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NOTES TO THE COURSE ON ORDINARY DIFFERENTIAL EQUATIONS

This is a supplement to the text in [BN], short for F. Brauer and J. Nohel: "The qualitative theory of differential equations", Dover 1989.

S1. Existence- and uniqueness theorems

Here are some comments to Section 1.6 in [BN], and also to Sections 2.1 and 2.3.

The formulation of the existence- and uniqueness theorems Theorem 1.1–3 is somewhat unclear, to the point of being tautological. Better insight can be gained if we speak of maximal solutions.

Definition S1. *Consider a solution* $\boldsymbol{\varphi}$ *of the equation*

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{S1.1}$$

in the open set $D \subset \mathbb{R}^{n+1}$*, with* $\boldsymbol{\varphi}$ *defined on an open interval* I*; i.e., for* $t \in I$*,*

$$(t, \boldsymbol{\varphi}(t)) \in D \text{ and } \boldsymbol{\varphi}'(t) = \mathbf{f}(t, \boldsymbol{\varphi}(t)). \tag{S1.2}$$

$\boldsymbol{\varphi}$ *is said to be* **maximal***, if it cannot be extended to a solution on a strictly larger open interval* $I' \supset I$*.*

It is the maximal solutions that are unique under the hypotheses in Theorem 1.1. A proof is given later in Chapter 3. It is shown there in Theorem 3.3 that for any $(t_0, \boldsymbol{\eta}) \in D$, there exists an open interval around t_0 where there exists a solution $\boldsymbol{\varphi}$ with

$$\boldsymbol{\varphi}(t_0) = \boldsymbol{\eta}. \tag{S1.3}$$

Uniqueness and maximality is dealt with in Sections 3.3 and 3.4.

The tautological statement in Theorem 1.1 "The solution $\boldsymbol{\varphi}$ exists on any interval I containing t_0 for which the points $(t, \boldsymbol{\varphi}(t))$, with $t \in I$, lie in D" is probably meant to indicate the fact that if $\mathbf{f}$ is bounded, then "a solution can be continued until its graph hits the boundary of D", or it can escape to infinity, cf. Theorem 3.6ff. It cannot stop at a point $(t', \boldsymbol{\eta}')$ inside D, because there is existence of solution in an open interval around t'.

The discussion of such statements can wait until we reach Chapter 3. For the time being, we make the following replacements:

Theorem S1. *(This replaces* [BN, Th. 1.1, 1.2, 1.3].*)*

1° *Let* $\mathbf{f}$ *be a vector function with* n *components defined on an open set* $D \subset \mathbb{R}^{n+1}$*, such that* $\mathbf{f}$ *and* $\partial \mathbf{f}/\partial y_i$*,* $i = 1, \dots n$*, are continuous on* D*. For any point* $(t_0, \boldsymbol{\eta}) \in D$ *there exists a unique maximal solution* $\boldsymbol{\varphi}$ *of* (S1.1) *satisfying* (S1.3)*.*

2° *Let* h *be a function on an open set* $D \subset \mathbb{R}^{n+1}$ *such that* h *and its first partial derivatives* $\partial h/\partial y_1, \dots, \partial h/\partial y_n$ *are continuous on* D*. For any point* $(t_0, \eta_1, \dots, \eta_n) \in D$*, there exists a unique maximal solution* φ *of the problem*

$$y^{(n)} = h(t, y, y', \dots, y^{(n-1)}), \tag{S1.4}$$

satisfying

$$(\varphi(t_0), \dots, \varphi^{(n-1)}(t_0)) = (\eta_1, \dots, \eta_n). \tag{S1.5}$$

The solution depends continuously on the data t_0 and $\boldsymbol{\eta}$, in a sense that is explained and proved in Section 3.5.

In Section 2.1, the proof of Theorem 2.1 and its corollary relies on the statement in Theorem 3.6, so we shall not do it completely it at this stage.

In connection with fundamental matrices in Section 2.3 it is worth noting the following fact, derived directly from Theorem 2.2. The solutions and matrices are here taken to be complex valued.

Theorem S2. *Let* $\boldsymbol{\varphi}_1(t), \dots, \boldsymbol{\varphi}_n(t)$ *be solutions (defined for* $t \in \mathbb{R}$*) of the linear system*

$$\mathbf{y}' = A(t)\mathbf{y}. \tag{S1.6}$$

The following statements (i)–(iii) *are equivalent:*

(i) *The vector functions* $\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n$ *span the vector space* V *of solutions (i.e., form a fundamental set of solutions).*

(ii) *At each* $t \in \mathbb{R}$*, the vectors* $\boldsymbol{\varphi}_1(t), \dots, \boldsymbol{\varphi}_n(t)$ *are linearly independent.*

(iii) *There is a* $t_0 \in \mathbb{R}$ *such that the vectors* $\boldsymbol{\varphi}_1(t_0), \dots, \boldsymbol{\varphi}_n(t_0)$ *are linearly independent.*

Proof. It is clear that (ii) $\Longrightarrow$ (iii). That (iii) $\Longrightarrow$ (i) was shown in the proof of Theorem 2.2, where it was shown that the complex vector space of solutions V has dimension n, and that a linearly independent set of n initial data $\boldsymbol{\sigma}_j \in \mathbb{C}^n$, $j = 1, \dots, n$, gave a linearly independent set of solutions $\boldsymbol{\varphi}_j$ satisfying $\boldsymbol{\varphi}_j(t_0) = \boldsymbol{\sigma}_j$, $j = 1, \dots, n$. In fact, the mapping $T_{t_0} \colon \mathbb{C}^n \to V$ that sends a set of n linearly independent vectors $\boldsymbol{\sigma}_j$, $j = 1, \dots, n$, over into the set of solutions $\boldsymbol{\varphi}_j$ satisfying $\boldsymbol{\varphi}_j(t_0) = \boldsymbol{\sigma}_j$, is a vector space isomorphism. Finally, (i) $\Longrightarrow$ (ii), since, at any point $t \in \mathbb{R}$, $T_t \colon \mathbb{C}^n \to V$ is a vector space isomorphism. □

Theorem 2.4 follows as an immediate corollary, without use of Abel's theorem. Note also:

Theorem S3. *If* $\boldsymbol{\varphi}$ *solves* (S1.6) *and is zero at a point* t_0 *(i.e.,* $\boldsymbol{\varphi}(t_0) = \mathbf{0}$*), then* $\boldsymbol{\varphi}$ *is zero everywhere.*

Proof. The function $\boldsymbol{\psi} \equiv \mathbf{0}$ solves (S1.6) and takes the value $\mathbf{0}$ at t_0; then by the uniqueness of maximal solutions, $\boldsymbol{\varphi} = \boldsymbol{\psi}$. □

This can also be seen as a consequence of the proof of Theorem S2, where T_{t_0} maps $\mathbf{0}$ to the function $\boldsymbol{\psi} \equiv \mathbf{0}$, since T_{t_0} is linear.

S2. Real solutions to first-order constant-coefficient linear systems

We here give some comments on the problem treated in Section 2.5 of [BN],

$$\mathbf{y}' = A\mathbf{y}, \tag{S2.1}$$

where A is a constant $(n \times n)$-matrix, and the solutions are sought as n-vector functions of $t \in \mathbb{R}$.

We know from the general linear theory that there exists a fundamental solution $\Phi(t)$ defined for $t \in \mathbb{R}$, and that the general solution can be expressed in the form $\Phi(t)\mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector. In the constant coefficient case, we have seen that a fundamental solution $\Phi(t)$ of (S2.1) can always be constructed as the matrix function

$$e^{At} = \sum_{j=0}^{\infty} \tfrac{1}{j!} A^j t^j. \tag{S2.2}$$

Then the solution taking the value $\mathbf{v}$ at $t = 0$ is

$$\mathbf{f}(t) = e^{At}\mathbf{v}. \tag{S2.3}$$

This holds both when we work with real and when we work with complex numbers. In general, when A is a complex $(n \times n)$-matrix, e^{At} is a complex matrix function. However, if A is real-valued, e^{At} will be so too, so we conclude that when A is real and $\mathbf{v}$ is a real n-vector, then the solution (S2.3) is real.

The detailed analysis of how e^{At} looks more precisely was carried out for complex matrices. It was used that the complex polynomial $p_A(\lambda) = \det(A - \lambda E)$ has n complex roots, counted with multiplicities. They can also be described as the set of *mutually different* roots $\{\lambda_1, \dots, \lambda_k\}$, with multiplicities $\{n_1, \dots, n_k\}$, respectively, so that $n_1 + \dots + n_k = n$. Now $\mathbb{C}^n$ can be decomposed in the direct sum of the generalized eigenspaces:

$$\mathbb{C}^n = X_1 \oplus X_2 \oplus \cdots \oplus X_k, \quad X_j = \{\mathbf{x} \mid (A - \lambda_j E)^{n_j}\mathbf{x} = 0\}; \tag{S2.4}$$

each X_j has dimension n_j and is mapped into itself by A. This was used to show that the general solution $\mathbf{f}(t) = e^{At}\mathbf{v}$ has the form

$$\mathbf{f}(t) = \sum_{j=1}^{k} e^{\lambda_j t}[E + \tfrac{1}{1!}(A - \lambda_j E)t + \cdots + \tfrac{1}{(n_j-1)!}(A - \lambda_j E)^{n_j-1}t^{n_j-1}]\mathbf{v}_j, \tag{S2.5}$$

when $\mathbf{v}$ is decomposed into $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$ according to (S2.4).

The description is simplified when A is diagonalizable. This means that all the X_j are genuine eigenspaces $X_j = \{\mathbf{x} \mid (A - \lambda_j E)\mathbf{x} = 0\}$, so there are no powers (≥ 1) of t in the formulas (S2.5) then.

But A is not always diagonalizable, so we do need the general representation in (S2.5). It can be made still more explicit by use of a reduction of A to Jordan canonical form, but the calculatory effort needed to find appropriate transformation matrices may not be worthwhile.

Now let A be given as a real matrix; $A = (a_{ij})_{i,j=1,\dots n}$ with real entries a_{ij}. Then one can ask how the real solutions look. If $\mathbf{f}(t)$ solves (S2.1), we find by taking the real and imaginary parts that

$$\operatorname{Re}\mathbf{f}'(t) = A\operatorname{Re}\mathbf{f}(t), \quad \operatorname{Im}\mathbf{f}'(t) = A\operatorname{Im}\mathbf{f}(t), \tag{S2.6}$$

so we get real solutions by taking real parts of expressions (S2.5).

Let us consider two simple cases.

1) If λ is a real eigenvalue of A, then there exists an associated real eigenvector $\mathbf{v} \neq 0$ (found by solving the real problem $(A - \lambda E)\mathbf{v} = 0$). Then $e^{\lambda t}\mathbf{v}$ is a real solution of (S2.1).

2) If $\lambda = \sigma + i\nu$ is a complex eigenvalue of A with imaginary part $\nu \neq 0$, and $\mathbf{v}$ is an associated (complex) eigenvector, then $\overline{\lambda}$ is also an eigenvalue of A, and it has $\overline{\mathbf{v}}$ as associated eigenvector; this is seen by complex conjugation of the equation $(A - \lambda E)\mathbf{v} = 0$. Moreover, since $\overline{\lambda} \neq \lambda$, $\mathbf{v}$ and $\overline{\mathbf{v}}$ are linearly independent. Then the solutions $\mathbf{f}(t) = e^{\lambda t}\mathbf{v}$ and $\overline{\mathbf{f}}(t) = e^{\overline{\lambda}t}\overline{\mathbf{v}}$ are linearly independent (since they take linearly independent values at $t = 0$). Now the following linear combinations of them:

$$\operatorname{Re}\mathbf{f}(t) = \tfrac{1}{2}(\mathbf{f}(t) + \overline{\mathbf{f}}(t)), \quad \operatorname{Im}\mathbf{f}(t) = \tfrac{1}{2i}(\mathbf{f}(t) - \overline{\mathbf{f}}(t)), \tag{S2.7}$$

are likewise solutions of (S2.1), and we get $\mathbf{f}(t)$ and $\overline{\mathbf{f}}(t)$ back as linear combinations of the latter. We conclude that the two functions in (S2.7) are *linearly independent real solutions* of (S2.1). When $\mathbf{v}$ is written $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ with real vectors $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{f}(t) = e^{\sigma t}(\cos\nu t + i\sin\nu t)(\mathbf{u} + i\mathbf{w})$, so these real solutions equal

$$\operatorname{Re}\mathbf{f}(t) = e^{\sigma t}(\cos\nu t\,\mathbf{u} - \sin\nu t\,\mathbf{w}), \quad \operatorname{Im}\mathbf{f}(t) = e^{\sigma t}(\cos\nu t\,\mathbf{w} + \sin\nu t\,\mathbf{u}). \tag{S2.8}$$

In these cases we get explicit formulas for some real solutions in a straightforward manner. If A is diagonalizable, one can express all real solutions in this way. If not, the situation is more complicated; one can use the Jordan canonical form, but with possibly *complex* transition matrices. We shall not try to include a discussion here, except for the case $n = 2$, which will be treated in the next section.

S3. The two-dimensional case

The following is a supplement to Section 2.8 in [BN]. Consider real (2×2)-matrices A, assuming $\det A \neq 0$. We shall show how to classify the matrices in the six categories on page 90 in [BN], which states that there exist *real* transformation matrices T making A is similar to one of the types:

(i) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\mu < \lambda < 0$ or $0 < \mu < \lambda$,

(ii) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, where $\lambda > 0$ or $\lambda < 0$,

(iii) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\mu < 0 < \lambda$,

(iv) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where $\lambda > 0$ or $\lambda < 0$,

(v) $\begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix}$, where $\sigma, \nu \neq 0$, $\sigma > 0$ or $\sigma < 0$,

(vi) $\begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$, where $\nu \neq 0$.

The characteristic polynomial is

$$p_A(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}); \tag{S3.1}$$

note that $a_{11} + a_{22}$ is the trace of A and $a_{11}a_{22} - a_{12}a_{21}$ is $\det A$, both real. There are three possibilities, according to whether the discriminant $D = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$ is > 0, $= 0$ or < 0:

(I) $p_A(\lambda)$ has two different real roots λ_1 and λ_2,
(II) $p_A(\lambda)$ has one real root λ_1 of multiplicity 2,
(III) $p_A(\lambda)$ has two complex roots $\lambda_1 = \sigma + i\nu$, $\lambda_2 = \sigma - i\nu$, with $\sigma, \nu \in \mathbb{R}$ and $\nu \neq 0$.

Case (I). In this case, there is a real eigenvector $\mathbf{v}_1$, $\mathbf{v}_2$, for each of the roots, and A is diagonalized to the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = T^{-1}AT, \tag{S3.2}$$

by use of the matrix T with $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns. This gives the cases (i) and (iii).

Case (II). (a) If $A - \lambda_1 E = 0$, then $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$, and we are in the case (ii); A is already diagonalized. All vectors in $\mathbb{R}^2 \setminus \{0\}$ are eigenvectors; any basis of $\mathbb{R}^2$ is a basis of eigenvectors.

(b) If $A - \lambda_1 E \neq 0$, it has rank 1 (the rank cannot be 2 since $\det(A - \lambda_1 E) = 0$). Then the eigenspace is one-dimensional, and there is a real eigenvector $\mathbf{v}$ and another real vector $\mathbf{w} \neq 0$, such that $\mathbf{w}$ is not an eigenvector, and $\mathbf{v}$ and $\mathbf{w}$ together span $\mathbb{R}^2$ (and $\mathbb{C}^2$). Now define $\mathbf{u}$ to be

$$\mathbf{u} = (A - \lambda_1 E)\mathbf{w}, \tag{S3.3}$$

it is $\neq 0$ since $\mathbf{w}$ is not an eigenvector, and we shall show that $\mathbf{u}$ is an eigenvector. It has a unique decomposition

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w},$$

and we have that

$$(A - \lambda_1 E)\mathbf{u} = (A - \lambda_1 E)(a\mathbf{v} + b\mathbf{w}) = b(A - \lambda_1 E)\mathbf{w} = b\mathbf{u},$$

since $\mathbf{v}$ is an eigenvector. This implies

$$A\mathbf{u} = (\lambda_1 + b)\mathbf{u},$$

so $\mathbf{u}$ is a eigenvector with eigenvalue $\lambda_1 + b$. Since λ_1 is the only eigenvalue, b must equal 0, so in fact $\mathbf{u}$ is an eigenvector for λ_1.

Now take T with $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ as columns; it is nonsingular. Then we find using (S3.3):

$$AT = A(\mathbf{u}, \mathbf{w}) = (\lambda_1 \mathbf{u}, \mathbf{u} + \lambda_1 \mathbf{w}) = T \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix},$$

which shows that this T reduces A to the form in (iv).

Remark. Note that since $0 = (A - \lambda_1 E)\mathbf{u} = (A - \lambda_1 E)^2\mathbf{w}$, we have for a general $\mathbf{x} \in \mathbb{C}^2$, written as $x = c_1\mathbf{v} + c_2\mathbf{w}$, that $(A - \lambda E)^2\mathbf{x} = 0$; this confirms that the whole space is the generalized eigenspace belonging to this eigenvalue.

Case (III). Let $\mathbf{v}$ be an eigenvector for $\lambda_1 = \sigma + i\nu$, then $\overline{\mathbf{v}}$ is an eigenvector for $\lambda_2 = \sigma - i\nu$. Recall that $\nu \neq 0$. We can write $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, where $\mathbf{u}$ and $\mathbf{w}$ are real; then $\overline{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$; and they must be different since the eigenvalues are so, so $\mathbf{w} \neq 0$. Note that

$$A(\mathbf{u} + i\mathbf{w}) = (\sigma + i\nu)(\mathbf{u} + i\mathbf{w}) = (\sigma\mathbf{u} - \nu\mathbf{w}) + i(\nu\mathbf{u} + \sigma\mathbf{w}),$$

which implies, by taking real and imaginary parts:

$$A\mathbf{u} = \sigma\mathbf{u} - \nu\mathbf{w}, \quad A\mathbf{w} = \nu\mathbf{u} + \sigma\mathbf{w}. \tag{S3.4}$$

The first equation implies that $\mathbf{u} \neq 0$, since $\nu\mathbf{w} \neq 0$.

We shall take $T = (\mathbf{u}, \mathbf{w}) = \begin{pmatrix} u_1 & w_1 \\ u_2 & w_2 \end{pmatrix}$. To check that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent, note that if $\mathbf{u} = c\mathbf{w}$ (c necessarily real), then the second equation in (S3.4) would give $A\mathbf{w} = (\nu c + \sigma)\mathbf{w}$, so that $\nu c + \sigma$ would be a real eigenvalue, but all eigenvalues are nonreal.

Now

$$\begin{aligned} AT = (A\mathbf{u}, A\mathbf{w}) = (\sigma\mathbf{u} - \nu\mathbf{w}, \nu\mathbf{u} + \sigma\mathbf{w}) &= \begin{pmatrix} \sigma u_1 - \nu w_1 & \nu u_1 + \sigma w_1 \\ \sigma u_2 - \nu w_2 & \nu u_2 + \sigma w_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & w_1 \\ u_2 & w_2 \end{pmatrix} \begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix} = T \begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix}. \end{aligned}$$

This gives case (v) when $\sigma \neq 0$ and case (vi) when $\sigma = 0$.

S4. On the proofs of existence- and uniqueness theorems

Here are some comments to Chapter 3 of [BN]. There is a list of misprints further below.

The statement top of page 114, that a solution cannot cross the lines with slope $\pm M$ follows easily from (3.3) by use of (3.7):

$$|\varphi(t) - y_0| = \left|\int_{t_0}^{t} f(s, \varphi(s))\, ds\right| \leq M|t - t_0|. \tag{S4.1}$$

The proofs on page 117 do not need the fine estimate of the remainder established in (3.14). For one thing, we know from analysis courses that when a sequence of continuous functions $\varphi_j(t)$ on an interval $[t_0 - \alpha, t_0 + \alpha]$ converges uniformly to a function $\varphi(t)$ (and that was what was obtained top of page 116):

$$\sup_{t \in [t_0 - \alpha, t_0 + \alpha]} |\varphi(t) - \varphi_j(t)| \to 0 \text{ for } j \to \infty, \tag{S4.2}$$

then the limit function $\varphi(t)$ is also continuous. And now (3.15) follows, using (3.7) again, from

$$\text{(S4.3)} \quad \Big| \int_{t_0}^{t} [f(s,\varphi(s)) - f(s,\varphi_j(s))]\, ds \Big| \le \Big| \int_{t_0}^{t} M|\varphi(s)-\varphi_j(s)|\, ds \Big| \le M\alpha \sup_{s\in[t_0-\alpha,t_0+\alpha]} |\varphi(s)-\varphi_j(s)| \to 0 \text{ for } j\to\infty,$$

in view of (S4.2). This convergence is also uniform in t.

However, the estimate (3.14) is interesting for the last statement on page 117.

Theorem 3.4 can be stated more sharply as follows:

Theorem S4.1. *Suppose that $\boldsymbol{f}$ and $\partial \boldsymbol{f}/\partial y_j$ $(j = 1,\dots,n)$ are continuous on the "box" $B = \{(t,\boldsymbol{y}) \mid |t-t_0| \le a, |\boldsymbol{y}-\boldsymbol{\eta}| \le b\}$.*

(i) *If $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are two solutions of*

$$\text{(S4.4)} \qquad \boldsymbol{y}' = \boldsymbol{f}(t,\boldsymbol{y}), \quad \boldsymbol{y}(t_0) = \boldsymbol{\eta},$$

defined on open intervals J_1 resp. J_2 and running in B, then $\boldsymbol{\varphi}_1(t) = \boldsymbol{\varphi}_2(t)$ for $t \in J_1 \cap J_2$, and the solutions extend to a solution on $J_1 \cup J_2$.

(ii) *There is one and only one maximal solution running in B.*

Proof. The first statement in (i) is proved in the book: Since J_1 and J_2 are open intervals containing t_0, so is $J = J_1 \cap J_2$. By use of the Grönwall inequality one finds that $\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2$ is 0 on J. Now let $J_1 \ne J_2$. Then the left endpoints or the right endpoints are not the same; take for example the case where $a_1 < a_2$, a_1 and a_2 being the left endpoints of J_1 resp. J_2. Then $\boldsymbol{\varphi}_2$ can be extended to a solution on $]a_1, t_0] \cup J_2$ by using $\boldsymbol{\varphi}_1$ on $]a_1, t_0]$. The other possible cases are treated similarly; in this way $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are extended to a (unique) solution on $J_1 \cup J_2$.

For (ii), we can now consider two maximal solutions $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ running in B, defined on open intervals J_1 resp. J_2. By (i), they agree on $J_1 \cap J_2$, and they both extend to $J_1 \cup J_2$. But then this interval must be equal to J_1 and J_2, for otherwise the maximality is contradicted. So there is *at most* one maximal solution.

That there *exists* a maximal solution running in B is assured as follows: There does exist a solution defined on an interval $]t_0-\alpha, t_0+\alpha[$, by Theorem 3.1. Consider all solutions running in B and defined on open intervals; their domains of definition are intervals $]c,d[$ with $t_0 - a \le c < t_0 < d \le t_0 + b$, and they differ only by the values of c and d. Let

$$\text{(S4.5)} \qquad c^* = \inf\{c\}, \quad d^* = \sup\{d\},$$

where c and d run through the endpoints entering in the definitions of the solutions. Clearly, $t_0 - a \le c^* < t_0 < d^* \le t_0 + a$. Now the solutions extend, as in (i), to a solution on $]c^*, d^*[$, since there are values c arbitrarily close to c^*, and values d arbitrarily close to d^*. The solution on $]c^*, d^*[$ is maximal (think of why). □

Concerning the proof of Theorem 3.5: The mere fact that φ_1 coincides with φ_2 at $t = t_0$ but $\varphi_2 > \varphi_1$ somewhere to the right of t_0 does not imply that the number t_1 defined in the book is the left endpoint of an open interval where $\varphi_1 > \varphi_2$. (It is correct that $\varphi_2 = \varphi_1$

to the left of t_1, but one might have $\varphi_2 > \varphi_1$ only on a disjoint union of infinitely many open intervals to the right of t_1, with lengths going to 0 when they approach t_1.) Instead we can proceed as follows: Assume that $t^* > t_0$ is a point ($< t_0 + \alpha_1$) where φ_1 and φ_2 differ; we can assume that $\varphi_2(t^*) > \varphi_1(t^*)$. Then in view of the continuity, $\varphi_2(t) > \varphi_1(t)$ holds on some open interval around t^*. Let

$$t_1 = \inf\{t_2 \mid \varphi_2(t) > \varphi_1(t) \text{ for } t \in]t_2, t^*[\,\}. \tag{S4.6}$$

Clearly, $t_1 \geq t_0$. We cannot have $\varphi_2(t_1) - \varphi_1(t_1) > 0$, for then $\varphi_2 - \varphi_1$ would also be positive in an open interval around t_1, contradicting its definition. We cannot have $\varphi_2(t_1) - \varphi_1(t_1) < 0$, for then since $\varphi_2(t^*) - \varphi_1(t^*) > 0$, there would be a point in $\bar{t} \in]t_1, t^*[$ where $\varphi_2(\bar{t}) - \varphi_1(\bar{t}) = 0$. Thus $\varphi_2(t_1) - \varphi_1(t_1) = 0$, while $\varphi_2 - \varphi_1$ is positive on $]t_1, t^*[$. Now one can reason as in the book with $t_1 + h = t^*$, to see that $\varphi_2 - \varphi_1 > 0$ on $]t_1, t^*[$ leads to a contradiction. Thus the assumption that $\varphi_2 - \varphi_1 > 0$ at some $t^* \in]t_0, t_0 + \alpha_1[$ cannot hold.

Theorem 3.6 and its corollaries (the text in [BN] Section 3.4 from page 132 on) will be replaced by the following more precise account, where the most important ideas have been taken from the textbook (in Swedish) of K. G. Andersson and L.-C. Böiers: Ordinära Differentialekvationer, Studentlitteratur, Lund 1992. It will be referred to as [AB].

Theorem S4.2. *Suppose that $\boldsymbol{f}$ and $\partial \boldsymbol{f}/\partial y_j$ ($j = 1, \dots, n$) are continuous for $(t, \boldsymbol{y})$ in an open set $D \subset \mathbb{R}^{n+1}$. Then there is for each $(t_0, \boldsymbol{\eta}) \in D$ a unique maximal solution $\boldsymbol{\varphi}(t)$ to* (S4.4) *running in D. Its domain is an interval $]c^*, d^*[$, where $-\infty \leq c^* < t_0$, $t_0 < d^* \leq \infty$.*

For any compact subset K of D, the curve $(t, \boldsymbol{\varphi}(t))$ leaves K when $t \to d^*$, and it leaves K when $t \to c^*$.

Proof. First let us establish the properties of a maximal solution through $(t_0, \boldsymbol{\eta})$. Consider all solutions through $(t_0, \boldsymbol{\eta})$. There exist some, since there is a closed "box" around $(t_0, \boldsymbol{\eta})$, where Theorem 3.1 applies. If $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are two solutions defined on open intervals J_1 resp. J_2 and running in D, let $J = J_1 \cap J_2$. If the solutions differ at some point in $t^* \in J$, say $t^* > t_0$, let t_1 be the infimum of the points $t > t_0$ where they differ. Then $\boldsymbol{\varphi}_1(t) = \boldsymbol{\varphi}_2(t)$ for $t \in [t_0, t_1]$, whereas $\boldsymbol{\varphi}_1(t) \neq \boldsymbol{\varphi}_2(t)$ at points $t > t_1$ arbitrarily close to t_1. Using Theorem S4.1 on a "box" around $(t_1, \boldsymbol{\varphi}_1(t_1))$, we see that the solutions must agree also on an interval to the right of t_1, contradicting the definition of t_1. This shows that $\boldsymbol{\varphi}_1(t) = \boldsymbol{\varphi}_2(t)$ for $t \in J_1 \cap J_2$. Now we can go on as in the proof of Theorem S4.1 (i) to see that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ extend to a unique solution on $J_1 \cup J_2$. Finally, we see in a similar way as in the proof of Theorem S4.1 (ii) that there exists a unique maximal solution running in D. The maximal solution $\boldsymbol{\varphi}$ is defined on an interval $J =]c^*, d^*[$, where c^* can possibly be $-\infty$, d^* possibly $+\infty$.

Let K be a compact subset of D; we shall show that there is a point $t' \in [t_0, d^*[$ such that $(t, \boldsymbol{\varphi}(t)) \notin K$ for $t \in]t', d^*[$. Assume the contrary: For any $t' \in [t_0, d^*[$, there is a point $t'' > t'$ such that $(t'', \boldsymbol{\varphi}(t'')) \in K$. Letting t' go through a sequence converging to d^*, we find a sequence of points t_j, where $(t_j, \boldsymbol{\varphi}(t_j)) \in K$ and $t_j \to d^*$ for $j \to \infty$. In view of the compactness, there is a subsequence $(t'_l, \boldsymbol{\varphi}(t'_l))$ that converges to a point $(d^*, \boldsymbol{\zeta})$ in K. Let $B = \{(t, \boldsymbol{y}) \mid |t - d^*| \leq a, |\boldsymbol{y} - \boldsymbol{\zeta}| \leq b\}$ be a "box" around the point, contained in D. Let

$$M = \sup_{(t,\boldsymbol{y}) \in B} |\boldsymbol{f}(t, \boldsymbol{y})|. \tag{S4.7}$$

For each point $(s, \boldsymbol{\xi})$ in the interior of B we can choose a box $B_{s,\boldsymbol{\xi}} = \{(t, \boldsymbol{y}) \mid |t - s| \leq a', |\boldsymbol{y} - \boldsymbol{\xi}| \leq b'\}$ around the point contained in B. Note that for $(s, \boldsymbol{\xi})$ in the "half-box"

$$B_{1/2} = \{(t, \boldsymbol{y}) \mid |t - d^*| \leq a/2, |\boldsymbol{y} - \boldsymbol{\zeta}| \leq b/2\},$$

we can take $a' = a/2$, $b' = b/2$, i.e.,

$$B_{s,\boldsymbol{\xi}} = \{(t, \boldsymbol{y}) \mid |t - s| \leq a/2, |\boldsymbol{y} - \boldsymbol{\xi}| \leq b/2\}.$$

Considering just the points $(s, \boldsymbol{\xi})$ in $B_{1/2}$, we have by Theorem 3.1 that there is a solution through $(s, \boldsymbol{\xi})$ which lives on the interval $[s - \alpha, s + \alpha]$ with

$$\alpha = \min\{a/2, b/(2M)\}. \tag{S4.8}$$

Note that α is independent of $(s, \boldsymbol{\xi})$! Now take t'_l so close to d^* that $d^* - t'_l < \alpha$ and $|\boldsymbol{\zeta} - \boldsymbol{\varphi}(t'_l)| < b/2$. Then an application with $(s, \boldsymbol{\xi}) = (t'_l, \boldsymbol{\varphi}(t'_l))$ shows the existence of a solution on the interval $[t'_l - \alpha, t'_l + \alpha]$ that contains d^* in its interior; this contradicts the maximality of $\boldsymbol{\varphi}$.

It is seen in a similar way that the solution leaves K for $t \to c^*$. □

With this theorem we have finally proved Theorem S1.1. Moreover, we can account for the behavior of maximal solutions:

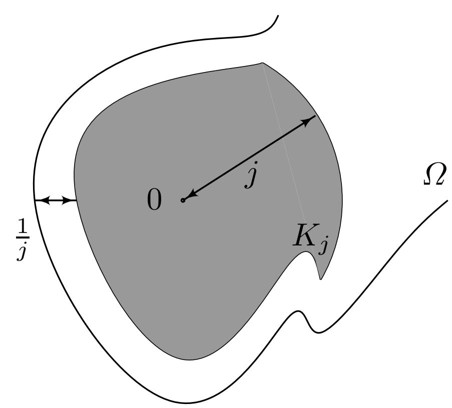
Corollary S4.3. *Under the hypotheses of Theorem* S4.2*, a maximal solution can behave in one of the following ways when $t \to d^*$, or $t \to c^*$:*

(a) *$(t, \boldsymbol{\varphi}(t))$ converges to a point of the boundary ∂D of D,*
(b) *$|t| + |\boldsymbol{\varphi}(t)| \to \infty$,*
(c) *d^* (resp. c^*) is finite, and $(t, \boldsymbol{\varphi}(t))$ approaches ∂D without having a limit point.*

Proof. We can fill out D with a sequence of compact sets K_j,

$$K_j = \{(t, \boldsymbol{y}) \in D \mid \operatorname{dist}((t, \boldsymbol{y}), \partial D) \geq 1/j \text{ and } \|(t, \boldsymbol{y})\| \leq j\}, \quad j = 1, 2, \ldots; \tag{S4.9}$$

here we take the euclidean distance and norm in order to have a nice geometric picture.



When $D \neq \emptyset$, the K_j will be $\neq \emptyset$ from a certain step. They have the convenient property that any compact subset of D is contained in one of them. (This follows from the covering property: The interiors K_j° of the K_j,

$$K_j^\circ = \{(t, \boldsymbol{y}) \in D \mid \operatorname{dist}((t, \boldsymbol{y}), \partial D) > 1/j \text{ and } \|(t, \boldsymbol{y})\| < j\}, \quad j = 1, 2, \ldots$$

form an open cover of D. For any compact $K \subset D$, a finite set of them covers K, hence so does the one with the largest index j in the set.)

Consider $t \to d^*$. The behavior under (a) is consistent with Theorem S4.2, since, for each K_j, $(t, \boldsymbol{\varphi}(t))$ will be outside K_j for t sufficiently close to d^* (which is finite in this case).

The behavior under (b) is consistent with Theorem S4.2 in a similar way.

Now let us show that (c) describes the remaining possibilities. When (b) does not hold, there exists a j_0 such that $\|(t, \boldsymbol{\varphi}(t))\| \le j_0$ for all $t \in [t_0, d^*[$. In particular, $d^* \le j_0$. Then for $j > j_0$, the maximal solution can only escape K_j by having $\operatorname{dist}((t, \boldsymbol{\varphi}(t)), \partial D) < 1/j$ for sufficiently large t. This is the behavior described in (c), when (a) does not hold. □

Example S4.4. The behavior (a) is found for example in linear constant-coefficient equations $\boldsymbol{y}' = A\boldsymbol{y}$, if we take as D a set $]a, b[\times \mathbb{R}^n$. The solutions defined for $t \in \mathbb{R}$ restrict to solutions on $]a, b[$ that are maximal with respect to D, and we find convergence to the boundary as under (a).

The behavior (b) is seen for example in linear constant-coefficient equations $\boldsymbol{y}' = A\boldsymbol{y}$, if we take $D = \mathbb{R} \times \mathbb{R}^n$; here t runs in the unbounded set $\mathbb{R}$. A case with unbounded $\varphi(t)$ and finite d^* is found e.g. in Example 1.3.1.

As an example of the behavior (c), consider the equation

$$y' = -\frac{1}{t^2}\cos\frac{1}{t}, \quad (t, y) \in D =]-\infty, 0[\times \mathbb{R}, \tag{S4.10}$$

it has the maximal solution $\varphi(t) = \sin\frac{1}{t}$ defined on $]-\infty, 0[$. Here $(t, \sin\frac{1}{t})$ approaches the boundary $\partial D = \{0\} \times \mathbb{R}$ without converging to a point, when $t \nearrow 0$. A still stranger behavior is found in $\varphi(t) = \frac{1}{t}\sin\frac{1}{t}$, whose graph approaches the boundary with points near *any point* $(0, \zeta)$, $\zeta \in \mathbb{R}$; it solves

$$y' = -\frac{1}{t^2}\sin\frac{1}{t} + \frac{1}{t^3}\cos\frac{1}{t}, \quad (t, y) \in D =]-\infty, 0[\times \mathbb{R}. \tag{S4.11}$$

To comment on the way the corollary applies, let us first note that case (b) is of course excluded when D is bounded.

By Lemma 3.3, case (c) is excluded when $\boldsymbol{f}$ is bounded on D.

Remark S4.5. Observe that for autonomous systems $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, the domain in $(t, \boldsymbol{y})$-space is $D = \mathbb{R} \times D'$, where D' is the domain of $\boldsymbol{f}$ in $\mathbb{R}^n$ (called D in Section 2.8), so here case (b) can certainly occur. It does so in the linear two-dimensional cases we have studied.

Here are some further observations:

Corollary S4.6.

(i) *Under the hypotheses of Theorem* S4.2, *let* $D = \mathbb{R}^{n+1}$. *Then the maximal solution* $\boldsymbol{\varphi}$ *behaves as in* (b). *If* $|\boldsymbol{\varphi}(t)|$ *is known to be bounded for* $t \to d^*$, d^* *must equal* ∞.

There is a similar result for c^*.

(ii) *Under the hypotheses of Theorem* S4.2, *let* $D =]a, b[\times \mathbb{R}^n$. *Assume that* $\boldsymbol{f}$ *is continuous at the boundary* $(\{a\} \times \mathbb{R}^n) \cup (\{b\} \times \mathbb{R}^n)$. *Then if the maximal solution* $\boldsymbol{\varphi}$ *of* (S4.4) *in D is known to be bounded for $t \to d^*$, d^* must equal b, and $(t, \boldsymbol{\varphi}(t))$ is convergent for $t \to b$, the continuous extension of $\boldsymbol{\varphi}$ solving* (S4.4) *on* $]c^*, b]$.

There is a similar result for c^*.

Proof. (i). If $D = \mathbb{R}^{n+1}$, there is no boundary, and only case (b) occurs. In fact, in the proof of Corollary S4.3, the K_j are the balls with radius j, so $|t| + |\boldsymbol{\varphi}(t)| \geq \|(t, \boldsymbol{\varphi}(t))\|$ goes to ∞ for $t \to \pm\infty$.

If $\boldsymbol{\varphi}(t)$ stays bounded for $t \to d^*$, it is $|t|$ that goes to ∞, so $d^* = \infty$.

For (ii), let A be a constant such that $|\boldsymbol{\varphi}(t)| \leq A$ for all $t \in [t_0, d^*[$. Replace D by $D_1 =]a, b[\times \{|\boldsymbol{y}| < A + 1\}$, then the solutions runs in D_1 for $t \geq t_0$. This D_1 is bounded, and $\boldsymbol{f}$ is continuous on $\overline{D}_1$, hence bounded on D_1, so only (a) can occur, in view of Lemma 3.3.

By hypothesis, $(t, \boldsymbol{\varphi}(t))$ has for $t \geq t_0$ a distance ≥ 1 (in the length-norm) to the part of ∂D_1 where $|\boldsymbol{y}| = A + 1$, so the solution can for $t \to d^*$ only come close to the "vertical" part of the boundary where $|\boldsymbol{y}| \leq A$, namely $\{b\} \times \{|\boldsymbol{y}| \leq A\}$. It does so, so d^* must equal b, and $\boldsymbol{\varphi}(t)$ converges to a value $\boldsymbol{\zeta}$ when $t \to b$. We extend $\boldsymbol{\varphi}$ to $]c^*, b]$ to be continuous, taking the value $\boldsymbol{\zeta}$ at b. Then the differential equation also holds at b (for the derivative from the left), since $\boldsymbol{f}$ is continuous at $(b, \boldsymbol{\zeta})$. □

Since we now have Theorem 1.1 available in the form of the present Theorem S4.2, we can also obtain a complete proof of Theorem 2.1.

Proof of Theorem 2.1: First use the proof details for Theorem 2.1 in the book to see that the maximal solution $\boldsymbol{\varphi}$ through $(t_0, \boldsymbol{\eta})$ is bounded on its domain. Next, use Corollary S4.6 (ii) to see that $\boldsymbol{\varphi}(t)$ is indeed defined on all of $]a, b[$. Moreover, the corollary tells us that $\boldsymbol{\varphi}$ extends to a continuous function on $[a, b]$, which satisfies the equation also at the endpoints. □

The corollaries to Theorem 2.1 likewise follow. For example, to find a solution on $]c, d[$, where $A(t)$ and $\boldsymbol{g}(t)$ are given to be continuous on $]c, d[$, we can apply Theorem 2.1 to any subinterval $[a, b]$ with $c < a < t_0 < b < d$, obtaining a solution on $[a, b]$. Since the definitions on various subintervals are consistent, we get a solution on $]c, d[$ by letting $a \searrow c$, $b \nearrow d$ (this includes cases where $c = -\infty$ or $d = \infty$).

Without further hypotheses on $A(t)$ and $\boldsymbol{g}(t)$ near the endpoints one cannot say more about the behavior of the solution there. (Example S4.5 shows that a behavior as in (c) is possible, e.g. by certain choices of $\boldsymbol{g}(t)$.)

We have hereby completed the background results for the development in Section 2.3 and onwards.

List of misprints and misformulations in [BN]

Page 14, line 8: Replace **Example 3** by **Example 4**.

Page 15, line 2: Replace (1.9) by (1.8).

Page 32, line 11: Replace $\alpha \leq tp \leq \beta$ by $\alpha \leq t \leq \beta$.

Page 52, line 6 from below: This is not the only solution, but it is the one that is $\mathbf{0}$ at t_0. (2.16) does not "become" (2.17), but it gives (2.17) if we require $\boldsymbol{v}(t_0) = \mathbf{0}$.

Page 62, Exercise 12: The eigenvectors do not always form a basis, see e.g. Example 5 on p. 65. The conclusion is true, however, if one adds the hypothesis that the a_{ii} are distinct.

Page 81, lines 4–7: Instead of allowing $\operatorname{Re}\lambda_j \leq \varrho$ for simple eigenvalues, one can allow this for those eigenvalues whose *algebraic multiplicity* (the multiplicity as a root in $p_A(\lambda) = \det(A - \lambda E)$, called n_j) equals the *geometric multiplicity* (the dimension of the associated eigenspace). For, in those cases the roots contribute to the fundamental solution with terms bounded by $Ce^{t \operatorname{Re}\lambda_j}$.

Page 82, line 3: Replace $\boldsymbol{y}$ by $\boldsymbol{\eta}$.

Page 83, line 12: When $a = \varrho$, one needs a value $> a$.

Page 86, line 9 from below: It should be a column vector $\begin{pmatrix} e^{-2t}\eta_1 \\ e^{-3t}\eta_2 \end{pmatrix}$.

Page 105, line 13: Replace 8 by 9.

Page 110, Exercise 3: The hint refers to a case where g does not depend on y, and cannot immediately be used.

Page 112, line 7: Replace t_1 by t.

Page 114, line 5 from below: Replace $m = 1$ by $m = 0$.

Page 116, line 6: Replace $e^{K\alpha}$ by $(e^{K\alpha} - 1)$.

Page 126, line 1: Replace $\leq$ by $| \leq$.

Page 135, line 6 from below: Replace "corresponds" by "exists". Include also the condition $|t_0 - \hat{t}_0| < \delta$. — The statement on the existence of δ can be sharpened: There exists $C > 0$ depending only on M, K and $\beta - \alpha$ such that for each $\varepsilon > 0$ one has with $\delta = \varepsilon / C$ that if $|t_0 - \hat{t}_0| < \delta$, $|t - \hat{t}| < \delta$ and $|\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}| < \delta$, then (3.28) holds. This uniformity is important since the theorem speaks only of two particular solutions.

Page 148, line 2: Replace $t \leq \infty$ by $t < \infty$.

Page 149, line 17: Replace $|y_0|$ by y_0.

Page 152, line 3: To get $\sigma = 0$, one can also allow the eigenvalues whose algebraic multiplicity equals the geometric multiplicity to have zero real part. (See the comment to page 81.)

Page 158, line 6: Replace "$|B(t)| < \frac{k}{\sigma}$" by "$|B(t)| \leq \eta$ for some $\eta < \frac{k}{\sigma}$".

Page 162, line 6 from below: Replace "for which $|\boldsymbol{\psi}(t)| \leq \alpha$" by "for which $|\boldsymbol{\psi}(s)| \leq \alpha$ for $t_0 \leq s \leq t$".

Page 192, line 17 from below: Replace "If $\boldsymbol{\eta}$ is any point of D that is not a critical point of (5.6)," by "If $\boldsymbol{\eta}$ is any point of D,". (The proof is the same, and the full statement is useful for the proof of Lemma 5.2.)

Page 205, line 1 from below: Replace $\boldsymbol{\varphi}(\mathbf{0})$ by $\boldsymbol{\varphi}(0)$.

Page 206, line 11 from below: Replace $0 < \delta < \mu$ by $0 < \delta < \varepsilon$.

Page 206, comments to the proof: We are considering a maximal solution through $(0, \boldsymbol{y}_0)$. For $t \geq 0$ it exists on an interval $[0, t_1[$, and part of the problem is to show that $t_1 = \infty$. This will be achieved by showing (for suitable $\boldsymbol{y}_0$) that there is an à priori bound on $\|\boldsymbol{\varphi}(t)\|$. Note that the inequalities in (5.22) and (5.23) can only be used for the values t such that $\|\boldsymbol{\varphi}(s)\| \leq r$ for $s \in [0, t]$. In line 11 from below, $\delta \in]0, \varepsilon[$ is chosen so small that $V(\boldsymbol{y}_0) < \mu$ for $\|\boldsymbol{y}_0\| \leq \delta$. If the orbit of the solution $\boldsymbol{\varphi}$ starting at such a $\boldsymbol{y}_0$ reaches S, let $\bar{t}$ be the first point where this happens (so that for $0 \leq t < \bar{t}$, $\|\boldsymbol{\varphi}(t)\| < \varepsilon$. Then since (5.23) is valid for $t \in [0, \bar{t}]$, we get the contradiction $V(\boldsymbol{\varphi}(\bar{t})) \leq V(\boldsymbol{y}_0) < \mu \leq V(\boldsymbol{\varphi}(\bar{t}))$. Hence there is a bound: $\|\boldsymbol{\varphi}(t)\| < \varepsilon$ for all $t \in [0, t_1[$. Now Corollary S4.3 in the notes can be used to see that $t_1 = \infty$ (check it yourself!). Thus we have found that the solutions starting at points $\boldsymbol{y}_0 \in B(\mathbf{0}, \delta)$ exist for $t \in [0, \infty[$ and have their orbits in the ball $B(\mathbf{0}, \varepsilon)$.

Page 207, lines 2 and 3, and line 4 from below: This δ should be labeled δ_0, to distinguish it from the other δ used in the middle of the proof.

Page 207, lines 4-5 from below, proof that $V(\boldsymbol{\varphi}(t)) \to 0$ implies $\boldsymbol{\varphi}(t) \to \mathbf{0}$: If $\boldsymbol{\varphi}(t)$ does not converge to $\mathbf{0}$ for $t \to \infty$, then there is an $\varepsilon_0 > 0$ and a sequence $t_j \to \infty$ such that $\|\boldsymbol{\varphi}(t_j)\| \geq \varepsilon_0$ for all j. Now V, considered on $\{\varepsilon_0 \leq \|\boldsymbol{y}\| \leq r\}$, has a positive lower bound a there. But this contradicts that $V(\boldsymbol{\varphi}(t_j)) \to 0$ for $j \to \infty$.

Page 282, line 6 from below: The indexation by j is ill chosen, since j is already used as an index on the different eigenvalues, and the matrices of this particular form are blocks in the matrix C associated with λ_j. Moreover, in the lower right corner of the matrix, λ^j should be λ_j.

Extra exercises

The exercises 8–10 were used for the 3 hour in-class test November 2007. The exercises 11–13 were used for the 3 hour in-class test January 2008 (reexamination). In in-class tests it is allowed to bring written material of all kinds (books and printed papers, personal notes etc.). Electronic equipment is not allowed. Answers can be formulated in Danish or English, as preferred.

Exercise E1. Consider the n'th order equation from Exercise 2.3.21, with constant coefficients and $a_0 = 1$. Show that for the corresponding first order system $\boldsymbol{y}' = A\boldsymbol{y}$, $\det(A - zE)$ is $(-1)^n$ times the polynomial you get by replacing $y^{(k)}$ by z^k (for each $k = 0, \dots, n$) in the left-hand side of the equation.

Exercise E2. Find the real eigenvalues and eigenfunctions of the following problem:

$$(e^{2x}y')' + \lambda e^{2x}y = 0,$$
$$y(0) = 0, \quad y(\pi) = 0.$$

(*Hint:* Set $y = e^{-x}u$.)

Exercise E3. Consider the differential equation

(a)
$$\boldsymbol{y}' = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \boldsymbol{y}.$$

Find the constants ϱ for which every solution $\boldsymbol{\varphi}(t)$ satisfies an inequality

(b)
$$|\boldsymbol{\varphi}(t)| \le Ce^{\varrho t} \text{ for } t \ge 0$$

(with C depending on the solution).

Same question for the solutions of

(c)
$$\boldsymbol{y}' = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} e^{-2t} \\ e^{-3t} \\ 0 \end{pmatrix}.$$

Exercise E4. Find a fundamental matrix for the system

$$\boldsymbol{y}' = \begin{pmatrix} -2 & 0 \\ 4 & -4 \end{pmatrix} \boldsymbol{y}.$$

Sketch the phase portrait and determine whether the origin is a node, saddle point, spiral point or center. Is it an attractor?

In the sketch you should indicate both the vector field and some typical orbits of solutions. Here you can either use Maple, or work it out by explicit calculations.

Exercise E5. Consider the equation

$$y' = y^2 - y, \text{ for } (t, y) \in D = \mathbb{R}^2. \tag{E5.1}$$

(It is a simple case of the type $y' = c(y-a)(y-b)$, which is important in Economics theory.)

(a) Show that the functions $\varphi_0(t) \equiv 0$ and $\varphi_1(t) \equiv 1$ are solutions, defined on $\mathbb{R}$.

(b) Show (by separation of variables) that all other solutions are of the form

$$\varphi(t) = \frac{1}{1 - Ce^t}, \text{ with } C \in \mathbb{R} \setminus \{0\}, \tag{E5.2}$$

and find the domains (the intervals of definition) for the maximal solutions of this form.

(c) Show that the maximal solutions with $C < 0$ take values in $]0, 1[$ and that the maximal solutions with $C > 0$ take values either in $]1, \infty[$ or in $]-\infty, 0[$. Make a drawing of the graphs, including one of each type. (You are welcome to use Maple.)

(d) For each of the solutions φ_0 and φ_1 defined in **(a)**, find out whether it is stable, asymptotically stable, globally asymptotically stable (for $t \to \infty$).

Exercise E6. Consider the system of equations

$$\begin{aligned} y_1' &= -y_1 + y_2, \\ y_2' &= y_2^2 - y_2, \end{aligned} \tag{E6.1}$$

for $(t, y_1, y_2) \in D = \mathbb{R}^3$.

(a) Find the maximal solutions (you can use results from Exercise E5).

(b) Consider the solutions $\boldsymbol{\varphi}_0(t) = (e^{-t}, 0)$ and $\boldsymbol{\varphi}_1(t) = (e^{-t} + 1, 1)$. Find out whether they are stable, asymptotically stable, globally asymptotically stable (for $t \to \infty$).

Exercise E7. Consider the system (E6.1) from Exercise E6; note that it is autonomous.

(a) Show that $\boldsymbol{\psi}_0(t) \equiv (0, 0)$ is a solution to (E6.1) defined for $t \in \mathbb{R}$.

(b) Show that for the linear system

$$\begin{aligned} y_1' &= -y_1 + y_2, \\ y_2' &= -y_2, \end{aligned} \tag{E7.1}$$

the point $(0, 0)$ is an attractor.

(c) Can we use Theorem 4.2 or Theorem 4.3 to show that the solution $\boldsymbol{\psi}_0$ defined in **(a)** is asymptotically stable, as a solution of (E6.1)?

Exercise E8. Consider the autonomous differential equation

$$y' = \tfrac{1}{2}(y^2 - 1), \tag{E8.1}$$

for $(t, y) \in D = \mathbb{R} \times D'$, where $D' = \mathbb{R}$ (note that D' is one-dimensional, being a line).

(a) Find the two critical points and define the corresponding equilibrium solutions.

(b) Find all other maximal solutions, in particular their domains (intervals of definition). (*Hint:* One can use separation of variables.)

(c) Find out, for each of the two critical points, whether the equilibrium solution is stable (Def. 4.1), asymptotically stable (Def. 4.2), or globally asymptotically stable (as defined on page 149 in the textbook).

Exercise E9. Consider the differential equation

$$\boldsymbol{y}' = 2tA\boldsymbol{y}, \quad (t, \boldsymbol{y}) \in D = \mathbb{R}^{n+1}, \tag{E9.1}$$

where A is a real $(n \times n)$-matrix.

(a) Which type of equation is (E9.1)? Can one prove without solving the equation that all maximal solutions have $\mathbb{R}$ as domain (interval of definition)?

(b) How is the structure of the space formed of the full set of maximal solutions?

(c) Show that $\Phi(t) = e^{t^2 A}$ is a fundamental matrix for (E9.1), defined for $t \in \mathbb{R}$.

(d) Calculate $e^{t^2 A}$ in the cases (with $n = 2$):

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{E9.2}$$

Exercise E10. Consider the third order differential equation

$$y''' + 3y'' - 4y = 0 \text{ on } \mathbb{R}. \tag{E10.1}$$

(a) Denoting $y = y_1$, $y' = y_2$, $y'' = y_3$, write (E10.1) as a first-order equation for the column vector $\boldsymbol{y} = (y_1, y_2, y_3)$:

$$\boldsymbol{y}' = A\boldsymbol{y}. \tag{E10.2}$$

(b) The eigenvalues of A are $\lambda_1 = 1$ (simple) and $\lambda_2 = -2$ (double). Find the eigenspace X_1 for λ_1 and the generalized eigenspace X_2 for λ_2 (the latter as a span of two linearly independent vectors). Is X_2 an eigenspace for A?

(c) For a general vector $\boldsymbol{v} \in \mathbb{R}^3$, decomposed as $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with $\boldsymbol{v}_1 \in X_1$, $\boldsymbol{v}_2 \in X_2$, write down the solution $\boldsymbol{\varphi}(t)$ of (E10.2) satisfying the condition

$$\boldsymbol{\varphi}(0) = \boldsymbol{v}.$$

(d) Find the solution of (E10.1) with $y(0) = y'(0) = y''(0) = 1$.

Exercise E11. Consider the system of differential equations

$$y_1' = y_2, \quad y_2' = y_2^2. \tag{E11.1}$$

(a) Show by reference to appropriate theorems that there for any $t_0 \in \mathbb{R}$, $(\eta_1, \eta_2) \in \mathbb{R}^2$, exists a unique maximal solution $(\varphi_1(t), \varphi_2(t))$ such that $(\varphi_1(t_0), \varphi_2(t_0)) = (\eta_1, \eta_2)$.

(b) Find all maximal solutions of (E11.1), including the description of their domain of definition.
(*Hint.* Begin with the second equation.)

(c) What is the solution through $(0, 0)$?

(d) Is the solution in (c) stable?

Exercise E12. Consider the third order differential equation

$$y''' - y'' - y' + y = 0 \text{ on } \mathbb{R}. \tag{E12.1}$$

(a) Denoting $y = y_1$, $y' = y_2$, $y'' = y_3$, write (E12.1) as a first-order equation for the column vector $\boldsymbol{y} = (y_1, y_2, y_3)$:

$$\boldsymbol{y}' = A\boldsymbol{y}. \tag{E12.2}$$

(b) You are informed that A has the eigenvalue 1. Find the full set of eigenvalues and their multiplicities.

(c) Find the general solution of (E12.1).

(d) Find the solution of (E12.1) with $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.

Exercise E13. Consider the autonomous differential equation on $\mathbb{R}^2$:

$$\boldsymbol{y}' = A\boldsymbol{y}, \tag{E13.1}$$

where $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and A is a 2×2-matrix.

(a) For each of the three following choices of A, find out what kind of point, **0** is (node, saddle point, spiral point or center), and whether it is an attractor.

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix}. \tag{E13.2}$$

(b) Give a rough sketch of the phase portrait, in each case. (Do not spend much time on this.)