψdo boundary operator calculus containing elliptic differential boundary problems and their solution operators is the Boutet de Monvel calculus. Can we show similar formulas for such operators?

Problematic fact: Even for $P_T = (-\Delta)_{\mathrm{Dirichlet}}$, the complex powers $(P_T)^s$ and the logarithm $\log(P_T)$ are not in the BdM calculus. But the resolvent $(P_T - \lambda)^{-1}$ does belong to a parameter-dependent version of the BdM calculus.

Subquestion: Can we prove the formulas (a) and (b) using only resolvent information?

III. Some applications of res, TR and $C_0(A,P)$.

Recall: $\operatorname{res} A$ is proportional to the residue of $\zeta(A,P,s)$ at $s = 0$, so

$$\operatorname{res} A = 0 \iff \zeta(A,P,s) \text{ is regular at } 0.$$

Holds when A is a diff. op., in particular for $\zeta(I,P,s) \equiv \zeta(P,s)$.

The *eta function* of a selfadjoint, not semibounded elliptic ψdo:

$$\eta(P,s) = \textstyle\sum_{\lambda \text{ ev. } \neq 0} \operatorname{sign} \lambda \, |\lambda|^{-s} = \zeta(P|P|^{-1}, |P|, s),$$

is not covered by this. Deep result by Atiyah-Patodi-Singer and Gilkey:

$$(*) \quad \operatorname{res}(P|P|^{-1}) = 0,$$

i.e., $\eta(P,s)$ is regular at $s = 0$. For a Dirac operator D, with

$$D = \sigma(\partial_{x_n} + A) \text{ near } \partial X,\ A \text{ tangential selfadjoint},$$

the value $\eta(A,0)$ for A on ∂X is a nonlocal term entering in the index formula for the APS realization of D.

From (*) one can moreover deduce that $\operatorname{res} \Pi = 0$ for any classical ψdo projection Π, a fact with various applications.

Concerning TR and $C_0(A,P)$:

Some people call $C_0(A,P)$ a *regularized trace* of A, with notation e.g. $\overline{\mathrm{Tr}}(A)$ (Melrose). Enters in an index formula for A: When B is an approximate inverse (a parametrix),

$$\begin{aligned}\operatorname{ind} A &= \operatorname{Tr}(AB-I) - \operatorname{Tr}(BA-I)\\ &= C_0(AB-I,P) - C_0(BA-I,P) = C_0([A,B],P)\\ &= -\tfrac{1}{m}\operatorname{res}(A[B,\log P]),\end{aligned}$$

by the trace defect formula. This is a point of departure for further calculations.

IV. Manifolds with boundary.

Now let X be a compact n-dimensional manifold with smooth boundary $X' = \partial X$ (itself a closed manifold).

Typical operators when $X \subset \mathbb{R}^n$:

$$\begin{pmatrix} 1-\Delta \\ \gamma_0 \end{pmatrix} \text{ and its inverse } \begin{pmatrix} Q_+ + G & K \end{pmatrix};$$

γ_0 is the trace operator $u \mapsto u|_{X'}$,

$Q = (1-\Delta)^{-1} = \mathrm{Op}(\frac{1}{1+|\xi|^2})$ on $\mathbb{R}^n$,

$Q_+ = r^+ Q e^+$ (e^+ extends by 0, r^+ restricts to X),

G is a singular Green operator (the "boundary correction"),

K is a Poisson operator.

Here $R = Q_+ + G$ and K solve the respective semi-homogeneous problems:

$$\begin{cases} (1-\Delta)u &= f \text{ in } X, \\ \gamma_0 u &= 0 \text{ on } X'; \end{cases} \qquad \begin{cases} (1-\Delta)u &= 0 \text{ in } X, \\ \gamma_0 u &= g \text{ on } X'. \end{cases}$$

Boutet de Monvel '71 defined pseudodifferential boundary operators (ψdbo's) in general as systems (Green operators):

$$\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X,E) \\ \times \\ C^\infty(X',F) \end{array} \to \begin{array}{c} C^\infty(X,E') \\ \times \\ C^\infty(X',F') \end{array},$$

where

P is a ψdo on a closed manifold $\widetilde{X} \supset X$, $\quad P_+ = r^+ P e^+$,

G is a singular Green operator,

T is a trace operator from X to X',

K is a Poisson operator from X' to X,

S is a ψdo on X'.

P must satisfy the *transmission condition* at X', assuring that P_+ preserves smoothness on X. Consider operators of order ν with polyhomogeneous symbols of suitable types.

Traces can be studied when $E = E'$, $F = F'$; the new object is $A = P_+ + G : C^\infty(X,E) \to C^\infty(X,E)$. Transmission requires integer order. For G alone one can study all real orders.

Technical condition: G should be *of class* 0 (well-defined on $L_2(X)$), for otherwise, order $< -n$ does not assure trace-class.

The noncommutative residue was defined by Fedosov, Golse, Leichtnam and Schrohe '96 for $A = P_+ + G$ by:

$$\operatorname{res}(A) = \int_X \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\, đS(\xi) dx \\ + \int_{X'} \int_{|\xi'|=1} \operatorname{tr}(\operatorname{tr}_n g)_{1-n}(x',\xi')\, đS(\xi') dx';$$

here tr_n takes the trace in the normal direction to X'; in fact $\operatorname{tr}_n G$ is a classical ψdo on X'.

That this is indeed a residue was shown by G-Schrohe '01: As auxiliary operator we can take an elliptic differential operator P_1 of order $m > n + \nu$ on $\widetilde{X}$ having a sector V around $\mathbb{R}_-$ in its resolvent set. Then

$$\operatorname{Tr}(A(P_1 - \lambda)_+^{-1}) \sim \sum_{0 \le j \le n+\nu} c_j (-\lambda)^{\frac{n+\nu-j}{m} - 1} \\ + (c_0' \log(-\lambda) + c_0'')(-\lambda)^{-1} + O(\lambda^{-1-\varepsilon}), \text{ for } \lambda \to \infty \text{ in } V.$$

There is a corresponding expansion for $\Gamma(s)\,\mathrm{Tr}(A(P_1^{-s})_+)$. In particular,

$$\mathrm{Tr}(A(P_1^{-s})_+) = \tfrac{1}{s}C_{-1}(A,P_{1,+}) + C_0(A,P_{1,+}) + O(s)$$

for $s \to 0$, with $C_{-1}(A,P_{1,+}) = c_0'$, $C_0(A,P_{1,+}) = c_{n+\nu} + c_0''$ (as usual, we set $c_{n+\nu} = 0$ if $\nu + n \notin \mathbb{N}$). By G-Schrohe '01,

$$\mathrm{res}(A) = m \cdot C_{-1}(A,P_{1,+}).$$

Searching for a *canonical trace*, G-Schrohe '04 showed:

(i) $C_0(A,P_{1,+})$ is a quasi-trace, in the sense that

$C_0(A,P_{1,+}) - C_0(A,P_{2,+})$ and $C_0([A,A'],P_{1,+})$ are *local.*

(ii) The value of $C_0(A,P_{1,+})$ is a finite part integral

$$\int_X ⨍ \mathrm{tr}\, p(x,\xi)\, đ\xi dx + \int_{X'} ⨍ \mathrm{tr}(\mathrm{tr}_n\, g)(x',\xi')\, đ\xi' dx',$$

modulo local contributions.

But $C_0(A,P_{1,+})$ is rarely a canonical trace. Yes, if $\nu < -n$. Yes, if $\nu \notin \mathbb{Z}$, but then only G enters. When $\nu \in \mathbb{Z}$ and $P \neq 0$, parity does not help much, for *both dimensions* n *and* $n-1$

enter at the same time. Cf. the closed manifold conditions:

(3) n odd and A even-even, **or** (4) n even and A even-odd.

So, $C_0(A, P_{1,+})$ itself becomes the important object!

Trace defect formulas? (See slide 9 for closed manifolds.)

By a proof that completely avoids the issue of how the operators $(P_1^{-s})_+$ and $(\log P_1)_+$ really act, relying instead on resolvent formulations, we have managed to show (G '05):

Theorem. *Let $A = P_+ + G$, $A' = P'_+ + G'$ be given, with two auxiliary elliptic differential operators P_1 and P_2.*

One can construct ψdo's S and S' on X' in a specific way from the given operators such that

(a) $\quad C_0(A, P_{1,+}) - C_0(A, P_{2,+})$

$$= -\tfrac{1}{m} \operatorname{res}_X((P(\log P_1 - \log P_2))_+) - \tfrac{1}{m} \operatorname{res}_{X'}(S),$$

(b) $\quad C_0([A, A'], P_{1,+})$

$$= -\tfrac{1}{m} \operatorname{res}_X((P[P', \log P_1])_+) - \tfrac{1}{m} \operatorname{res}_{X'}(S').$$

V. Ingredients in the proofs.

P is assumed elliptic with $\mathbb{R}_-$ in the resolvent set.

$Q_\lambda = (P-\lambda)^{-1}$ defined in sector V around $\mathbb{R}_-$, symbol $q(x,\xi,\lambda) \sim \sum_{j\geq 0} q_{-m-j}(x,\xi,\lambda)$. Then

P^{-s} is a classical ψdo of order $-ms$ (Seeley '67), symbol $p^{(-s)}(x,\xi) \sim \sum_{j\geq 0} p^{(-s)}_{-ms-j}(x,\xi)$, where

$$p^{(-s)}_{-ms-j}(x,\xi) = \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} q_{-m-j}(x,\xi,\lambda)\, d\lambda;$$

$\mathcal{C}$ a closed curve in $\mathbb{C}\setminus\mathbb{R}_-$ encircling the eigenvalues of $p_m(x,\xi)$.

$\log P = \mathrm{Op}(m \log[\xi] + b(x,\xi)); \quad [\xi] = |\xi|$ for $|\xi| \geq 1$;

$b(x,\xi) \sim \sum_{j\geq 0} b_{-j}(x,\xi)$ classical of order 0,

$$b_{-j}(x,\xi) = \tfrac{i}{2\pi} \int_{\mathcal{C}} \log \lambda q_{-m-j}(x,\xi,\lambda)\, d\lambda \text{ for } j > 0.$$

Note that $\mathrm{res}(\log P) = \int_X \int_{|\xi|=1} \operatorname{tr} b_{-n}(x,\xi)\, đS(\xi) dx$

$= \int_X \int_{|\xi|=1} \frac{i}{2\pi} \int_{\mathcal{C}} \log\lambda \operatorname{tr} q_{-m-n}(x,\xi,\lambda)\, d\lambda đS(\xi) dx.$

Remarkable fact (Scott '04, partially known earlier):

$$C_0(I,P) = -\tfrac{1}{m}\operatorname{res}(\log P).$$

We can show it without using complex powers, by observing:

Lemma 1. *The strictly homogeneous symbol $q^h_{-m-n}(x,\xi,\lambda)$ is integrable at $\xi = 0$ and ∞, and*

$$C_0(I,P) = \textstyle\int_X c_n(x), \text{ with } c_n(x) = \int_{\mathbb{R}^n} q^h_{-m-n}(x,\xi,-1)\, đ\xi.$$

Lemma 2. *When $f(x,\xi,\lambda)$ is holomorphic in λ on a nbd. of $\overline{\mathbb{R}}_-$, with suitable bounds, then (with a curve $\mathcal{C}$ in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$)*

$$\tfrac{1}{2\pi i}\textstyle\int_{\mathcal{C}} \log\lambda f(x,\xi,\lambda)\,d\lambda = \int_{-\infty}^0 f(x,\xi,t)\,dt.$$

For, $\log\lambda$ gives a jump of $2\pi i$ at $\mathbb{R}_-$; the contributions from $\log|\lambda|$ cancel out.

Combine this with homogeneity, polar coordinates:

$$\begin{aligned}\textstyle\int_{\mathbb{R}^n} q^h_{-m-n}(x,\xi,-1)\,đ\xi &= \tfrac{1}{m}\textstyle\int_{|\eta|=1}\int_{-\infty}^0 q^h_{-m-n}(x,\eta,t)\,dt\,đS(\eta)\\ &= -\tfrac{1}{m}\textstyle\int_{|\eta|=1}\tfrac{i}{2\pi}\int_{\mathcal{C}}\log\lambda\, q^h_{-m-n}(x,\eta,\lambda)\,d\lambda\,đS(\eta)\\ &= -\tfrac{1}{m}\textstyle\int_{|\eta|=1} b_{-n}(x,\eta)\,đS(\eta),\end{aligned}$$

the inner integral in the residue of $\log P$!

The trace defect formulas in the closed manifold case can be proved by calculations where this type of argument is central; we never need to consider P^{-s}, and $\log P$ enters only in a very rudimentary way.

Finally, for the case with boundary, this argument is again central, but a lot of extra efforts are needed to master the contributions from the boundary.

Application e.g. to index formulas:

If $A = P_+ + G\colon C^\infty(X,E) \to C^\infty(X,F)$ is elliptic of order and class 0, and B is a parametrix, then with auxiliary elliptic operators P_1 in E and P_2 in F,

$$\begin{aligned}\operatorname{ind} A &= C_0(AB - I, P_{2,+}) - C_0(BA - I, P_{1,+})\\ &= C_0(AB, P_{2,+}) - C_0(BA, P_{1,+})\\ &\quad + \tfrac{1}{m}\operatorname{res}((\log P_2)_+) - \tfrac{1}{m}\operatorname{res}((\log P_1)_+).\end{aligned}$$

Here $C_0(AB, P_{2,+}) - C_0(BA, P_{1,+})$ is a res with ψdo part $\operatorname{res}((B\log P_2 A - BA\log P_1)_+)$.

References

[B] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[FGLS] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe, *The noncommutative residue for manifolds with boundary*, J. Funct. Anal. **142** (1996), 1–31.

[G] G. Grubb, *On the logarithm component in trace defect formulas*, Comm. Part. Diff. Equ. (2005), arXiv: math.AP/0411483, to appear.

[GSc1] G. Grubb and E. Schrohe, *Trace expansions and the noncommutative residue for manifolds with boundary*, J. Reine Angew. Math. **536** (2001), 167–207.

[GSc2] ———, *Traces and quasi-traces on the Boutet de Monvel algebra*, Ann. Inst. Fourier **54** (2004), 1641–1696.

[GS1] G. Grubb and R. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), 481–529.

[Gu] V. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. **102** (1985), 184–201.

[KV] M. Kontsevich and S. Vishik, *Geometry of determinants of elliptic operators*, Functional Analysis on the Eve of the 21'st Century (Rutgers Conference in honor of I. M. Gelfand 1993), Vol. I (S. Gindikin et al., eds.), Progr. Math. 131, Birkhäuser, Boston, 1995, pp. 173–197.

[MN] R. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary*, manuscript, arXiv: funct-an/9606005.

[O] K. Okikiolu, *The multiplicative anomaly for determinants of elliptic operators*, Duke Math. J. **79** (1995), 723–750.

[Sco] S. Scott, *The residue determinant*, arXiv: math.AP/0406268, to appear in Comm. Part. Diff. Equ.

[S] R. T. Seeley, *Complex powers of an elliptic operator*, AMS Proc. Symp. Pure Math. X, 1966, Amer. Math. Soc., Providence, 1967, pp. 288–307.

[W] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), 143-178.