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The local and global parts of the basic zeta coefficient for operators on manifolds with boundary

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Abstract For operators on a compact manifold X with boundary ∂X, the basic zeta coefficient $C_0(B, P_{1,T})$ is the regular value at $s = 0$ of the zeta function $\operatorname{Tr}(BP_{1,T}^{-s})$, where $B = P_+ + G$ is a pseudodifferential boundary operator (in the Boutet de Monvel calculus)—for example the solution operator of a classical elliptic problem—and $P_{1,T}$ is a realization of an elliptic differential operator P_1, having a ray free of eigenvalues. Relative formulas (e.g., for the difference between the constants with two different choices of $P_{1,T}$) have been known for some time and are local. We here determine $C_0(B, P_{1,T})$ itself (with even-order P_1), showing how it is put together of local residue-type integrals (generalizing the noncommutative residues of Wodzicki, Guillemin, Fedosov–Golse–Leichtnam–Schrohe) and global canonical trace-type integrals (generalizing the canonical trace of Kontsevich and Vishik, formed of Hadamard finite parts). Our formula generalizes a formula shown recently by Paycha and Scott for manifolds without boundary. It leads in particular to new definitions of noncommutative residues of expressions involving $\log P_{1,T}$. Since the complex powers of $P_{1,T}$ lie far outside the Boutet de Monvel calculus, the standard consideration of holomorphic families is not really useful here; instead we have developed a resolvent parametric method, where results from our calculus of parameter-dependent boundary operators can be used.

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1 Introduction

The value of the zeta function at $s = 0$ plays an important role in the analysis of geometric invariants of operators on manifolds. For the zeta function $\zeta(P_1, s) = \operatorname{Tr} P_1^{-s}$ (extended meromorphically to $\mathbb{C}$) defined from a classical elliptic pseudodifferential operator (ψdo) P_1 on a closed manifold X, having a ray free of eigenvalues, the value at $s = 0$ is a fundamental ingredient in index formulas. For the generalized zeta function $\zeta(A, P_1, s) = \operatorname{Tr}(AP_1^{-s})$, there is a pole at $s = 0$ and the regular value behind it serves as a "regularized trace" or "weighted trace" (cf. e.g., [2,3,18,20]); it is likewise important in index formulas.

Much is known for the case of closed manifolds: The residue of $\zeta(A, P_1, s)$ at 0 is proportional to Wodzicki's noncommutative residue of A ([24], see also Guillemin [15]). The regular value at 0 (which we call *the basic zeta value*) equals the Kontsevich–Vishik [16] canonical trace in special cases, and in general there are defect formulas for it (formulas relating two different choices of the auxiliary operator P_1, and formulas where A is a commutator), in terms of noncommutative residues of related expressions involving $\log P_1$. The basic zeta coefficient itself has recently been shown by Paycha and Scott [22] to satisfy a formula with elements of canonical trace-type integrals (finite-part integrals in the sense of Hadamard) defined from A, as well as noncommutative residue-type integrals defined from $A \log P_1$. The finite-part integral contributions are *global*, in the sense that they depend on the full operator; the residue-type contributions are *local*, in the sense that they depend only on the strictly homogeneous symbols down to a certain order.

For a compact manifold X with boundary $\partial X = X'$, one can investigate the analogous questions. P_1 is here replaced by a realization $P_{1,T}$ of an elliptic differential operator P_1 provided with a boundary condition $Tu = 0$ at X' such that the resolvent $(P_{1,T} - \lambda)^{-1}$ exists on a ray in $\mathbb{C}$, say $\mathbb{R}_-$. A suitable framework for these operators is the Boutet de Monvel calculus [1,5], which contains the direct operators as well as their inverses and is closed under composition of the various operator types (acting between bundles over X and X'). Now A is replaced by an operator B in the Boutet de Monvel calculus, typically of the form $B = P_+ + G$, where P is a ψdo on a larger manifold $\widetilde{X} \supset X$, truncated to X, and G is a so-called singular Green operator (s.g.o.). For example, B can be the operator Δ_D^{-1} solving the Dirichlet problem for a strongly elliptic differential operator Δ.

The Boutet de Monvel calculus (the calculus of pseudodifferential boundary operators, ψdbo's) is a narrow calculus in the sense that it has just what it takes to include elliptic differential boundary problems and their solution operators in an "algebra". It requires the ψdo component to satisfy the *transmission condition* at the boundary (assuring that $C^\infty(X)$ is mapped into $C^\infty(X)$), and is limited to ψdo's of integer order. There exist other calculi for manifolds with boundary, such as e.g., the b- or c-calculi of Melrose [19,20], but these calculi involve a degeneracy of the operators at the boundary, and do not contain Δ_D^{-1}.

One could say that the Boutet de Monvel calculus is more "noncommutative" than the other calculi, because it contains operators that do *not* have the property (typical of pseudodifferential calculi) that the commutator of two scalar operators A_1 and A_2 of orders m_1 resp. m_2 is of lower order than $m_1 + m_2$.

The celebrated method for determining spectral invariants such as noncommutative residues and weighted traces, in the case of operators on closed manifolds, or operators with a degeneracy at the boundary, has been to study *holomorphic families*, i.e., families of operators depending holomorphically on their order $z \in \mathbb{C}$ [2,15,16,20,22,24]. This does not work in the Boutet de Monvel calculus, simply because the transmission condition in its strict form allows integer orders only. Instead, we have had to establish an alternative method: *the resolvent parametric method*.

The most frequently used holomorphic family is the family of powers; in the present case it would be $BP_{1,T}^{-s}$. It can be defined by Cauchy integrals from the resolvent parametric family $B(P_{1,T}-\lambda)^{-1}$, and the invariants related to the holomorphic family $BP_{1,T}^{-s}$ match certain invariants associated with the family $B(P_{1,T}-\lambda)^{-1}$. In our method, we can bypass the detailed analysis of the family $P_{1,T}^{-s}$ and carry out all the work on $B(P_{1,T}-\lambda)^{-1}$ and its iterates $B(P_{1,T}-\lambda)^{-N}$; the only other operator we do have to involve is $\log P_{1,T}$. A detailed study of this operator is worked out in [10], joint with Gaarde. (In the holomorphic calculi, the log-operator comes into the picture when z-differentiation is performed; the reasons for its appearance in the resolvent considerations are more subtle, connected with homogeneity properties of symbols).

Previous results on noncommutative residues and zeta values for pseudodifferential boundary operators are as follows: The noncommutative residue $\operatorname{res}(B)$ was introduced by Fedosov et al. [4] as a sum of integrals of symbol terms of order "minus the dimension". The zeta function $\zeta(B, P_{1,T}, s)$ was defined as the meromorphic extension of $\operatorname{Tr}(BP_{1,T}^{-s})$ to $\mathbb{C}$ by Grubb and Schrohe [11] (under restrictive hypotheses on $P_{1,T}$), and the residue at $s = 0$ identified with $\operatorname{res} B$. "Weighted trace" properties (defect formulas) for the regular value at zero (with $P_{1,T}^{-s}$ replaced by its ψdo part) were established in [8,12], including the fact that the value *modulo local terms* is a finite-part integral defined from B.

In the present paper we shall derive an explicit formula for the regular value $C_0(B, P_{1,T})$ itself, with ingredients of the form of finite-part integrals (canonical trace-type terms) as well as residue-type integrals involving $\log P_{1,T}$. The formula generalizes that of [22], but has several more terms due to the presence of the boundary.

The study leads to the introduction of a number of new noncommutative residue formulas, generalizing those of [4,15,24]. In particular, we show that the new residue definitions have a certain traciality, vanishing on (suitable) commutators.

Plan of the paper.
Section 2 explains the problem in more detail, describing the asymptotic expansion of $\operatorname{Tr} B(P_{1,T}-\lambda)^{-N}$, and the decomposition of the relevant coefficient in five terms (where $B = P_+ + G$ and $(P_{1,T}-\lambda)^{-1} = Q_{\lambda,+} + G_\lambda$)

$$C_0(B, P_{1,T}) = l_0((PQ_\lambda)_+) - l_0(L(P, Q_\lambda)) + l_0(GQ_{\lambda,+}) + l_0(P_+G_\lambda) + l_0(GG_\lambda). \tag{1.1}$$

Section 3 establishes two general theorems describing the "positive regularity" method initiated in [8] (based on [5]), and applies them to complete the proofs of the needed asymptotic trace expansions. In Sect. 4, the nonlocal term $l_0((PQ_\lambda)_+)$ is found by a method for closed manifolds originating from [9], and the local term $l_0(L(P, Q_\lambda))$ is determined by use of Sect. 3 combined with a precise analysis. Section 5 treats the difficult nonlocal term $l_0(GQ_{\lambda,+})$, for which a new method involving Laguerre expansions is introduced. Section 6 shows the residual nature of $l_0(P_+G_\lambda)$ and $l_0(GG_\lambda)$. In Sect. 7, we collect the results in a theorem exhibiting the ingredients of canonical trace-type and residue-type, extended from the case where P_1 is of order 2 to general even-order cases. Here we moreover establish some new defect formulas, relating two different choices of P_1, or two different choices of branch cut of the logarithm, and show the vanishing of the residues on certain commutators.

2 Presentation of the problem and notation

Consider a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$, and a hermitian C^∞ M-dimensional vector bundle E over X. Let $B = P_+ + G$ be an operator of order σ belonging to the calculus of Boutet de Monvel [1], acting on sections of E. Here P is a classical pseudodifferential operator, given on a larger boundaryless n-dimensional manifold $\widetilde{X}$ in which X is smoothly imbedded, and acting in a bundle $\widetilde{E}$ extending E. P satisfies the transmission condition at ∂X, assuring that the truncation $P_+ = r^+Pe^+$ preserves $C^\infty(X)$ (here r^+ restricts to X° and e^+ extends by zero on $\widetilde{X}\setminus X^\circ$). G is a singular Green operator (s.g.o.) of class 0 with polyhomogeneous symbol. (More details on the calculus can be found e.g., in Boutet de Monvel [1] and Grubb [5]). When $P \neq 0$, we must assume $\sigma \in \mathbb{Z}$ because of the requirements of the transmission condition; when $P = 0$, all $\sigma \in \mathbb{R}$ are allowed.

Along with B we consider an auxiliary elliptic differential system $\{P_1, T\}$ where P_1 is an elliptic differential operator of order $m > 0$ acting in $\widetilde{E}$, and T is a differential trace operator, defining the realization $P_{1,T}$ in $L_2(X, E)$; its domain $D(P_{1,T})$ consists of the sections u in the Sobolev space $H^m(X, E)$ with $Tu = 0$. Here we assume that, in local coordinates, the principal symbol $p_1^0(x, \xi)$ has no eigenvalues on $\overline{\mathbb{R}}_-$, and the principal boundary symbol operator $\{p_1^0(x', 0, \xi', D_n) - \lambda,\ t^0(x', \xi', D_n)\}$ is bijective for $\lambda \in \overline{\mathbb{R}}_-$. Then $(p_1^0(x, \xi) - \lambda)^{-1}$ and the inverse of $\{p_1^0(x', 0, \xi', D_n) - \lambda,\ t^0(x', \xi', D_n)\}$ are defined for λ in a sector V around $\mathbb{R}_-$, for all x, all ξ' with $|\xi'| + |\lambda| \neq 0$. It can be assumed that $\widetilde{X}$ is compact (cf. e.g., [6, Proof of Theorem 7.4]). The resolvent is

$$\begin{aligned} R_\lambda &= (P_{1,T} - \lambda)^{-1} = Q_{\lambda,+} + G_\lambda, \text{ where} \\ Q_\lambda &= (P_1 - \lambda)^{-1} \text{ on } \widetilde{X}. \end{aligned} \tag{2.1}$$

These operators are defined except for λ in a discrete subset of $\mathbb{C}$; in particular they exist for large λ in the sector V. We can assume that no eigenvalues lie on $\mathbb{R}_-$ (by a rotation if necessary), so that $\mathbb{R}_-$ is a so-called spectral cut.

The composed operator BR_λ is trace-class when $m > \sigma + n$, and we are interested in the expansion of its trace in powers of λ (with logarithmic factors). If one does not want to assume that P_1 has a high order, one can instead work with Nth powers of the resolvent. Here we write

$$R_\lambda^N = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} R_\lambda = Q_{\lambda,+}^N + G_\lambda^{(N)}, \text{ with}$$
$$Q_{\lambda,+}^N = (Q_\lambda^N)_+ = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} Q_{\lambda,+}, \quad G_\lambda^{(N)} = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} G_\lambda. \tag{2.2}$$

For certain calculations, it is convenient to take P_1 of order $m = 2$, considering $\mathrm{Tr}(BR_\lambda^N)$ for N so large that BR_λ^N is trace-class, namely $N > (\sigma + n)/2$.

The point of departure for the analysis of Laurent coefficients of $\zeta(B, P_{1,T}, s)$ is the existence of trace expansions for $\lambda \to \infty$ on rays in V, with $\delta > 0$:

$$\mathrm{Tr}(BR_\lambda^N) = \sum_{0 \le j < n+\sigma} a_j^{(N)} (-\lambda)^{\frac{n+\sigma-j}{m} - N}$$
$$+ (a_0^{(N)\prime} \log(-\lambda) + a_0^{(N)\prime\prime})(-\lambda)^{-N} + O(\lambda^{-N-\delta}), \tag{2.3}$$

when $N > (\sigma + n)/m$. The expansion (2.3) was established by Grubb and Schrohe in [11] when $m = 2$, P_1 is principally scalar near X' and T defines the Dirichlet condition. (In fact, a full expansion with powers of $-\lambda$ going to $-\infty$ was shown there, but we shall not need the lower order terms in the present paper.) We shall show that it holds for general second-order realizations $P_{1,T}$ with a spectral cut (Corollary 3.7). For $m > n + \sigma$, the expansion with $N = 1$ was partially established in [8], namely with R_λ replaced by its ψdo part $Q_{\lambda,+}$:

$$\mathrm{Tr}(BQ_{\lambda,+}) = \sum_{0 \le j < n+\sigma} c_j (-\lambda)^{\frac{n+\sigma-j}{m} - 1}$$
$$+ (c_0' \log(-\lambda) + c_0'')(-\lambda)^{-1} + O(\lambda^{-1-\delta}), \tag{2.4}$$

and we shall show the supplementing result for the s.g.o.-part further below in Sect. 3 (Corollary 3.5). The expansion (2.4) (as well as expansions with $Q_{\lambda,+}$ replaced by $(Q_\lambda^N)_+$), has an interest in itself.

The coefficients of $(-\lambda)^{-N}$ are particularly interesting. In Eq. (2.3), one has for all $N > (\sigma + n)/m$:

$$a_0^{(N)\prime} = a_0', \quad a_0^{(N)\prime\prime} = a_0'' - \alpha_N a_0', \text{ with } \alpha_N = \textstyle\sum_{1 \le j < N} \frac{1}{j}, \tag{2.5}$$

where a_0' and a_0'' are constants independent of N. A brief explanation of why α_N enters is that it comes from derivatives of the log-term when N is increased (cf. (2.2)); a detailed account is given in [9, Lemma 2.1]. It is the coefficient a_0'' that we are searching for in particular.

The main efforts are made in the case where P_1 is of order 2; the results are then extended to general cases of even m in Sect. 7.

We do not here give a complete treatment of cases where m is odd. Some results for such cases are included without extra effort (e.g., those in Sect. 6), for other aspects we only give partial results; some indications of what may be done in the first-order case are given in Remark 3.8 below. (Some difficulties are connected with the fact that the classical part of $\log P_1$ does not satisfy the transmission condition when m is odd).

To explain the connection with zeta functions, note that the resolvents can be used to define complex powers P_1^{-s} on $\widetilde{X}$ and $P_{1,T}^{-s}$ on X, by Cauchy integral formulas such as

$$P_{1,T}^{-s} = \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} (P_{1,T} - \lambda)^{-1} \, d\lambda, \tag{2.6}$$

where $\mathcal{C}$ is a curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the nonzero spectrum of $P_{1,T}$ in the positive direction (e.g., a Laurent loop around $\overline{\mathbb{R}}_-$). By the transition formulas accounted for e.g., in [14], the above resolvent trace expansions imply that the zeta functions

$$\zeta(B, P_{1,T}, s) = \operatorname{Tr}(B P_{1,T}^{-s}), \quad \zeta(B, P_{1,+}, s) = \operatorname{Tr}(B(P_1^{-s})_+), \tag{2.7}$$

holomorphic for $\operatorname{Re} s > (\sigma + n)/m$, extend meromorphically to $\operatorname{Re} s > -\delta$, with simple poles at the real points $(\sigma + n - j)/m$ and in particular a Laurent expansion at $s = 0$:

$$\begin{aligned} \zeta(B, P_{1,T}, s) &= C_{-1}(B, P_{1,T}) s^{-1} + [C_0(B, P_{1,T}) - \operatorname{Tr}(B \Pi_0(P_{1,T}))] s^0 + O(s), \\ \zeta(B, P_{1,+}, s) &= C_{-1}(B, P_{1,+}) s^{-1} + [C_0(B, P_{1,+}) - \operatorname{Tr}(B \Pi_0(P_1)_+)] s^0 + O(s), \end{aligned} \tag{2.8}$$

where $\Pi_0(P_1)$ resp. $\Pi_0(P_{1,T})$ is the generalized eigenprojection for the zero eigenvalue of P_1 resp. $P_{1,T}$. Here, in relation to (2.3), (2.4) with (2.5),

$$C_{-1}(B, P_{1,T}) = a_0' = a_0^{(N)\prime}, \quad C_{-1}(B, P_{1,+}) = c_0', \tag{2.9}$$

$$C_0(B, P_{1,T}) = a_0'' = a_0^{(N)\prime\prime} + \alpha_N a_0', \quad C_0(B, P_{1,+}) = c_0''. \tag{2.10}$$

The Π_0-terms enter only in the zeta-expansions (2.8), not in the resolvent expansions (2.3) and (2.4); they stem from the fact that 0 is excluded from the contour integration (2.6).

It is known from [11] for (2.3) with $m = 2$, from [8] for (2.4), and will follow easily for general cases from the proofs given below, that

$$C_{-1}(B, P_{1,T}) = C_{-1}(B, P_{1,+}) = \tfrac{1}{m} \operatorname{res} B, \tag{2.11}$$

where $\operatorname{res} B = \operatorname{res}(P_+ + G)$ is the noncommutative residue; the linear functional defined by Fedosov et al. [4]:

$$\operatorname{res}(P_+) = \int_X \int_{|\xi|=1} \operatorname{tr} p_{-n}(x, \xi) \, đS(\xi), \quad \operatorname{res} G = \operatorname{res}_{X'}(\operatorname{tr}_n G)$$

(expressed in local coordinates). Here $\operatorname{res}_{X'}$ is the Wodzicki residue of the ψdo $\operatorname{tr}_n G$ over X' (the normal trace of G, see (3.2) below). The indication tr means fiber trace,

$d\!\!{}^-\!\xi$ stands for $(2\pi)^{-n}\,d\xi$, and $d\!\!{}^-\!S(\xi)$ is $(2\pi)^{-n}$ times the usual surface measure. This coefficient is completely independent of P_1 or T, and is local. It is zero when $\sigma + n \notin \mathbb{N} = \{0, 1, 2, \dots\}$.

We shall focus the attention on the coefficient $C_0(B, P_{1,T})$, called *the basic zeta coefficient*, as well as its variant $C_0(B, P_{1,+})$, and we search for explicit formulas for these constants.

In the papers [8,12] we have studied the trace defect $C_0(B, P_{1,+}) - C_0(B, P_{2,+})$, which compares the coefficients for two different auxiliary operators P_1 and P_2. It was shown to be local in [12]. Moreover, [8] showed that it can be expressed as a noncommutative residue in the sense of [4]:

$$C_0(B, P_{1,+}) - C_0(B, P_{2,+}) = -\tfrac{1}{m}\operatorname{res}(B(\log P_1 - \log P_2)_+), \tag{2.12}$$

when m is even. In [8], $m > \sigma + n$ or $m = 2$; general values of m are covered by Theorem 3.6 below. When m is odd, the residue definition of [4] is not directly applicable, but there is a more complicated residue-like interpretation of the right-hand side in (2.12).

The operator $\log P_1$ can be defined (on smooth functions) by a Cauchy integral and approximation:

$$\log P_1 = \lim_{s\searrow 0} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} \log\lambda\, (P_1 - \lambda)^{-1}\, d\lambda, \tag{2.13}$$

where $\mathcal{C}$ is a curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the nonzero spectrum of P_1. Its symbol in local coordinates is of the form

$$\begin{aligned} &\operatorname{symb}(\log P_1) = m\log[\xi] + l(x,\xi), \text{ with } l(x,\xi) \text{ classical of order } 0,\\ &[\xi] \text{ is a } C^\infty \text{ function } \geq \tfrac{1}{2} \text{ with } [\xi] = |\xi| \text{ for } |\xi| \geq 1, \end{aligned} \tag{2.14}$$

cf. e.g., Okikiolu [21], and $l(x,\xi)$ satisfies the transmission condition when m is even (details in [10, Lemma 2.1]).

In the present paper, we shall prove a formula for $C_0(B, P_{1,+})$ itself, showing how it is, in local coordinates, the sum of a finite-part integral (in the sense of Hadamard) defined from B and a residue-type integral defined from $B(\log P_1)_+$, suitably interpreted. The value *modulo local terms* was found in [12]; the present description is much more precise. Moreover, we shall extend the description to $C_0(B, P_{1,T})$.

Our result is a generalization of, and is inspired from, the recent result of Paycha and Scott [22] on Laurent coefficients in the boundaryless case, further analyzed in our note [9]. They showed the following:

Let A and P_1 be classical pseudodifferential operators of order $\sigma \in \mathbb{R}$ resp. $m \in \mathbb{R}_+$ on a compact n-dimensional manifold $\widetilde{X}$ without boundary, P_1 being elliptic with $\mathbb{R}_-$ as a spectral cut. Then all the coefficients in the Laurent expansions of the generalized zeta function $\zeta(A, P_1, s) = \operatorname{Tr}(AP_1^{-s})$ around the poles can be expressed as combinations of finite-part integrals and residue-type integrals of associated logarithmic symbols. Denote (similarly to (2.8)) the Laurent coefficient of s^0 at zero (the regular value at zero) by $C_0(A, P_1) - \operatorname{Tr}(A\Pi_0(P_1))$, where $\Pi_0(P_1)$ is the generalized zero-eigenprojection of P_1. Then $C_0(A, P_1)$ satisfies

$$C_0(A, P_1) = \int_{\widetilde{X}} \big(\mathrm{TR}_x(A) - \tfrac{1}{m}\,\mathrm{res}_{x,0}(A \log P_1)\big)\, dx. \tag{2.15}$$

Here the function $\mathrm{TR}_x(A) - \frac{1}{m}\,\mathrm{res}_{x,0}(A \log P_1)$ is defined in a local coordinate system by:

$$\mathrm{TR}_x(A) = \,⨍\, \mathrm{tr}\, a(x,\xi)\, đ\xi, \tag{2.16}$$

$$\mathrm{res}_{x,0}(A \log P_1) = \int_{|\xi|=1} \mathrm{tr}\, r_{-n,0}(x,\xi)\, đS(\xi). \tag{2.17}$$

For (2.16), the expression $⨍\, \mathrm{tr}\, a(x,\xi)\, đ\xi$ is defined for each x as a Hadamard finite-part integral, namely as the constant term in the asymptotic expansion of $\int_{|\xi|\le R} \mathrm{tr}\, a(x,\xi)\, đ\xi$ in powers $R^{\sigma+n-j}$ ($j \in \mathbb{N}$), R^0 and $\log R$ (cf. Lesch [17], see also [12, (3.12)ff.]). The notation TR_x is inspired from the notation of [16]; in fact, as pointed out in [17], $\mathrm{TR}_x\, A$ integrates in suitable cases to the canonical trace $\mathrm{TR}\, A$ (see also [7]).

For (2.17), the symbol of $R = A \log P_1$ is denoted $r(x,\xi)$; it is log-polyhomogeneous of the form (cf. (2.14))

$$r(x,\xi) \sim \sum_{j\in\mathbb{N}} (r_{\sigma-j,0}(x,\xi) + r_{\sigma-j,1}(x,\xi)\log[\xi]), \tag{2.18}$$

where each $r_{\sigma-j,l}$ is homogeneous in ξ of degree $\sigma - j$ for $|\xi| \ge 1$. So $r_{-n,0}$ is the log-free term of order $-n$ (taken equal to 0 when $\sigma + n \notin \mathbb{N}$).

We find it interesting that the two "competing" trace functionals res and TR both enter here, in localized versions (only the collected expression $(\mathrm{TR}_x(A) - \frac{1}{m}\,\mathrm{res}_{x,0}(A \log P_1))dx$ has an invariant meaning as a density on $\widetilde{X}$). Note that $\mathrm{res}_{x,0}(A \log P_1)$ is *local* (depends on homogeneous symbols down to order $-n$), whereas $\mathrm{TR}_x(A)$ is *global* (depends on the full structure).

The formula was shown in [22] by use of holomorphic families of ψdo's (depending holomorphically on their complex order z). We showed in [9] how the formula could be derived by methods relying directly on the knowledge of the resolvent $(P_1 - \lambda)^{-1}$, as a preparation for the present generalization to manifolds with boundary.

Like [22], we shall use the notation

$$\mathrm{res}_x(Q) = \int_{|\xi|=1} \mathrm{tr}\, q_{-n}(x,\xi)\, đS(\xi), \tag{2.19}$$

when Q is a classical ψdo on $\widetilde{X}$ with symbol $q(x,\xi)$ in local coordinates. Here $\mathrm{res}_x(Q)$ has a meaning only in local coordinates, but the integral of $\mathrm{res}_x(Q)$ over $\widetilde{X}$ can be given an invariant meaning as the Wodzicki noncommutative residue $\mathrm{res}\, Q$.

Remark 2.1 In view of (2.14), we can in local coordinates define $(\log P_1)_{\mathrm{cl}} = \mathrm{OP}(l(x,\xi))$ as "the classical part of $\log P_1$". Then, as is easily checked from the composition rules,

$$\mathrm{res}_{x,0}(A\log P_1) = \mathrm{res}_x(A(\log P_1)_{\mathrm{cl}}), \tag{2.20}$$

when we use the notation (2.19). But this generally has a meaning only in local coordinates.

Our goal is to generalize (2.15) to a characterization of $C_0(B, P_{1,T})$, via a study of the coefficient of $(-\lambda)^{-N}$ in the trace expansion of $BR^N_\lambda = \frac{\partial_\lambda^{N-1}}{(N-1)!}BR_\lambda$. The operator breaks up in five terms:

$$\begin{aligned} BR_\lambda &= (P_+ + G)(Q_{\lambda,+} + G_\lambda) \\ &= P_+Q_{\lambda,+} + GQ_{\lambda,+} + (P_+ + G)G_\lambda, \\ &= (PQ_\lambda)_+ - L(P, Q_\lambda) + GQ_{\lambda,+} + P_+G_\lambda + GG_\lambda, \end{aligned} \tag{2.21}$$

$$BR^N_\lambda = (PQ^N_\lambda)_+ - L(P, Q^N_\lambda) + GQ^N_{\lambda,+} + P_+G^{(N)}_\lambda + GG^{(N)}_\lambda, \tag{2.22}$$

where we use the notation (2.2), and the notation $L(P, P')$ for the "leftover term" in the truncation of a product: $L(P, P') = (PP')_+ - P_+P'_+$ (more on this in (4.17)ff.). The part $BQ^N_{\lambda,+}$ (the first three terms in (2.22)) has been studied in [11,12] and later [8], the last paper giving the trace defect formula (2.12). For this part, the strategy will be:

1. *Find $C_0(B, P_{1,+})$ for one particularly manageable choice of P_1.*
2. *Extend to more general P_2 by combination with the trace defect formula* (2.12).

This program is carried out in Sects. 4 and 5.

The part $BG^{(N)}_\lambda$ (the last two terms in (2.22)) has been treated under special assumptions and with only qualitative results in [11,12]. An exact trace defect formula was not shown, so we would need to find this. In fact, we shall aim directly for a precise description of the coefficient in question (from which a defect formula follows), since this can be done with the methods established in [8]. The coefficient is *local*—as shown under restrictive hypotheses in [11]. Part of the study is carried out in Sect. 3, showing the existence of an expansion with a qualitative description of the coefficients, and the connection with our main problem is worked out in Sect. 6.

It will be practical to introduce a general notation for the relevant coefficient in various trace expansions (cf. (2.5)).

Definition 2.2 Let M_λ be an operator family defined for large λ in a sector V of $\mathbb{C}$, with $M_{\lambda,N} = \frac{\partial_\lambda^{N-1}}{(N-1)!}M_\lambda$ defined there for all positive integers N. If the $M_{\lambda,N}$ for all $N > N_0$ have trace expansions of the form

$$\begin{aligned} \mathrm{Tr}\, M_{\lambda,N} = &\sum_{0\le j<n+\sigma} a_{N,j}\,(-\lambda)^{\frac{n+\sigma-j}{2}-N} \\ &+ (a'_0\log(-\lambda) + a''_0 - \alpha_N a'_0)(-\lambda)^{-N} + O(\lambda^{-N-\delta}), \end{aligned} \tag{2.23}$$

some $\delta > 0$, for $\lambda \to \infty$ on rays in V, with a_0' and a_0'' independent of N, $\alpha_N = \sum_{1\le j<N} \frac{1}{j}$, we define

$$l_0(M_\lambda) = a_0''. \tag{2.24}$$

The main result is the following theorem with P_1 of even order m, whose ingredients will be explained in the next sections:

Theorem 2.3 *The basic zeta coefficient $C_0(B, P_{1,T})$ is a sum of terms, calculated in local coordinates*:

$$\begin{aligned} C_0(B, P_{1,T}) = &\int_X \left[\mathrm{TR}_x\, P - \tfrac{1}{m}\,\mathrm{res}_{x,0}(P \log P_1)\right] dx + \tfrac{1}{m}\,\mathrm{res}\, L(P, \log P_1) \\ &+ \int_{X'} \left[\mathrm{TR}_{x'}\, \mathrm{tr}_n\, G - \tfrac{1}{m}\,\mathrm{res}_{x',0}\, \mathrm{tr}_n'(G(\log P_1)_+)\right] dx' \\ &- \tfrac{1}{m}\,\mathrm{res}\left(P_+ G_1^{\log}\right) - \tfrac{1}{m}\,\mathrm{res}\left(G G_1^{\log}\right). \end{aligned} \tag{2.25}$$

Here $C_0(B, P_{1,+})$ is the sum of the first three terms.

The five terms are found as the coefficient l_0 in trace expansions of the five terms in (2.22), as written in (1.1). The residues appearing in (2.25) are various generalizations of the definition of [4], and will be suitably explained in the process of deduction of the formula. Both $\mathrm{res}\, L(P, \log P_1)$ and $\mathrm{res}(GG_1^{\log})$ are Wodzicki-type residues over X' of ψdo's $\mathrm{tr}_n\, L(P, \log P_1)$ resp. $\mathrm{tr}_n(GG_1^{\log})$, whereas $\mathrm{res}(P_+G_1^{\log})$, as well as $\mathrm{res}_{x',0}\, \mathrm{tr}_n'(G(\log P_1)_+)$ are more delicate to define. The theorem is shown in Sect. 7, based on the results from Sects. 4, 5 (with $m = 2$) and 6 (with $m > 0$); moreover, new defect formulas are derived. In particular, it is proved that the new residue definition is tracial in the sense that

$$\mathrm{res}([B, \log P_{1,T}]) = 0, \tag{2.26}$$

when B is of order and class 0.

3 A strategy for local contributions

In this section we show the supplementary results needed to get the trace expansions (2.3) and (2.4) in the stated generality. The terms $L(P, Q_\lambda^N)$, $P_+G_\lambda^{(N)}$ and $GG_\lambda^{(N)}$ in (2.22) will be seen to have trace expansions (2.23) with $a_0' = 0$, and a_0'' *local*. To show this result, we can apply a method introduced in [8], relying on the "positive regularity" (in the sense of [5]) of the ψdo family on X' obtained by taking normal traces.

We shall explain the method here, establishing the main technical points in a form applicable to general situations, and give its application to $P_+G_\lambda^{(N)} + GG_\lambda^{(N)} = BG_\lambda^{(N)}$ in the present section, treating $L(P, Q_\lambda^N)$ later in Sect. 4.

The trace expansions are derived as finite sums of trace expansions worked out in local coordinates, where the situation is carried over to $\mathbb{R}^n_+$, with coordinates

$x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$. We recall that when G_0 is a singular Green operator on $\mathbb{R}^n_+$ of order $< 1-n$ and class 0 with a kernel having compact (x', y')-support (hence is trace-class), then $\operatorname{Tr} G_0 = \operatorname{Tr}_{\mathbb{R}^{n-1}}(\operatorname{tr}_n G_0)$, where tr_n indicates the normal trace. To explain this further, recall from [5] that when G_0 has the symbol $g_0(x', \xi', \xi_n, \eta_n)$ and symbol-kernel

$$\tilde{g}_0(x', x_n, y_n, \xi') = r^+_{x_n} r^+_{y_n} \mathcal{F}^{-1}_{\xi_n \to x_n} \overline{\mathcal{F}}^{-1}_{\eta_n \to y_n} g_0(x', \xi', \xi_n, \eta_n),$$

the action of G_0 is defined for $u \in r^+ \mathcal{S}(\mathbb{R}^n)$ by

$$G_0 u = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{g}_0(x', x_n, y_n, \xi') \acute{u}(\xi', y_n)\, dy_n đ\xi'; \tag{3.1}$$

here $\acute{u} = \mathcal{F}_{x' \to \xi'} u$, and r^+ restricts to $\{x_n > 0\}$. Then the normal trace $\operatorname{tr}_n G_0$ is a ψdo S_0 on $\mathbb{R}^{n-1}$ with symbol

$$\begin{aligned} s_0(x', \xi') = (\operatorname{tr}_n g_0)(x', \xi') &= \int_0^\infty \tilde{g}_0(x', x_n, x_n, \xi')\, dx_n \\ &= \int_0^\infty r^+_{x_n} r^+_{y_n} \int^+ \int^+ e^{ix_n\xi_n - ix_n\eta_n} g_0(x', \xi', \xi_n, \eta_n)\, đ\xi_n đ\eta_n dx_n \\ &= \int^+ g_0(x', \xi', \xi_n, \xi_n)\, đ\xi_n. \end{aligned} \tag{3.2}$$

The plus-integral stands for an extension of the usual integral (cf. e.g., [5, p. 166]), and the last formula follows since the factor $e^{ix_n\xi_n - ix_n\eta_n}$ together with the integrations in η_n and x_n give rise to a backwards and a forwards Fourier transform.

We denote $(1 + |\xi'|^2)^{\frac{1}{2}} = \langle \xi' \rangle$, $(1 + |\xi'|^2 + \mu^2)^{\frac{1}{2}} = \langle \xi', \mu \rangle$, and use the sign $\dot{\leq}$ in inequalities to indicate "$\leq$ a constant times".

The idea is to show that taking the normal trace of $BG^{(N)}_\lambda$, in local coordinates, leads to a λ-dependent family of ψdo's on $\mathbb{R}^{n-1}$, where a kernel expansion and a formula for the basic coefficient can be deduced exactly as in the proof of [8, Theorem 4.5]. For the sake of general applicability, the crucial steps in that proof will be formulated in the following two general theorems, where we outline the proofs with reference to [8].

Theorem 3.1 *Let $\sigma \in \mathbb{R}$, let $\delta \in]0, 1[$ such that $\sigma + \delta \notin \mathbb{Z}$, let m and N be positive integers, and let $\mathcal{S}^{(N)}_\lambda = \operatorname{OP}(\mathfrak{s}^{(N)}(x', \xi', \lambda))$ be a family of ψdo's on $\mathbb{R}^{n-1}$ depending holomorphically on λ in a keyhole region $V' = V \cup \{0 < |\lambda| < r\}$, V being a sector around $\mathbb{R}_-$, of order $\sigma - mN$ and regularity $\sigma - \delta$ in terms of μ on each ray $\lambda = -\mu^m e^{i\theta}$ in V. Assume moreover that $(-\lambda)^N \mathcal{S}^{(N)}_\lambda$ is of order σ and regularity*

$\sigma+\delta$, and that the symbol satisfies

$$\Big|\partial^{\beta,\alpha}_{x',\xi'}[\mathfrak{s}^{(N)}(x',\xi',\lambda)-\sum_{j<J}\mathfrak{s}^{(N)}_{\sigma-mN-j}(x',\xi',\lambda)]\Big|\dot{\le}\langle\xi'\rangle^{\sigma-\delta-|\alpha|-J}\langle\xi',\mu\rangle^{-mN+\delta},$$
$$\Big|\partial^{\beta,\alpha}_{x',\xi'}[\mathfrak{s}^{(N)}(x',\xi',\lambda)-\sum_{j<J}\mathfrak{s}^{(N)}_{\sigma-mN-j}(x',\xi',\lambda)]\Big|\dot{\le}\langle\xi'\rangle^{\sigma+\delta-|\alpha|-J}\langle\xi',\mu\rangle^{-\delta}\mu^{-mN},\tag{3.3}$$

on the rays in V, for all α, β, J.

If $N>(\sigma+n-1)/m$, the kernel on the diagonal has an expansion

$$K(\mathcal{S}^{(N)}_\lambda,x',x')=\sum_{0\le l\le\sigma+n-1}\tilde{\mathfrak{s}}^{(N)}_l(x')(-\lambda)^{\frac{n-1+\sigma-l}{m}-N}+O\left(\lambda^{-N-\frac{\delta}{m}}\right),$$
$$\text{with}\quad\tilde{\mathfrak{s}}^{(N)}_l(x')=\int_{\mathbb{R}^{n-1}}\mathfrak{s}^{(N)h}_{\sigma-mN-l}(x',\xi',-1)\,đ\xi',\tag{3.4}$$

where the strictly homogeneous symbols $\mathfrak{s}^{(N)h}_{\sigma-mN-l}$ are integrable at $\xi'=0$ for $l\le\sigma+n-1$. If $\sigma+n-1\in\mathbb{N}$, the coefficient of $(-\lambda)^{-N}$ is

$$\tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x')=\int_{\mathbb{R}^{n-1}}\mathfrak{s}^{(N)h}_{-mN-n+1}(x',\xi',-1)\,đ\xi'.\tag{3.5}$$

If $\sigma+n-1\notin\mathbb{N}$, there is no term with $(-\lambda)^{-N}$; we include one trivially by setting $\tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x')=0$.

Proof The strictly homogeneous version $\mathfrak{s}^{(N)h}_j$ of $\mathfrak{s}^{(N)}_j$ is the extension of $\mathfrak{s}^{(N)}_j|_{|\xi'|\ge1}$ by homogeneity into $\{1>|\xi'|>0\}$.

The expansion down to the term with $l=n-2+\sigma$ is assured by the general theory of [5], cf. e.g., Lemmas 3.1–5 in [8]. To include the next term, one proceeds as in the proof of [8, (4.35)ff.], if $\sigma+n-1\in\mathbb{N}$ (with the number $\frac14$ replaced by $\delta>0$):

The estimates (3.3) imply that the symbol with $l=\sigma+n-1$ satisfies

$$\Big|\mathfrak{s}^{(N)}_{-mN-n+1}(x',\xi',\lambda)\Big|\dot{\le}\langle\xi'\rangle^{-\delta-n+1}\langle\xi',\mu\rangle^{-mN+\delta},$$
$$\Big|\mathfrak{s}^{(N)}_{-mN-n+1}(x',\xi',\lambda)\Big|\dot{\le}\langle\xi'\rangle^{\delta-n+1}\langle\xi',\mu\rangle^{-\delta}\mu^{-mN},\tag{3.6}$$

and the remainder $\mathfrak{s}^{(N)\prime}=\mathfrak{s}^{(N)}-\sum_{l<\sigma+n}\mathfrak{s}^{(N)}_{\sigma-mN-l+1}$ after this term satisfies

$$|\mathfrak{s}^{(N)\prime}|\dot{\le}\langle\xi'\rangle^{-\delta-n}\langle\xi',\mu\rangle^{-mN+\delta},$$
$$|\mathfrak{s}^{(N)\prime}|\dot{\le}\langle\xi'\rangle^{\delta-n}\langle\xi',\mu\rangle^{-\delta}\mu^{-mN}.\tag{3.7}$$

From (3.6) follows as in [5, Lemma 2.1.9] that similar estimates are valid for the strictly homogeneous symbols:

$$\begin{aligned}\left|\mathfrak{s}^{(N)h}_{-mN-n+1}(x',\xi',\lambda)\right| &\dot{\le} |\xi'|^{-\delta-n+1}|(\xi',\mu)|^{-mN+\delta},\\ \left|\mathfrak{s}^{(N)h}_{-mN-n+1}(x',\xi',\lambda)\right| &\dot{\le} |\xi'|^{\delta-n+1}|(\xi',\mu)|^{-\delta}\mu^{-mN},\end{aligned}\tag{3.8}$$

so $\mathfrak{s}^{(N)h}_{-mN-n+1}$ is integrable at $\xi'=0$ (besides being so for $|\xi'|\to\infty$) when $\lambda\neq 0$. Then

$$\begin{aligned}K\left(\mathrm{OP}\left(\mathfrak{s}^{(N)h}_{-mN-n+1}\right),x',x'\right) &= \tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x')\,(-\lambda)^{-1},\ \text{with}\\ \tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') &= \int\limits_{\mathbb{R}^{n-1}} \mathfrak{s}^{(N)h}_{-mN-n+1}(x',\xi',-1)\,đ\xi',\end{aligned}\tag{3.9}$$

as desired. This gives the needed extra term. The remainder estimate is obtained (like the preceding lines) exactly as in [8, Theorem 4.5]; we shall not repeat the details. If $\sigma+n-1\notin\mathbb{N}$, there is no term with $(-\lambda)^{-N}$, so only remainder estimates have to be checked. The condition $\sigma+\delta\notin\mathbb{Z}$ is imposed in order to make [8, Lemma 3.2] applicable without an ε-reservation.

The coefficients found for different rays in V coincide with those for the ray $\mathbb{R}_-$ in view of the holomorphy (as in [13, Lemma 2.3]). □

Theorem 3.2 *Let $\mathcal{S}^{(N)}_\lambda$ be defined as in Theorem 3.1 (with the stated regularity properties and estimates) for each each $N=1,2,\dots,$ and assume moreover that $\mathcal{S}^{(N)}_\lambda=\frac{\partial_\lambda^{N-1}}{(N-1)!}\mathcal{S}^{(1)}_\lambda$ for all N. Then for $N>(\sigma+n-1)/m$,*

$$\begin{aligned}\tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') &= \int\limits_{\mathbb{R}^{n-1}} \mathfrak{s}^{(N)h}_{-mN-n+1}(x',\xi',-1)\,đ\xi'\\ &= \int\limits_{\mathbb{R}^{n-1}} \mathfrak{s}^{(1)h}_{-m-n+1}(x',\xi',-1)\,đ\xi'.\end{aligned}\tag{3.10}$$

Define the "log-transform" S of $\mathcal{S}^{(1)}_\lambda$ as an operator whose symbol $s(x',\xi')$ for $|\xi'|\ge 1$ is deduced from the symbol $\mathfrak{s}^{(1)}(x',\xi',\lambda)$ of $\mathcal{S}^{(1)}_\lambda$ by

$$\begin{aligned}s(x',\xi') &= \tfrac{i}{2\pi}\int\limits_{\mathcal{C}} \log\lambda\,\mathfrak{s}^{(1)}(x',\xi',\lambda)\,d\lambda,\\ s_{\sigma-j}(x',\xi') &= \tfrac{i}{2\pi}\int\limits_{\mathcal{C}} \log\lambda\,\mathfrak{s}^{(1)}_{\sigma-m-j}(x',\xi',\lambda)\,d\lambda\end{aligned}\tag{3.11}$$

(*with a curve* $\mathcal{C}$ *in* $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ *encircling* $\mathbb{C}\setminus V'$). *It is a classical* ψ*do symbol of order* σ, *and for* $N > (\sigma + n - 1)/m$,

$$\begin{aligned} \tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') &= -\tfrac{1}{m} \int_{|\xi'|=1} s_{1-n}(x',\xi')\, đS(\xi'), \ \textit{hence} \\ \operatorname{tr} \tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') &= -\tfrac{1}{m} \operatorname{res}_{x'} S. \end{aligned} \tag{3.12}$$

Proof The first equality in (3.10) repeats (3.5), and the last equality follows in the way explained in [8, Remark 3.12] from the fact that $\mathfrak{s}^{(N)}(x',\xi',\lambda) = \frac{\partial_\lambda^{N-1}}{(N-1)!}\mathfrak{s}^{(1)}(x',\xi',\lambda)$.

Now consider the "log-transform"; the integrability in λ is assured by the second line in (3.3). The verification that s is a classical ψdo symbol of order σ with homogeneous terms $s_{\sigma-j}$ goes exactly as in [8, Theorem 4.5, (4.40–4.42)ff.] (with $\sigma + \sigma'$ replaced by σ, $\frac{1}{4}$ replaced by δ); we shall spare a repetition of details. The first formula in (3.12) represents the fact that

$$\int_{\mathbb{R}^{n-1}} \mathfrak{s}^{(1)h}_{-m-n+1}(x',\xi',-1)\, đ\xi' = -\tfrac{1}{m} \int_{|\xi'|=1} \tfrac{i}{2\pi} \int_{\mathcal{C}} \log\lambda\, \mathfrak{s}^{(1)h}_{-m-n+1}(x',\xi',\lambda)\, d\lambda đS(\xi'),$$

where the log-integral is turned into an integral along $\mathbb{R}_-$ and the homogeneity is used in the application of polar coordinates, by [8, Lemmas 1.2 and 1.3] in dimension $n-1$. Taking the fiber trace, we get the second formula in (3.12) by definition, using (2.19) in dimension $n-1$. □

We shall now see how these theorems apply to show that $BG^{(N)}_\lambda$ has a trace expansion (2.23) with $a'_0 = 0$ and a''_0 equal to a residue. In [11] the special case where G_λ is defined from a principally scalar Laplacian with Dirichlet condition was studied (giving a full expansion in log-powers of orders going to $-\infty$); we now allow more general $\{P_1, T\}$ and just show the expansion down to and including the crucial term with $(-\lambda)^{-N}$.

The order of P_1 is here a positive integer m. B is of the form $B = P_+ + G$, of order σ and with G of class 0; here $\sigma \in \mathbb{Z}$ when $P \neq 0$, and σ can be any real number when $P = 0$.

In order to apply Theorems 3.1 and 3.2 we shall show that G_λ can be rewritten in a form that shows a better fall-off in λ, at the cost of augmenting the order of the operator.

Lemma 3.3 (1) *The s.g.o.* G_λ *and its derivatives* $G^{(N)}_\lambda = \frac{\partial_\lambda^{N-1}}{(N-1)!}G_\lambda$ *may be written in the form*

$$\begin{aligned} G_\lambda &= \lambda^{-1} P_1 G_\lambda, \\ G^{(N)}_\lambda &= (-\lambda)^{-N} P_1 G^{(N)\prime}_\lambda, \end{aligned} \tag{3.13}$$

where $G^{(N)\prime}_\lambda$, *described in the proof below, is a singular Green operator of order* $-m$, *class* 0 *and regularity* $+\infty$.

(2) *For each* $N \geq 1$, *the s.g.o.* $BG_\lambda^{(N)} = (P_+ + G)G_\lambda^{(N)}$ *has order* $\sigma - mN$, *class* 0 *and regularity* σ. *Moreover, it can be written as*

$$BG_\lambda^{(N)} = (-\lambda)^{-N} BP_1 G_\lambda^{(N)\prime}, \tag{3.14}$$

where $BP_1 G_\lambda^{(N)\prime}$ *has order* σ, *class* 0 *and regularity* $\sigma + \frac{1}{2}$.

(3) *In local coordinates, the normal trace*

$$\mathcal{U}_\lambda^{(N)} = \operatorname{tr}_n\left(BG_\lambda^{(N)}\right) = \tfrac{\partial_\lambda^{N-1}}{(N-1)!}\mathcal{U}_\lambda^{(1)} \tag{3.15}$$

is a ψ*do family on* $\mathbb{R}^{n-1}$ *of order* $\sigma - mN$ *and regularity* $\sigma - \frac{1}{4}$. *It can also be written as*

$$\mathcal{U}_\lambda^{(N)} = (-\lambda)^{-N}\mathcal{U}_\lambda^{(N)\prime}, \quad \mathcal{U}_\lambda^{(N)\prime} = \operatorname{tr}_n\left(BP_1G_\lambda^{(N)\prime}\right), \tag{3.16}$$

where $\mathcal{U}_\lambda^{(N)\prime}$ *has order* σ *and regularity* $\sigma + \frac{1}{4}$.

(4) *The symbol* $\mathfrak{u}^{(N)}(x', \xi', \lambda)$ *of* $\mathcal{U}_\lambda^{(N)}$, *with the expansion in (quasi-) homogeneous terms* $\mathfrak{u}^{(N)}(x', \xi', \lambda) \sim \sum_{j\in\mathbb{N}} \mathfrak{u}_{\sigma-mN-j}^{(N)}(x', \xi', \lambda)$ (*the* $\mathfrak{u}_r^{(N)}$ *being homogeneous of degree* r *in* (ξ', μ) *on each ray* $\lambda = -\mu^m e^{i\theta}$, $\mu > 0$), *satisfies*:

$$\begin{aligned}
&\left|\partial_{x',\xi'}^{\beta,\alpha}\left[\mathfrak{u}^{(N)}(x', \xi', \lambda) - \sum_{j<J}\mathfrak{u}_{\sigma-mN-j}^{(N)}(x', \xi', \lambda)\right]\right| \dot{\leq} \langle\xi'\rangle^{\sigma-\frac{1}{4}-|\alpha|-J}\langle\xi', \mu\rangle^{-mN+\frac{1}{4}},\\
&\left|\partial_{x',\xi'}^{\beta,\alpha}\left[\mathfrak{u}^{(N)}(x', \xi', \lambda) - \sum_{j<J}\mathfrak{u}_{\sigma-mN-j}^{(N)}(x', \xi', \lambda)\right]\right| \dot{\leq} \langle\xi'\rangle^{\sigma+\frac{1}{4}-|\alpha|-J}\langle\xi', \mu\rangle^{-\frac{1}{4}}\mu^{-mN},
\end{aligned} \tag{3.17}$$

on the rays in V, *for all* α, β, J.

Proof We start by noting that, since P_1 is a differential operator,

$$\begin{aligned}
Q_\lambda + \lambda^{-1} &= Q_\lambda + \lambda^{-1}(P_1 - \lambda)Q_\lambda = \lambda^{-1}P_1Q_\lambda \text{ on } \widetilde{X},\\
R_\lambda + \lambda^{-1} &= R_\lambda + \lambda^{-1}(P_1 - \lambda)R_\lambda = \lambda^{-1}P_1(Q_{\lambda,+} + G_\lambda)\\
&= \lambda^{-1}[(P_1Q_\lambda)_+ + P_1G_\lambda] = Q_{\lambda,+} + \lambda^{-1} + \lambda^{-1}P_1G_\lambda \text{ on } X,
\end{aligned} \tag{3.18}$$

which implies the first formula in (3.13). For the second formula, we calculate:

$$\begin{aligned}
G_\lambda^{(N)} &= P_1\tfrac{\partial_\lambda^{N-1}}{(N-1)!}(\lambda^{-1}G_\lambda) = P_1\sum_{j=0}^{N-1} c_j(-\lambda)^{-1-j}G_\lambda^{(N-j)}\\
&= (-\lambda)^{-N}P_1\sum_{j=0}^{N-1} c_j((P_1 - \lambda) - P_1)^{N-1-j}G_\lambda^{(N-j)}\\
&= (-\lambda)^{-N}P_1\sum_{j=0}^{N-1} c_j\sum_{k=0}^{N-1-j} c_k' P_1^k(P_1 - \lambda)^{N-1-j-k}G_\lambda^{(N-j)};
\end{aligned} \tag{3.19}$$

here the sums over j and k define an s.g.o. $G_\lambda^{(N)\prime}$ of order $-m$, class 0 and regularity $+\infty$.

For (2) we use the rules in [5, Proposition 2.3.14], which give that P_+ and G have order and regularity σ, whereas $P_+P_{1,+} = (PP_1)_+ - L(P, P_1)$ has order $\sigma + m$ with $(PP_1)_+$ of regularity $\sigma + m$, $L(P, P_1)$ an s.g.o. of class m and hence regularity $\sigma + \frac{1}{2}$, and likewise $GP_{1,+}$ of order $\sigma + m$ and class m and hence regularity $\sigma + \frac{1}{2}$. The compositions with $G_\lambda^{(N)}$ resp. $G_\lambda^{(N)\prime}$ inherit these regularities by the rules in [5, Sect. 2.7].

For (3), there is a loss of $\frac{1}{4}$ in the regularity when the general rule of [8, Lemma 3.4] is applied.

The information in (4) follows from the definition of the class of symbols of the stated regularity, when $\sigma < 0$. When $\sigma \geq 0$, the regularity information itself gives weaker estimates when $\sigma - |\alpha| - J \geq 0$. But here we can use the device introduced in the proof of [8, Proposition 4.3]: Compose $\mathcal{U}_\lambda^{(N)}$ to the left with $\Lambda^\varrho \Lambda^{-\varrho}$, $\Lambda = \mathrm{OP}(\langle \xi' \rangle)$, with $\varrho \geq \sigma + 1$. Taking $\Lambda^{-\varrho}$ together with B one finds that $\Lambda^{-\varrho}\mathcal{U}_\lambda^{(N)}$ satisfies the regularity statements with σ replaced by $\sigma - \varrho$, hence the estimates with the same replacement, and the desired estimates follow after composition with Λ^ϱ. □

Theorems 3.1 and 3.2 apply straightforwardly to the symbols described in this lemma:

Theorem 3.4 *Let $\mathcal{U}_\lambda^{(N)}$ be as in Lemma 3.3. When $N > (\sigma + n - 1)/m$, the kernel of $\mathcal{U}_\lambda^{(N)}$ on the diagonal has an expansion*

$$K\left(\mathcal{U}_\lambda^{(N)}, x', x'\right) = \sum_{0 \leq l \leq \sigma + n - 1} \tilde{\mathrm{u}}_l^{(N)}(x')(-\lambda)^{\frac{\sigma + n - 1 - l}{m} - N} + O\left(\lambda^{-N - \frac{1}{4m}(+\varepsilon)}\right), \tag{3.20}$$

where $(+\varepsilon)$ indicates the addition of an $\varepsilon > 0$ if $\sigma + \frac{1}{4} \in \mathbb{Z}$ (no addition otherwise). The coefficient of $(-\lambda)^{-N}$ is

$$\begin{aligned} \tilde{\mathrm{u}}_{\sigma+n-1}^{(N)}(x') &= \int_{\mathbb{R}^{n-1}} \mathrm{u}_{-mN-n+1}^{(N)h}(x', \xi', -1)\, đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} \mathrm{u}_{-m-n+1}^{(1)h}(x', \xi', -1)\, đ\xi' \end{aligned} \tag{3.21}$$

if $\sigma + n - 1 \in \mathbb{N}$, zero if $\sigma + n - 1 \notin \mathbb{N}$.

The "log-transform" U, defined as an operator with symbol $u(x', \xi')$ deduced for $|\xi'| \geq 1$ from the symbol $\mathrm{u}^{(1)}(x', \xi', \lambda)$ of $\mathcal{U}_\lambda^{(1)}$ by

$$\begin{aligned} u(x', \xi') &= \tfrac{i}{2\pi} \int_{\mathcal{C}} \log\lambda\, \mathrm{u}^{(1)}(x', \xi', \lambda)\, d\lambda, \\ u_{\sigma-j}(x', \xi') &= \tfrac{i}{2\pi} \int_{\mathcal{C}} \log\lambda\, \mathrm{u}_{\sigma-m-j}^{(1)}(x', \xi', \lambda)\, d\lambda \end{aligned} \tag{3.22}$$

(*and extended smoothly for* $|\xi'| \le 1$), *is a classical* ψ*do of order* σ *such that the coefficient of* $(-\lambda)^{-N}$ *in* (3.20) *satisfies*:

$$\begin{aligned} \tilde{u}^{(N)}_{\sigma+n-1}(x') &= -\tfrac{1}{m} \int\limits_{|\xi'|=1} u_{1-n}(x', \xi')\, đS(\xi'), \\ \operatorname{tr} \tilde{u}^{(N)}_{\sigma+n-1}(x') &= -\tfrac{1}{m} \operatorname{res}_{x'} U; \end{aligned} \tag{3.23}$$

here u_{1-n} *is defined as zero if* $\sigma + n - 1 \notin \mathbb{N}$.

Proof When $\sigma \in \mathbb{Z}$ (which must hold if $P \neq 0$), we apply Theorems 3.1 and 3.2 with $\delta = \frac{1}{4}$. When $\sigma \notin \mathbb{Z}$, we apply the theorems with $\delta = \frac{1}{4}$ if $\sigma + \frac{1}{4} \notin \mathbb{Z}$, $\delta < \frac{1}{4}$ if $\sigma + \frac{1}{4} \in \mathbb{Z}$. □

Integrating over the local coordinate patches and carrying the pieces back to the manifold, we find:

Corollary 3.5 *When* $N > (\sigma + n - 1)/m$, $\operatorname{Tr}(BG^{(N)}_\lambda)$ *has an expansion*

$$\operatorname{Tr}\left(BG^{(N)}_\lambda\right) = \sum_{0 \le l \le \sigma+n-1} \tilde{u}^{(N)}_l (-\lambda)^{\frac{\sigma+n-1-l}{m} - N} + O\left(\lambda^{-N-\frac{1}{4m}(+\varepsilon)}\right) \tag{3.24}$$

(*with* $(+\varepsilon)$ *understood as in Theorem* 3.4), *where the coefficient of* $(-\lambda)^{-N}$ *is*

$$\tilde{u}^{(N)}_{\sigma+n-1} = \int\limits_{X'} \operatorname{tr} \tilde{u}^{(N)}_{\sigma+n-1}(x')\, dx' = -\tfrac{1}{m} \int\limits_{X'} \operatorname{res}_{x'} U\, dx' = -\tfrac{1}{m} \operatorname{res} U,$$

if $\sigma + n - 1 \in \mathbb{N}$, *zero otherwise. The expansion may also be written as*:

$$\operatorname{Tr}(BG^{(N)}_\lambda) = \sum_{1 \le j \le \sigma+n} b^{(N)}_j (-\lambda)^{\frac{\sigma+n-j}{m} - N} + O(\lambda^{-N-\frac{1}{4m}(+\varepsilon)}), \tag{3.25}$$

with $b^{(N)}_{\sigma+n} = -\frac{1}{m} \operatorname{res} U$; *in other words,*

$$l_0(BG_\lambda) = -\tfrac{1}{m} \operatorname{res} U. \tag{3.26}$$

Proof We use here that $G^{(N)}_\lambda$ cut down to interior coordinate patches is of order $-\infty$ and rapidly decreasing in λ, so that such coordinate patches contribute only to the remainder in (3.24). The alternative formulation in (3.25) is obtained by denoting $l + 1 = j$ and relabeling $\tilde{u}^{(N)}_l = b^{(N)}_{l+1}$. □

In the case where $m > \sigma + n$, this result can simply be added to (2.4), implying the validity of (2.3) with $N = 1$ in this case.

We also want to establish (2.3) for general N, $m = 2$, which requires proving a version of [8, Theorem 3.6] where $B(Q_{1,\lambda} - Q_{2,\lambda})_+$ for auxiliary operators of order

$m > \sigma + n$ is replaced by $B(Q^N_{1,\lambda} - Q^N_{2,\lambda})_+$ for auxiliary operators of order 2. This can be done with the same methods as used for [8, Theorems 3.6, 3.10], and could also be done more mechanically with the technology of Sect. 5 there. Since there are no new difficulties in this, the explanation will be brief. Without extra effort, we can let the auxiliary differential operators have an arbitrary positive order m.

Theorem 3.6 *Let P_1 and P_2 be auxiliary elliptic differential operators on $\widetilde{X}$ of order $m > 0$ with no eigenvalues on $\mathbb{R}_-$, let $Q_{i,\lambda} = (P_i - \lambda)^{-1}$, and consider $B(Q^N_{1,\lambda} - Q^N_{2,\lambda})_+$ on X, decomposed in its ψdo part and s.g.o. part*

$$B\left(Q^N_{1,\lambda} - Q^N_{2,\lambda}\right)_+ = \left(P\mathcal{Q}^{(N)}_\lambda\right)_+ + \mathcal{G}^{(N)}_\lambda, \quad \textit{where } \mathcal{Q}^{(N)}_\lambda = Q^N_{1,\lambda} - Q^N_{2,\lambda}. \tag{3.27}$$

For $N > (\sigma + n)/m$, the ψdo part and s.g.o. parts have trace expansions

$$\begin{aligned} \operatorname{Tr}\left(P\mathcal{Q}^{(N)}_\lambda\right)_+ &= \sum_{0\le j\le n+\sigma} c^{(N)}_j (-\lambda)^{\frac{n+\sigma-j}{m}-N} + O\left(\lambda^{-N-\frac{1}{4m}(+\varepsilon)}\right), \\ \operatorname{Tr}\mathcal{G}^{(N)}_\lambda &= \sum_{1\le j\le n+\sigma} d^{(N)}_j (-\lambda)^{\frac{n+\sigma-j}{m}-N} + O\left(\lambda^{-N-\frac{1}{4m}(+\varepsilon)}\right), \end{aligned} \tag{3.28}$$

with $(+\varepsilon)$ understood as in Theorem 3.4. Here the coefficients of $(-\lambda)^{-N}$ are zero if $\sigma + n \notin \mathbb{N}$, and otherwise, in terms of local coordinates,

$$c^{(N)}_{\sigma+n} = -\tfrac{1}{m}\int_X \operatorname{res}_x(P(\log P_1 - \log P_2))\,dx, \tag{3.29}$$

$$d^{(N)}_{\sigma+n} = -\tfrac{1}{m}\int_{X'} \operatorname{res}_{x'} S\,dx', \ \textit{with } S = \tfrac{i}{2\pi}\int_{\mathcal{C}} \log\lambda\ \operatorname{tr}_n(\mathcal{G}^{(1)}_\lambda)\,d\lambda. \tag{3.30}$$

If m is even, (2.12) *holds with the residue defined in* [4].

Proof The ψdo part is dealt with by methods as in [8, Sect. 2]. The crucial fact is that the symbol of $\mathcal{Q}^{(N)}_\lambda$, as a difference between two iterated resolvent symbols, has homogeneous terms that are rational functions of (ξ, λ) with a one step better fall-off in λ than the individual symbols of the iterated resolvents $Q^N_{i,\lambda}$, as in [8, Proposition 2.1] (it is only the leading term that needs some thought). For the composition with P this implies that there is a strictly homogeneous symbol, integrable at $\xi = 0$, which produces the term with $(-\lambda)^{-N}$ in the first line of (3.28). Then a reformulation in polar coordinates, plus a comparison with log-formulas as in the proof of [8, Theorem 2.2], lead to a diagonal kernel expansion that integrates over X to give (3.28) with (3.29).

For the s.g.o. term, if $\sigma \in \mathbb{Z}$, the symbolic properties of $\mathcal{Q}^{(N)}_\lambda$ are used as in [8, Theorem 3.6] to see that there is a strictly homogeneous term in the symbol of $\operatorname{tr}_n \mathcal{G}^{(N)}_\lambda$, integrable at $\xi' = 0$, that produces the term with $(-\lambda)^{-N}$ in the second line of (3.28). It is interpreted as in [8, Theorem 3.10, Remark 3.12] to give (3.30). If $\sigma \notin \mathbb{Z}$, there is no such term, only the remainder.

When m is even, $\log P_1 - \log P_2$ has the transmission property, and

$$G' = -L(P, \log P_1 - \log P_2) + G(\log P_1 - \log P_2)_+$$

is a singular Green operator in the calculus. It is seen as in the proof of [8, Theorem 3.6] that $\operatorname{res}_{x'} S = \operatorname{res}_{x'} \operatorname{tr}_n G'$ (in the relevant term, the log-integration can be moved outside tr_n). Then indeed,

$$\begin{aligned} C_0(B, P_{1,+}) - C_0(B, P_{2,+}) &= -\tfrac{1}{m} \operatorname{res}(P(\log P_1 - \log P_2))_+ - \tfrac{1}{m} \operatorname{res}_{X'} S \\ &= -\tfrac{1}{m} \operatorname{res}(B(\log P_1 - \log P_2)_+). \end{aligned} \tag{3.31}$$

□

As remarked earlier, the interpretation in [8, Theorem 3.6] of the trace defect as a residue in the sense of [4] holds only when m is even; otherwise it can be regarded as a residue defined in a more general sense. We note in passing that there is a misprint in the statement of the theorem there; $\sigma - \frac{1}{4}$ should be replaced by $\sigma + \frac{1}{4}$ in line 6 of p. 1691.

Corollary 3.7 (1) *The expansion* (2.3) *holds with* $N = 1$ *for general systems* $\{P_1, T\}$ *of order* $m > \sigma + n$.

(2) *For general second-order elliptic operators* P_1*, the operator family* $BQ^N_{\lambda,+}$ *has trace expansions* (2.23) *when* $N > (\sigma + n)/2$.

(3) *The expansion* (2.3) *holds for general systems* $\{P_1, T\}$ *of order* 2*, with* $N > (\sigma + n)/2$.

Proof For (1) we have already observed that it follows by adding the result of Corollary 3.5 with $m > \sigma + n$, $N = 1$, to (2.4).

For (2), we consider a general operator P_1 together with a special choice P_2 as in [11]; then the result from there on the expansion of $\operatorname{Tr}(B(P_2 - \lambda)^{-N}_+)$ taken together with Theorem 3.6 on $\operatorname{Tr}(B((P_1 - \lambda)^{-N}_+ - (P_2 - \lambda)^{-N}_+))$ implies the statement.

Now (2.3) is obtained for a general second-order realization $P_{1,T}$ by combination with Corollary 3.5 for $m = 2$, $N > (\sigma + n)/2$. □

There is more information on the case of general even m in Sect. 7.

Remark 3.8 For auxiliary operators P_1 of order 1, Theorems 3.4–3.6 are valid. Results as in Corollary 3.7 can also be shown, under slightly restrictive circumstances:

Assume that P_1 can be chosen skew-selfadjoint of order 1, acting in the bundle $\widetilde{E}$ over $\widetilde{X}$. Then we can use the trick (found e.g., in [13]) of introducing a "doubled" operator (also skew-selfadjoint)

$$\mathcal{P}_1 = \begin{pmatrix} 0 & P_1 \\ P_1 & 0 \end{pmatrix}, \tag{3.32}$$

whose resolvent is

$$(\mathcal{P}_1-\lambda)^{-1}=\begin{pmatrix}\lambda(P_1^2-\lambda^2)^{-1} & P_1(P_1^2-\lambda^2)^{-1}\\ P_1(P_1^2-\lambda^2)^{-1} & \lambda(P_1^2-\lambda^2)^{-1}\end{pmatrix}, \tag{3.33}$$

for $\lambda\in\mathbb{C}\setminus i\mathbb{R}$. Define also

$$\mathcal{B}=\begin{pmatrix}B & 0\\ 0 & B\end{pmatrix}, \tag{3.34}$$

then if B is of so low order that the trace exists ($\sigma+n<1$),

$$\operatorname{Tr}(\mathcal{B}(\mathcal{P}_1-\lambda)_+^{-1})=2\lambda\operatorname{Tr}(B(P_1^2-\lambda^2)_+^{-1}).$$

Since P_1^2 is selfadjoint negative, the right-hand side has an expansion in powers of λ on suitable rays, by the preceding results. For the left-hand side, this gives a trace expansion for one special case of a first-order operator. Thanks to Theorem 3.6, we can then also get expansions for $\operatorname{Tr}(\mathcal{B}(\mathcal{P}'-\lambda)_+^{-1})$ for other choices of first-order operators $\mathcal{P}'$. Take e.g.,

$$\mathcal{P}'=\begin{pmatrix}P_1 & 0\\ 0 & P_1\end{pmatrix}, \tag{3.35}$$

then

$$\operatorname{Tr}(\mathcal{B}(\mathcal{P}'-\lambda)_+^{-1})=2\operatorname{Tr}(B(P_1-\lambda)_+^{-1}),$$

from which we obtain an expansion of $\operatorname{Tr}(B(P_1-\lambda)_+^{-1})$ as in (2.3), on suitable rays. Now Theorem 3.6 can be used again, to allow other first-order elliptic operators P_2 in the place of P_1.

If B is not of low order, one must work with derivatives $\frac{\partial_\lambda^{N-1}}{(N-1)!}$ of $\operatorname{Tr}(\mathcal{B}(\mathcal{P}_1-\lambda)_+^{-1})$ to get the result, which gives more complicated calculations.

Thanks to Corollary 3.5, the analysis extends to cases where $(P_2-\lambda)_+^{-1}$ is replaced by $(P_{2,T}-\lambda)^{-1}$, where P_2 is provided with an elliptic differential boundary condition $Tu=0$.

Let us remark that not all first-order elliptic operators have elliptic differential boundary conditions. For Dirac operators there are in any case the *pseudodifferential* Atiyah–Patodi–Singer boundary conditions (and their generalizations), but the present set-up would not include them; this requires different efforts (cf. e.g., [6]), since the singular Green part G_λ then has low regularity. (There is much other current literature on that subject, also from the author's side, but we shall not lengthen the reference list with this).

Having established (2.3) for general N, $m=2$, we go on to the detailed analysis of $C_0(B,P_{1,T})$, treating the contributions from the five terms in (2.22) one by one.

4 The constant coming from $P_+ Q_{\lambda,+}$; localization

We now begin the analysis of the contributions to $a_0'' = C_0(B, P_{1,T})$. For the contributions from $BQ^N_{\lambda,+} = P_+ Q^N_{\lambda,+} + GQ^N_{\lambda,+}$ in (2.22) (which together give the term $C_0(B, P_{1,+})$), it is an advantage to start with a very simple case where P_1 is similar to the Laplacian.

As in [9], we make a reduction to local coordinates and choose P_1 with essentially constant coefficients there. This goes as follows: X is covered by a system of open subsets U_j, $j = 1, \dots, J$, of $\widetilde{X}$, with trivializations $\Phi_j : \widetilde{E}|_{U_j} \to V_j \times \mathbb{C}^M$ ($M = \dim \widetilde{E}$) and base maps $\kappa_j : U_j \to V_j$, with V_j bounded in $\mathbb{R}^n$. Now $\{\psi_j\}_{1\le j\le J}$ is an associated partition of unity (with $\psi_j \in C_0^\infty(U_j)$), and $\varphi_j \in C_0^\infty(U_j)$ with $\varphi_j = 1$ on $\operatorname{supp}\psi_j$. Then

$$B = \sum_{1\le j\le J} \psi_j B = \sum_{1\le j\le J} \psi_j B\varphi_j + \sum_{1\le j\le J} \psi_j B(1-\varphi_j),$$

where the last sum is of order $-\infty$; for this part it is well-known that $C_0(\psi_j B(1-\varphi_j), P_{1,+}) = \operatorname{Tr}(\psi_j B(1-\varphi_j))$. So it remains to treat each of the terms $\psi_j B\varphi_j$.

For the present case of a manifold with boundary we can assume that, say, the sets with $j = 1, \dots, J_0$ intersect ∂X and the remaining sets with $j = J_0+1, \dots, J$ have closures lying in the interior of X, and we can for $j \le J_0$ take the V_j of the form $W_j \times\,]-c, c[\,$, $W_j \subset \mathbb{R}^{n-1}$ such that $W_j = \kappa_j(U_j \cap \partial X)$.

For $j > J_0$, $\psi_j B\varphi_j = \psi_j(P + \mathcal{R}_j)\varphi_j$ with $\mathcal{R}_j$ a ψdo of order $-\infty$ (since s.g.o.s are smoothing on the interior of X), so these terms are essentially covered by the analysis in [9], and by the account we give for the ψdo part below. Therefore we can restrict the attention to one of the terms with $j \le J_0$, say, $\psi_1 B\varphi_1$.

As noted in [9], one can assume that X is already covered by $U_{10}, U_2, \dots, U_J$ with $\overline{U}_{10}$ compact in U_1, ψ_1 and φ_1 supported in U_{10}, and introduce $U_1' = U_1$, $U_j' = U_j \setminus \overline{U}_{10}$ for $j \ge 2$, as a new cover of X with associated partition of unity ψ_j' and functions $\varphi_j' \in C_0^\infty(U_j')$ equal to 1 on $\operatorname{supp}\psi_j'$, $1 \le j \le J$. This allows us to choose

$$P_1 u = \sum_{1\le j\le J} \varphi_j'[(-\Delta) I_M((\psi_j' u)\circ \Phi_j^{*-1})]\circ \Phi_j^*, \tag{4.1}$$

as the auxiliary elliptic operator, such that when $\psi_j B\varphi_j$ is carried over to $\widetilde{B}$ in the coordinate system V_1, $\psi_j B\varphi_j(P_1-\lambda)^{-1}$ is carried over to $\widetilde{B}(-\Delta-\lambda)^{-1} I_M + \mathcal{R}_\lambda$ in V_1, where $\mathcal{R}_\lambda$ is of order $-\infty$ with a trace that is $O(\lambda^{-N'})$, any N' (with similar properties of λ-derivatives). I_M stands for the $M \times M$ identity matrix, not mentioned explicitly in the following (cf. [9] for more details).

This reduces the problem to a calculation of the trace expansion of operators of the form $\widetilde{B}(-\Delta-\lambda)^{-N}$ on $\mathbb{R}^n_+$, where we can work out the kernel explicitly. (In [9], $1-\Delta$ was used rather than $-\Delta$, since $\log(1-\Delta)$ makes sense on $\mathbb{R}^n$, but the lack of homogeneity is a disadvantage when we calculate more complicated boundary contributions; instead we shall modify the symbol near 0 when needed).

After the reduction, we again write $\widetilde{B}$ as $P_+ + G$, with symbols $p(x,\xi)$ resp. $g(x',\xi',\xi_n,\eta_n)$, functions on $\mathbb{R}^{2n}$.

In the rest of the present section we consider the contributions from $P_+ Q_{\lambda,+}$ and its iterated versions; recall that the order σ of P is an integer. The operator $P_+ Q^N_{\lambda,+}$ is written as a sum of two terms

$$P_+ Q^N_{\lambda,+} = (P Q^N_\lambda)_+ - L(P, Q^N_\lambda) = \tfrac{\partial_\lambda^{N-1}}{(N-1)!}[(P Q_\lambda)_+ - L(P, Q_\lambda)], \tag{4.2}$$

that are treated in different ways. The first term is the truncation to $\mathbb{R}^n_+$ of the ψdo PQ^N_λ on $\mathbb{R}^n$, whereas the second term, the "leftover term", is a composition of s.g.o.s; see (4.17)ff. later. We see in the following that for the first term one can use the method of [9], for the second term that of Sect. 3.

For the first term, the pointwise calculations are essentially as in [9], relative to the closed manifold $\widetilde{X}$, and we get the contribution to the basic zeta value as the integral over X of the pointwise contribution (pulled back from local coordinates). A short explanation will now be given.

Proposition 4.1 *When $N > (\sigma+n)/2$, the kernel of PQ^N_λ on the diagonal $\{x = y\}$ has an expansion*:

$$\begin{aligned} K(PQ^N_\lambda, x, x) &= \sum_{0\le j<\sigma+n} (-\lambda)^{\frac{\sigma+n-j}{2}-N} c_j(x) \\ &\quad + \tfrac{\partial_\lambda^{N-1}}{(N-1)!}[(-\lambda)^{-1}\log(-\lambda) c_0'(x) + (-\lambda)^{-1} c_0''(x)] + O\left(\lambda^{-N-\frac12+\varepsilon}\right) \\ &= \sum_{0\le j<\sigma+n} (-\lambda)^{\frac{\sigma+n-j}{2}-N} c_j(x) + (-\lambda)^{-N}\log(-\lambda) c_0'(x) \\ &\quad + (-\lambda)^{-N}(c_0''(x) - \alpha_N c_0'(x)) + O\left(\lambda^{-N-\frac12+\varepsilon}\right), \end{aligned} \tag{4.3}$$

any $\varepsilon > 0$, for $\lambda \to \infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. Here $\alpha_N = \sum_{1\le k<N} \frac{1}{k}$, and

$$\begin{aligned} c_j(x) &= \int_{\mathbb{R}^n} p^h_{\sigma-j}(x,\xi)(|\xi|^2+1)^{-N}\, đ\xi, \\ c_0'(x) &= \tfrac12 \int_{|\xi|=1} p_{-n}(x,\xi)\, đS(\xi), \quad c_0''(x) = ⨍ p(x,\xi)\, đ\xi; \end{aligned} \tag{4.4}$$

then (*cf.* (2.16), (2.19))

$$\operatorname{tr} c_0'(x) = \tfrac12 \operatorname{res}_x P, \quad \operatorname{tr} c_0''(x) = \operatorname{TR}_x P. \tag{4.5}$$

Proof This is a purely pseudodifferential situation, and the proof goes as in [9], except that we must in addition account for the effect of the λ-derivative. Using the expansion

$p \sim \sum_{j \in \mathbb{N}} p_{\sigma-j}(x,\xi)$ in terms $p_{\sigma-j}(x,\xi)$ homogeneous of degree $\sigma - j$ in ξ for $|\xi| \geq 1$, we write

$$p(x,\xi) = \sum_{0 \leq j < \sigma+n} p_{\sigma-j}(x,\xi) + p_{-n}(x,\xi) + p_{<-n}(x,\xi).$$

The finite-part integral $⨍ p(x,\xi)\, đ\xi$ is defined as recalled above after (2.16). Here (cf. e.g., [7, (1.18)])

$$\begin{aligned} ⨍ p_{\sigma-j}(x,\xi)\, đ\xi &= \int_{|\xi|\leq 1} (p_{\sigma-j}(x,\xi) - p^h_{\sigma-j}(x,\xi))\, đ\xi, \text{ for } \sigma - j > -n, \\ ⨍ p_{-n}(x,\xi)\, đ\xi &= \int_{|\xi|\leq 1} p_{-n}(x,\xi)\, đ\xi, \\ ⨍ p_{<-n}(x,\xi)\, đ\xi &= \int_{\mathbb{R}^n} p_{<-n}(x,\xi)\, đ\xi, \end{aligned} \tag{4.6}$$

where $p^h_{\sigma-j}$ denotes the strictly homogeneous version of $p_{\sigma-j}$ (the extension of $p_{\sigma-j}|_{|\xi|\geq 1}$ into $\{1 > |\xi| > 0\}$ that is homogeneous for $|\xi| > 0$).

The kernel of PQ^N_λ is the integral in ξ of its symbol times $e^{i(x-y)\cdot\xi}$. Its value for $x = y$ is then

$$K(PQ^N_\lambda, x, x) = K(P(-\Delta - \lambda)^{-N}, x, x) = \int_{\mathbb{R}^n} p(x,\xi)(|\xi|^2 - \lambda)^{-N}\, đ\xi, \tag{4.7}$$

which we shall expand in powers of λ.

The symbols $p^h_{\sigma-j}$ with $j < \sigma + n$ are integrable at $\xi = 0$, so these terms $p_{\sigma-j}$ produce, by homogeneity, the sum over $j < \sigma + n$ in (4.3) plus $(-\lambda)^{-N} ⨍ \sum_{j<\sigma+n} p_{\sigma-n}\, đ\xi$, as in the proof of [9, Lemma 4.1], with an error that is $O(\lambda^{-N-1})$.

For the symbol p_{-n}, one has for any $N \geq 1$ that

$$\int_{\mathbb{R}^n} p_{-n}(x,\xi)(|\xi|^2 - \lambda)^{-N}\, đ\xi = \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{\mathbb{R}^n} p_{-n}(x,\xi)(|\xi|^2 - \lambda)^{-1}\, đ\xi, \tag{4.8}$$

and the proof in [9, Lemma 4.1] can be used word for word, when it is furthermore remarked that the expansion of $\log(1 + s)$ allows taking derivatives; this gives

$$\int_{\mathbb{R}^n} p_{-n}(|\xi|^2 - \lambda)^{-N}\, đ\xi = \frac{\partial_\lambda^{N-1}}{(N-1)!} \Bigg[\tfrac{1}{2}(-\lambda)^{-1} \log(-\lambda) \int_{|\xi|=1} p_{-n}\, đS(\xi) + (-\lambda)^{-1} ⨍ p_{-n}\, đ\xi \Bigg] + O(\lambda^{-N-1})$$

$$= (-\lambda)^{-N}\log(-\lambda)c_0' + (-\lambda)^{-N}\left[⨍ p_{-n}\, đ\xi - \alpha_N c_0'\right]$$
$$+ O(\lambda^{-N-1}),$$

for any $N \geq 1$; α_N comes from differentiating the log-term.

Finally, since $p_{<-n}$ and $|\xi|^{1-\varepsilon'} p_{<-n}$ are integrable for $\varepsilon' \in]0,1]$, we find

$$\int_{\mathbb{R}^n} p_{<-n}(x,\xi)(|\xi|^2 - \lambda)^{-N}\, đ\xi = (-\lambda)^{-N}\int_{\mathbb{R}^n} p_{<-n}(x,\xi)\, đ\xi + O\left(\lambda^{-N-\frac{1}{2}+\varepsilon}\right), \tag{4.9}$$

any $N \geq 1$ and $\varepsilon > 0$, by insertion of an expansion of $(|\xi|^2 - \lambda)^{-N}$:

$$\begin{aligned}
(|\xi|^2 - \lambda)^{-1} &= (-\lambda)^{-1} - (-\lambda)^{-1}|\xi|^2(|\xi|^2-\lambda)^{-1}\\
(|\xi|^2 - \lambda)^{-N} &= (-\lambda)^{-N} - \sum_{0\leq k<N} c_k(-\lambda)^{-k}((-\lambda)^{-1}|\xi|^2(|\xi|^2-\lambda)^{-1})^{N-k}\\
&= (-\lambda)^{-N} - (-\lambda)^{-N}\sum_{0\leq k<N} c_k(|\xi|^2(|\xi|^2-\lambda)^{-1})^{N-k}\\
&= (-\lambda)^{-N} + O\left((-\lambda)^{-N-\frac{1}{2}+\varepsilon}|\xi|^{1-2\varepsilon}\right),\ \ \varepsilon \in]0, \tfrac{1}{2}[\,.
\end{aligned} \tag{4.10}$$

This shows (4.3). □

Proposition 4.2 *The operator family* $(PQ_\lambda^N)_+ = (P(-\Delta-\lambda)^{-N})_+$ *in* (4.2) *on* $\mathbb{R}^n_+$ *has for* $N > (\sigma+n)/2$ *expansions as in* (2.23)*, with* $\delta = \frac{1}{2} - \varepsilon$*, any* $\varepsilon > 0$*, for* $\lambda \to \infty$ *on rays in* $\mathbb{C}\setminus\mathbb{R}_+$*. Here* $a_0' = \frac{1}{2}\operatorname{res}(P_+)$ *and*

$$l_0((P(-\Delta-\lambda)^{-1})_+) = \int_{\mathbb{R}^n_+}\left[\mathrm{TR}_x\, P - \tfrac{1}{2}\operatorname{res}_{x,0}(P\log P_1')\right]dx; \tag{4.11}$$

with $\operatorname{res}_{x,0}(P\log P_1') = 0$.

Proof This is found by integrating the fiber trace of the expansion in (4.3) with respect to $x \in \mathbb{R}^n_+$ (recall that the symbol of P has compact x-support in this situation). This gives rise to the coefficients a_0' and l_0 as described. Here $\operatorname{res}_{x,0}(P\log P_1')$ is zero, because $\log P_1'$ has symbol $2\log[\xi]$, so that the log-free term of order $-n$ in the symbol of $P\log P_1'$ vanishes for $|\xi| \geq 1$. □

We have included the trivial term with $\operatorname{res}_{x,0}(P\log P_1')$ in (4.11) for the sake of generalizations to other auxiliary operators P_2.

As explained in the beginning of this section, we can choose P_1 on the manifold so that its resolvent is similar to $(-\Delta-\lambda)^{-1}$ in the specially selected local coordinates. Then, summing over the coordinate patches, we find that $(P(P_1-\lambda)^{-N})_+$ has a trace expansion (2.23), where l_0 is a sum of contributions of the form

$$\varphi\int\left[\mathrm{TR}_x\, P - \tfrac{1}{2}\operatorname{res}_{x,0}(P\log P_1)\right]dx\,\psi, \tag{4.12}$$

with cutoff functions φ, ψ. Also here, $\operatorname{res}_{x,0}(P \log P_1)$ is 0, since P_1 in the local coordinates has the same symbol terms as P_1' for $|\xi| \geq 1$. Now the formula will be extended to general choices of auxiliary operator as follows:

When P_2 is a general auxiliary elliptic operator of order 2, the kernel of $P((P_2 - \lambda)^{-N} - (P_1 - \lambda)^{-N})$ has an expansion on the diagonal (calculated in local coordinates):

$$K(P((P_2-\lambda)^{-N}-(P_1-\lambda)^{-N}), x, x)=\sum_{0\le j\le \sigma+n} s_j^{(N)}(x)(-\lambda)^{\frac{\sigma+n-j}{2}-N}+O\left(\lambda^{-N-\frac{1}{2}}\right), \tag{4.13}$$

with

$$s_{\sigma+n}^{(N)}(x) = s_{\sigma+n}(x) = -\tfrac{1}{2} \int_{|\xi|=1} \operatorname{symb}_{-n}(P(\log P_2 - \log P_1))(x,\xi)\, đS(\xi). \tag{4.14}$$

The general idea of proof of this is given in [8, Proposition 2.1, Theorem 2.2], and the adaptation to the situation with Nth powers is given in [9, Sect. 3]. Since $\log P_2 - \log P_1$ is a classical ψdo having the transmission property (cf. [10, Lemma 2.1]), so is $P(\log P_2 - \log P_1)$, so the integral over X of the fiber trace of (4.14) is precisely $-\frac{1}{2}\operatorname{res}([P(\log P_2 - \log P_1)]_+)$ defined as in [4]. Thus integration over X of the fiber trace of (4.13) shows that $[P((P_2 - \lambda)^{-N} - (P_1 - \lambda)^{-N})]_+$ is as in Definition 2.2 with $\delta = \frac{1}{2}$ and

$$\begin{aligned} l_0([P((P_2-\lambda)^{-1} - (P_1-\lambda)^{-1})]_+) &= -\tfrac{1}{2}\operatorname{res}([P(\log P_2 - \log P_1)]_+) \\ &= -\tfrac{1}{2}\int_X \operatorname{res}_x(P(\log P_2 - \log P_1))\, dx. \end{aligned} \tag{4.15}$$

Here the symbol of $P(\log P_2 - \log P_1)$ is classical, without logarithmic terms, so $\operatorname{res}_{x,0}$ of it coincides with res_x of it. Then $\operatorname{res}_{x,0}(P \log P_2) = \operatorname{res}_x(P(\log P_2 - \log P_1)) + \operatorname{res}_{x,0}(P \log P_1)$. Adding (4.15) to the result for $l_0((P(P_1-\lambda)^{-1})_+)$, we conclude:

Theorem 4.3 *For a ψdo P of order $\sigma \in \mathbb{Z}$ satisfying the transmission condition at X', together with a general elliptic differential operator P_2 of order 2 having $\mathbb{R}_-$ as a spectral cut, the operator family $(P(P_2-\lambda)^{-N})_+$ has for $N > (\sigma+n)/2$ expansions as in (2.23), with $\delta = \frac{1}{2} - \varepsilon$, any $\varepsilon > 0$, for $\lambda \to \infty$ on rays in the sector V around $\mathbb{R}_-$ where $p_2^0 - \lambda$ is invertible. Here*

$$l_0((P(P_2-\lambda)^{-1})_+) = \int_X \left[\operatorname{TR}_x P - \tfrac{1}{2}\operatorname{res}_{x,0}(P \log P_2)\right] dx. \tag{4.16}$$

Next, we study the second term $-L(P, Q_\lambda^N)$ in (4.2). In the localization we are considering, the operator is built up of s.g.o.s as follows:

$$
\begin{aligned}
L(P, Q_\lambda^N) &= G^+(P)G^-(Q_\lambda^N), \text{ where}\\
G^+(P) &= r^+Pe^-J, \quad G^-(Q_\lambda^N) = Jr^-Q_\lambda^N e^+, \quad \text{with}\\
J &: u(x', x_n) \mapsto u(x', -x_n);
\end{aligned}
\tag{4.17}
$$

cf. [10, (2.6.5)ff.]. (Here $r^\pm$ denotes restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ and $e^\pm$ denotes extension by 0 from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$, respectively.) This term contributes in a different way than $(PQ_\lambda^N)_+$. It was shown in [11, Sect. 3] that the trace expansion of $-G^+(P)G^-(Q_\lambda^N)$ has no term of the form $c(-\lambda)^{-N}\log(-\lambda)$ and that the coefficient of $(-\lambda)^{-N}$ is local. The arguments there were based on Laguerre expansions, which give quite complicated formulas (see e.g., Lemmas A.1 and A.2 in the Appendix), so it is preferable to find a more robust method giving a simpler information on the coefficient of $(-\lambda)^{-N}$. We shall here use the strategy of [8, Sect. 4] recalled in Sect. 3, invoking the *regularity* of parameter-dependent symbols introduced in [5]. In the notation of [5], the parameter is $\mu = |\lambda|^{\frac{1}{2}}$, where λ runs on a ray in $\mathbb{C}\setminus\mathbb{R}_+$, $-\lambda = \mu^2 e^{i\theta}$ (for some $\theta \in\,]-\pi, \pi[$).

Lemma 4.4 *Consider $G^+(P)G^-(Q_\lambda^N)$; the order σ of P is integer.*

(1) *For each $N \geq 1$, the s.g.o. $-G^+(P)G^-(Q_\lambda^N)$ has order $\sigma - 2N$, class 0 and regularity σ. Moreover, it can be written as*

$$
\begin{aligned}
-G^+(P)G^-(Q_\lambda) &= (-\lambda)^{-1}G^+(P)P_1G^-(Q_\lambda) \ \text{ if } N = 1,\\
-G^+(P)G^-(Q_\lambda^N) &= (-\lambda)^{-N}G^+(P)P_1G_\lambda^{-,N} \ \text{ in general},
\end{aligned}
\tag{4.18}
$$

where $G^+(P)P_1G_\lambda^{-,N}$ has order σ, class 0 and regularity $\sigma + \frac{1}{2}$. ($G_\lambda^{-,N}$ is described in (4.24) below).

(2) *The normal trace*

$$
\mathcal{S}_\lambda^{(N)} = \operatorname{tr}_n(-G^+(P)G^-(Q_\lambda^N)) = \frac{\partial_\lambda^{N-1}}{(N-1)!}\mathcal{S}_\lambda^{(1)}
\tag{4.19}
$$

is a ψdo on $\mathbb{R}^{n-1}$ of order $\sigma - 2N$ and regularity $\sigma - \frac{1}{4}$. It can also be written as

$$
\mathcal{S}_\lambda^{(N)} = (-\lambda)^{-N}\mathcal{S}_\lambda^{(N)\prime}, \quad \mathcal{S}_\lambda^{(N)\prime} = \operatorname{tr}_n[G^+(P)P_1G_\lambda^{-,N}],
\tag{4.20}
$$

where $\mathcal{S}_\lambda^{(N)\prime}$ has order σ and regularity $\sigma + \frac{1}{4}$.

(3) *The symbol $\mathfrak{s}^{(N)}(x', \xi', \lambda)$ of $\mathcal{S}_\lambda^{(N)}$, with the expansion in (quasi-) homogeneous terms $\mathfrak{s}^{(N)}(x', \xi', \lambda) \sim \sum_{j\in\mathbb{N}} \mathfrak{s}^{(N)}_{\sigma-2N-j}(x', \xi', \lambda)$ (homogeneous in (ξ', μ) on each*

ray $\lambda = -\mu^2 e^{i\theta}$, $\mu > 0$), *satisfies*:

$$\Big|\partial^{\beta,\alpha}_{x',\xi'}[\mathfrak{s}^{(N)}(x',\xi',\lambda) - \sum_{j<J} \mathfrak{s}^{(N)}_{\sigma-2N-j}(x',\xi',\lambda)]\Big| \dot{\le} \langle\xi'\rangle^{\sigma-\frac14-|\alpha|-J}\langle\xi',\mu\rangle^{-2N+\frac14},$$
$$\Big|\partial^{\beta,\alpha}_{x',\xi'}[\mathfrak{s}^{(N)}(x',\xi',\lambda) - \sum_{j<J} \mathfrak{s}^{(N)}_{\sigma-2N-j}(x',\xi',\lambda)]\Big| \dot{\le} \langle\xi'\rangle^{\sigma+\frac14-|\alpha|-J}\langle\xi',\mu\rangle^{-\frac14}\mu^{-2N}, \tag{4.21}$$

on the rays in $\mathbb{C}\setminus\mathbb{R}_+$, *for all* α, β, J.

Proof Consider first the case $N = 1$. Since Q_λ is strongly polyhomogeneous (the strictly homogeneous symbol is smooth in ξ and λ for $|\xi| + |\lambda| \neq 0$), it is of regularity $+\infty$, and so is the s.g.o. $G^-(Q_\lambda)$, of class 0. $G^+(P)$ is λ-independent of order σ and class 0, hence has regularity σ by [5, (2.3.54)], so by the composition rules (cf. e.g., [5, (2.6.5) 10°]), the composed operator has order $\sigma - 2$, class 0 and regularity σ. For $N > 1$ we have similarly, since Q^N_λ is of order $-2N$ and regularity $+\infty$, that $G^+(P)G^-(Q^N_\lambda)$ has order $\sigma - 2N$, class 0 and regularity σ. This shows the first statement in (1).

For the second statement in (1), we use that

$$\begin{aligned}(P_1-\lambda)^{-1} &= (-\lambda)^{-1} - (-\lambda)^{-1}P_1(P_1-\lambda)^{-1}\\ (P_1-\lambda)^{-N} &= (-\lambda)^{-N} - (-\lambda)^{-N}\sum_{0\le k<N} c_k(P_1(P_1-\lambda)^{-1})^{N-k}\\ &= (-\lambda)^{-N} - (-\lambda)^{-N}P_1\sum_{1\le j\le N} c_{N-j}P_1^{j-1}(P_1-\lambda)^{-j}\end{aligned} \tag{4.22}$$

as in (4.10); here $G^-(\lambda^{-N}) = 0$. Since P_1 is a differential operator, $G^-(P_1\, Q_\lambda) = P_1 G^-(Q_\lambda)$, and we find in the case $N = 1$:

$$-G^+(P)G^-(Q_\lambda) = (-\lambda)^{-1}G^+(P)P_1G^-(Q_\lambda), \tag{4.23}$$

where $G^+(P)P_1$ is a λ-independent s.g.o. of order $\sigma + 2$ and class 2, hence has regularity $\sigma + \frac12$ by [5, (2.3.55)]. Then the composed operator $G^+(P)P_1G^-(Q_\lambda)$ is of order σ, class 0 and regularity $\sigma + \frac12$. For general N we use that the last sum in (4.22) is of order -2 and regularity $+\infty$. Here (4.18) holds with

$$G^{-,N}_\lambda = G^-\Bigg(\sum_{1\le j\le N} c_{N-j}P_1^{j-1}(P_1-\lambda)^{-j}\Bigg). \tag{4.24}$$

The statements in (2) now follow by use of [8, Lemma 3.4], which shows a loss of regularity $\frac14$ in general when tr_n is applied.

The information in (3) follows directly from the definition of the class of symbols of the stated regularity, when $\sigma < 0$. When $\sigma \ge 0$, the regularity information itself gives weaker estimates when $\sigma - |\alpha| - J \geq 0$; here we use the device from the proof of [8, Proposition 4.3]: Compose $\mathcal{S}_\lambda^{(N)}$ to the left with $\Lambda^\varrho \Lambda^{-\varrho}$, $\Lambda = \mathrm{OP}(\langle \xi' \rangle)$, with $\varrho \geq \sigma + 1$. Taking $\Lambda^{-\varrho}$ together with $G^+(P)$ one finds that $\Lambda^{-\varrho}\mathcal{S}_\lambda^{(N)}$ satisfies the regularity statements with σ replaced by $\sigma - \varrho$, hence the estimates with the same replacement, and the desired estimates follow after composition with Λ^ϱ. □

The lemma makes it possible to use Theorems 3.1 and 3.2 as in Sect. 3.

Theorem 4.5 *Consider $\mathcal{S}_\lambda^{(N)}$ defined in Lemma 4.4. When $N > (\sigma + n - 1)/2$, the kernel of $\mathcal{S}_\lambda^{(N)}$ on the diagonal has an expansion*

$$K(\mathcal{S}_\lambda^{(N)}, x', x') = \sum_{0 \leq l \leq \sigma + n - 1} \tilde{\mathfrak{s}}_l^{(N)}(x')(-\lambda)^{\frac{\sigma+n-1-l}{2} - N} + O\left(\lambda^{-N-\frac{1}{8}}\right), \tag{4.25}$$

where in particular the coefficient of $(-\lambda)^{-N}$ is

$$\begin{aligned} \tilde{\mathfrak{s}}_{\sigma+n-1}^{(N)}(x') &= \int_{\mathbb{R}^{n-1}} \mathfrak{s}_{-2N-n+1}^{(N)h}(x', \xi', -1)\, đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} \mathfrak{s}_{-2-n+1}^{(1)h}(x', \xi', -1)\, đ\xi'. \end{aligned} \tag{4.26}$$

The "log-transform" S with symbol $s(x', \xi')$ deduced for $|\xi'| \geq 1$ from the symbol $\mathfrak{s}^{(1)}(x', \xi', \lambda)$ of $\mathcal{S}_\lambda^{(1)}$ by

$$\begin{aligned} s(x', \xi') &= \tfrac{i}{2\pi} \int_{\mathcal{C}} \log \lambda\, \mathfrak{s}^{(1)}(x', \xi', \lambda)\, d\lambda, \\ s_{\sigma-j}(x', \xi') &= \tfrac{i}{2\pi} \int_{\mathcal{C}} \log \lambda\, \mathfrak{s}^{(1)}_{\sigma-2-j}(x', \xi', \lambda)\, d\lambda \end{aligned} \tag{4.27}$$

(*with a curve $\mathcal{C}$ in $\mathbb{C} \setminus \overline{\mathbb{R}}_-$ around* $[1, \infty[$) *is a classical ψdo of order σ such that*

$$\begin{aligned} \tilde{\mathfrak{s}}_{\sigma+n-1}^{(N)}(x') &= -\tfrac{1}{2} \int_{|\xi'|=1} s_{1-n}(x', \xi')\, đS(\xi'), \\ \operatorname{tr} \tilde{\mathfrak{s}}_{\sigma+n-1}^{(N)}(x') &= -\tfrac{1}{2} \operatorname{res}_{x'} S. \end{aligned} \tag{4.28}$$

Proof By Lemma 4.4, the operator family $\mathcal{S}_\lambda^{(N)}$ defined there satisfies the hypotheses for Theorems 3.1 and 3.2 with $m = 2$ and $\delta = \frac{1}{4}$, $V = \mathbb{C} \setminus \overline{\mathbb{R}}_+$. For $|\xi'| \geq 1$, the symbols are holomorphic in $\lambda \in \mathbb{C} \setminus [1, \infty[$. □

Let us consider the role of S more closely. Since the coefficient we are studying is local, it makes no difference if S is modified for $|\xi'| < 1$. If we replace P_1 by

$P_1' = \mathrm{OP}([\xi]^2)$ (cf. (2.14)), its resolvent is defined for $\lambda \in \mathbb{C}\setminus[\frac12, \infty[$, and $\log P_1'$ is well-defined as $\mathrm{OP}(\log[\xi]^2)$. A simple calculation using [5, (2.6.44)] shows that $G^-(Q_\lambda) = G^-((P_1-\lambda)^{-1})$ has symbol-kernel and symbol

$$\begin{aligned}\tilde g^-(q) &= \frac{1}{2\kappa} e^{-\kappa(x_n+y_n)}, \quad \kappa = (|\xi'|^2-\lambda)^{\frac12};\\ g^-(q) &= \frac{1}{2\kappa(\kappa+i\xi_n)(\kappa-i\eta_n)};\end{aligned} \tag{4.29}$$

when $|\xi'| \ge 1$ this also holds when P_1 is replaced by P_1'.

The symbol terms in S are for $|\xi'|\ge 1$ equal to

$$\begin{aligned} s_{\sigma-j}(x',\xi') &= \tfrac{i}{2\pi}\int_{\mathcal C} \log\lambda\, \mathfrak{s}^{(1)}_{\sigma-2-j}(x',\xi',\lambda)\,d\lambda\\ &= \tfrac{-i}{2\pi}\int_{\mathcal C}\log\lambda \int_{\mathbb{R}^2} g^+(p)_{\sigma-j}(x',\xi',\xi_n,\eta_n)\tfrac{1}{2\kappa(\kappa+i\eta_n)(\kappa-i\xi_n)}\,đ\xi_n đ\eta_n d\lambda\\ &= \tfrac{-i}{2\pi}\int_{\mathcal C}\log\lambda \int_{x_n,y_n>0} \tilde g^+(p)_{\sigma-j}(x',x_n,y_n,\xi')\tfrac{1}{2\kappa}e^{-\kappa(x_n+y_n)}\,dx_n dy_n d\lambda.\end{aligned} \tag{4.30}$$

Since $\tilde g^+(p)_{\sigma-j}$ is bounded, and $\|\frac{1}{2\kappa}e^{-\kappa(x_n+y_n)}\|_{L_{1,x_n,y_n}(\mathbb{R}^2_{++})} \dot\le \kappa^{-3}$, hence is $O(\lambda^{-\frac32})$, the log-integral can be moved inside the (x_n,y_n)-integral. Then we can use, as shown in [10, Example 2.8], that in fact

$$\tfrac{i}{2\pi}\int_{\mathcal C}\log\lambda\,\tfrac{1}{2\kappa}e^{-\kappa(x_n+y_n)}\,d\lambda = -\tfrac{1}{x_n+y_n}e^{-|\xi'|(x_n+y_n)}, \text{ equal to } \tilde g^-(\log P_1') \text{ for } |\xi'|\ge 1, \tag{4.31}$$

where $\tilde g^-(\log P_1')$ is the symbol-kernel of the generalized s.g.o. $G^-(\log P_1')$. Note that it is integrable in $(x_n,y_n)\in\mathbb{R}^2_{++}$ when $\xi'\ne 0$.

The singularity of the symbol-kernel at $x_n=y_n=0$ is typical for $G^\pm(\log P_2)$ for a general elliptic differential operator P_2, as well as for the s.g.o.-like part of $\log P_{2,T}$ for a realization of P_2 defined by a differential elliptic boundary condition $Tu=0$, cf. [10].

Now we get, for $|\xi'|\ge 1$,

$$\begin{aligned} s_{\sigma-j}(x',\xi') &= \int_{x_n,y_n>0} \tilde g^+(p)_{\sigma-j}(x',x_n,y_n,\xi')\tfrac{1}{x_n+y_n}e^{-|\xi'|(x_n+y_n)}\,dx_n dy_n\\ &= -\operatorname{tr}_n(\tilde g^+(p)_{\sigma-j}\circ_n \tilde g^-(\log P_1'))\\ &= -\operatorname{symb}_{\sigma-j}\operatorname{tr}_n(G^+(P)G^-(\log P_1'));\end{aligned} \tag{4.32}$$

in short, when we recall (4.17),

$$S \sim -\operatorname{tr}_n(G^+(P)G^-(\log P_1')) = -\operatorname{tr}_n L(P,\log P_1'), \tag{4.33}$$

modulo a smoothing operator. With this, the formula for the coefficient of $(-\lambda)^{-N}$ becomes:

$$\tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') = \tfrac{1}{2} \int_{|\xi'|=1} \operatorname{symb}_{1-n}(\operatorname{tr}_n L(P, \log P'_1))(x', \xi')\, đS(\xi'), \tag{4.34}$$

i.e., for the fiber traces:

$$\operatorname{tr} \tilde{\mathfrak{s}}^{(N)}_{\sigma+n-1}(x') = -\tfrac{1}{2} \operatorname{res}_{x'} S = \tfrac{1}{2} \operatorname{res}_{x'}(\operatorname{tr}_n L(P, \log P'_1)). \tag{4.35}$$

We conclude:

Theorem 4.6 *The operator family $-L(P, (-\Delta - \lambda)^{-N})$ in (4.2) on $\mathbb{R}^n_+$ has for $N > (\sigma + n)/2$ expansions as in (2.23), with $\delta = \frac{1}{8}$, for $\lambda \to \infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. Here $a'_0 = 0$, and*

$$l_0(-L(P, (-\Delta - \lambda)^{-1})) = \tfrac{1}{2} \int_{\mathbb{R}^{n-1}} \operatorname{res}_{x'}(\operatorname{tr}_n L(P, \log P'_1))\, dx'. \tag{4.36}$$

Proof This is found by integrating the fiber trace of the expansion in (4.25) with respect to $x' \in \mathbb{R}^{n-1}$, replacing l by $j - 1$, and using the interpretation of S that we have just accounted for. □

Next, consider the manifold situation, choosing P_1 as indicated after Proposition 4.2. Then, summing over the coordinate patches, we get $l_0(-L(P, (P_1 - \lambda)^{-1}))$ as a sum of contributions of the form

$$\varphi \int \tfrac{1}{2} \operatorname{res}_{x'} \operatorname{tr}_n L(P, \log P_1)\, dx' \psi, \tag{4.37}$$

with cutoff functions φ, ψ; the integral vanishes if the local coordinate patch does not meet the boundary.

For a general auxiliary operator P_2, we have of course

$$\begin{aligned} l_0(L(P, (P_2 - \lambda)^{-1})) &= l_0(L(P, (P_1 - \lambda)^{-1})) \\ &\quad + [l_0(L(P, (P_2 - \lambda)^{-1})) - l_0(L(P, (P_1 - \lambda)^{-1}))]. \end{aligned}$$

Here the expression in [...] has been treated before, namely in the course of the proof of [8, Theorem 3.6, Remark 3.12]. Rather than taking up details from that long proof, we simply note that in view of those results and Theorem 4.3,

$$\begin{aligned} &l_0(-L(P, (P_2 - \lambda)^{-1})) - l_0(-L(P, (P_1 - \lambda)^{-1})) \\ &\quad = C_0(P_+, P_{2,+}) - l_0((P(P_2 - \lambda)^{-1})_+) \\ &\qquad - C_0(P_+, P_{1,+}) + l_0((P(P_1 - \lambda)^{-1})_+) \\ &\quad = -\tfrac{1}{2} \operatorname{res}(P_+(\log P_2 - \log P_1)_+) + \tfrac{1}{2} \operatorname{res}((P(\log P_2 - \log P_1))_+) \\ &\quad = \tfrac{1}{2} \operatorname{res}(L(P, \log P_2 - \log P_1)). \end{aligned} \tag{4.38}$$

The residues here are covered by [4] since $\log P_2 - \log P_1$ is classical having the transmission property, and the expression is calculated in local coordinates as

$$\tfrac{1}{2} \int_{X'} \operatorname{res}_{x'}(\operatorname{tr}_n L(P, \log P_2 - \log P_1))\, dx'. \tag{4.39}$$

Combining this with what we found for $l_0(-L(P, (P_1 - \lambda)^{-1}))$, we can conclude:

Theorem 4.7 *Assumptions as in Theorem 4.3. The operator family $-L(P, (P_2 - \lambda)^{-N})$ has for $N > (\sigma + n)/2$ trace expansions as in (2.23), with $\delta = \frac{1}{8}$, for $\lambda \to \infty$ on rays in V. Here $a_0' = 0$, and*

$$l_0(-L(P, (P_2 - \lambda)^{-1})) = \tfrac{1}{2} \int_{X'} \operatorname{res}_{x'}(\operatorname{tr}_n L(P, \log P_2))\, dx', \tag{4.40}$$

calculated in local coordinates.

Observe that $\log P_2$ in local coordinates can be written as a sum of $\log P_1'$ and a classical ψdo having the transmission property. Then $G^-(\log P_2) = G^-(\log P_1') + G^{-,0}$ where $G^{-,0}$ is a standard singular Green operator (a similar fact is observed in [10, Proposition 2.9]). Thus $\frac{1}{2} \operatorname{res}_{x'} \operatorname{tr}_n L(P, \log P_2) = \frac{1}{2} \operatorname{res}_{x'} \operatorname{tr}_n G^+(P) G^-(\log P_2)$ is a sum of a term with the special function in (4.31) and a term covered by the residue definition of [4].

For general P_2, it is therefore a minor extension of the definitions in [4] to *define*

$$\operatorname{res} L(P, \log P_2) = -2 l_0(L(P, (P_2 - \lambda)^{-1})). \tag{4.41}$$

This number is defined directly in terms of the manifold situation, so we have "for free" that it is independent of local coordinates, although we also have the description in local coordinates (4.40).

5 The constant coming from $G Q_{\lambda,+}$

We now study the coefficient of $(-\lambda)^{-N}$ in the expansion of $G Q_{\lambda,+}^N = \frac{\partial_\lambda^{N-1}}{(N-1)!} G Q_{\lambda,+}$. The order σ can here be any real number. Again we begin with the special localized situation explained in the beginning of Sect. 4.

This is the hardest term to treat, since its contribution is nonlocal and the homogeneity properties of the symbols of G and Q_λ do not play together in an easy way. We here take recourse to one more trace concept, that of $\mathbb{N} \times \mathbb{N}$-matrices, in the Laguerre expansion representation of the operator G.

Recall (e.g., from [5]) that g has a rapidly convergent Laguerre expansion

$$g(x', \xi', \xi_n, \eta_n) = \sum_{l,m \in \mathbb{N}} c_{lm}(x', \xi') \hat{\varphi}_l([\xi'], \xi_n) \bar{\hat{\varphi}}_m([\xi'], \eta_n), \tag{5.1}$$

with $c_{lm}(x', \xi')$ polyhomogeneous of order σ, and Fourier transformed Laguerre-type functions

$$\hat{\varphi}_l([\xi'], \xi_n) = (2[\xi'])^{\frac{1}{2}} \frac{([\xi'] - i\xi_n)^l}{([\xi'] + i\xi_n)^{l+1}}, \tag{5.2}$$

$[\xi']$ defined as in (2.14). In view of the orthonormality of the system $\hat{\varphi}_l$, $l \in \mathbb{N}$,

$$(\operatorname{tr}_n g)(x', \xi') = \sum_{l \in \mathbb{N}} c_{ll}(x', \xi') \tag{5.3}$$

(as used also e.g., in [11, (5.17)]), so res G is defined from the diagonal terms alone, by integration in x' of

$$\operatorname{res}_{x'} \operatorname{tr}_n G = \int_{|\xi'|=1} \operatorname{tr} \sum_{l \in \mathbb{N}} c_{ll}(x', \xi')\, đS(\xi'). \tag{5.4}$$

Not just the diagonal terms but also the off-diagonal terms will contribute to $l_0(GQ_{\lambda,+})$, so let us define

$$\begin{aligned} g_{\text{diag}} &= \sum_{l \in \mathbb{N}} c_{ll}(x', \xi') \hat{\varphi}_l([\xi'], \xi_n) \bar{\hat{\varphi}}_l([\xi'], \eta_n), \\ g_{\text{off}} &= \sum_{l,m \in \mathbb{N}, l \neq m} c_{lm}(x', \xi') \hat{\varphi}_l([\xi'], \xi_n) \bar{\hat{\varphi}}_m([\xi'], \eta_n) = g - g_{\text{diag}}, \end{aligned} \tag{5.5}$$

denoting the corresponding singular Green operators G_{diag} resp. G_{off}.

It is seen from (5.1) that

$$|g(x', \xi', \xi_n, \xi_n)| \dot{\leq} [\xi']^{\sigma+1}([\xi']^2 + \xi_n^2)^{-1}, \tag{5.6}$$

and there are analogous estimates for homogeneous terms and remainders. One can furthermore conclude from this that the strictly homogeneous version $g^h_{\sigma-j}$ (extending $g_{\sigma-j}$ by homogeneity into $\{1 > |\xi'| > 0\}$) satisfies

$$|g^h_{\sigma-j}(x', \xi', \xi_n, \xi_n)| \dot{\leq} |\xi'|^{\sigma-j+1}(|\xi'|^2 + \xi_n^2)^{-1}. \tag{5.7}$$

Let us denote $G(-\Delta - \lambda)_+^{-N} = G'$, it has symbol

$$\begin{aligned} g'(x', \xi', \xi_n, \eta_n, \lambda) &= g \circ_n (p_1 - \lambda)_+^{-N} \\ &= h^-_{-1,\eta_n}[g(x', \xi', \xi_n, \eta_n)(|(\xi', \eta_n)|^2 - \lambda)^{-N}], \end{aligned} \tag{5.8}$$

and symbol-kernel

$$\tilde{g}'(x', x_n, y_n, \xi', \lambda) = r^+_{x_n} r^+_{y_n} \mathcal{F}^{-1}_{\xi_n \to x_n} \overline{\mathcal{F}}^{-1}_{\eta_n \to y_n} g'(x', \xi', \xi_n, \eta_n, \lambda).$$

The normal trace $\operatorname{tr}_n G'$ is a ψdo on $\mathbb{R}^{n-1}$ with symbol

$$\begin{aligned}\operatorname{tr}_n g'(x',\xi',\lambda) &= \int^+ g'(x',\xi',\xi_n,\xi_n,\lambda)\,đ\xi_n \\ &= \int^+ \Big(h^-_{-1,\eta_n}[g(x',\xi',\xi_n,\eta_n)(|(\xi',\eta_n)|^2-\lambda)^{-N}]\Big)\,|_{\eta_n=\xi_n}đ\xi_n \quad (5.9)\end{aligned}$$

so the kernel of the operator $\operatorname{tr}_n G'$ is, for $x'=y'$,

$$\begin{aligned}&K(\operatorname{tr}_n G',x',x',\lambda)\\ &= \int_{\mathbb{R}^{n-1}}\int^+ \Big(h^-_{-1,\eta_n}[g(x',\xi',\xi_n,\eta_n)(|(\xi',\eta_n)|^2-\lambda)^{-N}]\Big)\,|_{\eta_n=\xi_n}\,đ\xi_n đ\xi'. \quad (5.10)\end{aligned}$$

We can here make an important simplification, as in [11, (3.9)ff.]:

Lemma 5.1 *Let $r(x',\xi)$ be a ψdo symbol in the calculus, of normal order* 0*. For the compositions on the boundary symbol level*

$$\begin{aligned} g\circ_n r_+ &= h^-_{-1,\eta_n}[g(x',\xi',\xi_n,\eta_n)r(x',\xi',\eta_n)],\\ r_+\circ_n g &= h^+_{\xi_n}[r(x',\xi',\xi_n)g(x',\xi',\xi_n,\eta_n)],\end{aligned}$$

one has that

$$\begin{aligned} &\int^+ \Big(h^+_{\eta_n}[g(x',\xi',\xi_n,\eta_n)r(x',\xi',\eta_n)]\Big)\,|_{\eta_n=\xi_n}\,đ\xi_n = 0,\\ &\int^+ \Big(h^-_{-1,\xi_n}[r(x',\xi',\xi_n)g(x',\xi',\xi_n,\eta_n)]\Big)\,|_{\eta_n=\xi_n}\,đ\xi_n = 0;\end{aligned} \quad (5.11)$$

and hence

$$\begin{aligned}\operatorname{tr}_n(g\circ_n r_+) &= \int^+ \Big(h^-_{-1,\eta_n}[g(x',\xi',\xi_n,\eta_n)r(x',\xi',\eta_n)]\Big)\,|_{\eta_n=\xi_n}đ\xi_n\\ &= \int g(x',\xi',\xi_n,\xi_n)r(x',\xi',\xi_n)\,đ\xi_n;\\ \operatorname{tr}_n(r_+\circ_n g) &= \int^+ \Big(h^+_{\xi_n}[r(x,\xi',\xi_n)g(x',\xi',\xi_n,\eta_n)]\Big)\,|_{\eta_n=\xi_n}đ\xi_n\\ &= \int r(x',\xi',\xi_n)g(x',\xi',\xi_n,\xi_n)\,đ\xi_n.\end{aligned} \quad (5.12)$$

Proof Assume first that r is of normal order -1, hence is $O(\langle\xi_n\rangle^{-1})$. We have for each term in the Laguerre expansion (5.1), denoting $[\xi'] = \varrho$:

$$\begin{aligned}&\int^+ \Big(h^+_{\eta_n}[c_{lm}(x',\xi')\hat\varphi_l([\xi'],\xi_n)\bar{\hat\varphi}_m([\xi'],\eta_n)r(x',\xi',\eta_n)]\Big)\,|_{\eta_n=\xi_n}\,đ\xi_n\\
&= c_{lm}\int^+ \hat\varphi_l(\varrho,\xi_n)\Big(h^+_{\eta_n}[\bar{\hat\varphi}_m(\varrho,\eta_n)r(x',\xi',\eta_n)]\Big)\,|_{\eta_n=\xi_n}\,đ\xi_n,\\
&= c_{lm}\int^+ \hat\varphi_l(\varrho,\xi_n)h^+_{\xi_n}[\bar{\hat\varphi}_m(\varrho,\xi_n)r(x',\xi',\xi_n)]\,đ\xi_n.\end{aligned}$$

Here the integrand is holomorphic in ξ_n on $\mathbb{C}_-$ and C^∞ on $\overline{\mathbb{C}}_-$, and is $O(\xi_n^{-2})$ for $|\xi_n|\to\infty$ in $\overline{\mathbb{C}}_-$, so that the integration curve can be transformed to a closed curve in $\mathbb{C}_-$ where it gives 0 (one may check this with [5, (2.2.42)ff.]). Summing over l and m, we find the first line in (5.11), and the first statement in (5.12) is an immediate consequence, since $h^+f(\xi_n) + h^-_{-1}f(\xi_n) = f(\xi_n)$ for $f\in\mathcal{H}_{-1}$.

Similarly, for the second line in (5.11), each Laguerre term is a standard integral of an L_2 function that is seen to be 0 by changing the integration curve to a closed curve in $\mathbb{C}_+$, and the second statement in (5.12) follows.

Now if r is of normal order 0, it can be written as a sum $r(x',\xi) = r_0(x',\xi') + r'(x',\xi)$, where r_0 is independent of ξ_n (and polynomial in ξ'), and r' is of normal order -1, thanks to the transmission condition. Here the $\circ_n$ composition with r_0 is simply a multiplication that goes outside the tr_n-integral, and the preceding considerations apply to r'. □

Thus h^-_{-1,η_n} can be omitted in (5.10), and we arrive at the more convenient formula

$$\begin{aligned}K(\mathrm{tr}_n G', x', x', \lambda) &= \int_{\mathbb{R}^{n-1}}\int g(x',\xi',\xi_n,\xi_n)(|(\xi',\xi_n)|^2-\lambda)^{-N}\,đ\xi_n đ\xi'\\
&= \int_{\mathbb{R}^n} g(x',\xi',\xi_n,\xi_n)(|(\xi',\xi_n)|^2-\lambda)^{-N}\,đ\xi. \qquad (5.13)\end{aligned}$$

For this, the analysis of the asymptotic expansion in $-\lambda$ will to some extent be modeled after the proof of Proposition 4.1; but it presents additional difficulties since the terms in g are given as homogeneous for $|\xi'|\ge 1$ only, and polar coordinates in ξ are not very helpful. In relation to the expansion $g\sim\sum_{j\in\mathbb{N}} g_{\sigma-j}$ in homogeneous terms (with $g_{\sigma-j}(x',\xi',\xi_n,\eta_n)$ homogeneous in (ξ',ξ_n,η_n) *of degree* $\sigma-j-1$ for $|\xi'|\ge 1$, hence of *order* $\sigma-j$, in the notation of [8]), we set $g_{1-n}=0$ if $\sigma+n-1\notin\mathbb{N}$, and we define $g_{<1-n} = g - \sum_{\sigma-j\ge 1-n} g_{\sigma-j}$.

The terms with $\sigma-j\ne 1-n$ are relatively easy to handle, and will be treated first.

Lemma 5.2 *For $g_{\sigma-j}(x',\xi',\xi_n,\eta_n)$ with $\sigma - j > 1-n$, one has:*

$$\int_{\mathbb{R}^n} g_{\sigma-j}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$= (-\lambda)^{\frac{\sigma+n-1-j}{2}-N}\int_{\mathbb{R}^n} g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n)(|\xi|^2+1)^{-N}\, đ\xi$$
$$+(-\lambda)^{-N}⨍(\operatorname{tr}_n g_{\sigma-j})(x',\xi')\, đ\xi' + O(\lambda^{-N-\frac{1}{2}+\varepsilon}), \tag{5.14}$$

for $\lambda\to\infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. For $g_{<1-n}(x',\xi',\xi_n,\eta_n)$ one has:

$$\int_{\mathbb{R}^n} g_{<1-n}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$= (-\lambda)^{-N}⨍(\operatorname{tr}_n g_{<1-n})(x',\xi')\, đ\xi' + O(\lambda^{-N-\frac{1}{2}+\varepsilon}). \tag{5.15}$$

Proof Let $\sigma - j > 1-n$. It follows from (5.7) that $g^h_{\sigma-j}$ is integrable on cylinders $\{\xi\in\mathbb{R}^n \mid |\xi'|\le a\}$. Multiplication by $(|\xi|^2-\lambda)^{-N}$ makes it integrable over $\mathbb{R}^n$ when $\lambda\notin\overline{\mathbb{R}}_+$. Write

$$\int_{\mathbb{R}^n} g_{\sigma-j}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$= \int_{\mathbb{R}^n} g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$+\int_{|\xi'|\le 1} (g_{\sigma-j}(x',\xi',\xi_n,\xi_n) - g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n))(|\xi|^2-\lambda)^{-N}\, đ\xi. \tag{5.16}$$

For the first term we have by homogeneity for λ on rays in $\mathbb{C}\setminus\overline{\mathbb{R}}_+$, writing $\lambda = -|\lambda|e^{i\theta}$, $|\theta|<\pi$, and replacing ξ by $|\lambda|^{\frac{1}{2}}\eta$:

$$\int_{\mathbb{R}^n} g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n)(|\xi|^2+|\lambda|e^{i\theta})^{-N}\, đ\xi$$
$$= |\lambda|^{\frac{\sigma-j+n-1}{2}-N}\int_{\mathbb{R}^n} g^h_{\sigma-j}(x',\eta',\eta_n,\eta_n)(|\eta|^2+e^{i\theta})^{-N}\, đ\eta. \tag{5.17}$$

This equals the first term in the right hand side of (5.14) if $\theta = 0$, and the identity extends analytically to general λ (as in [13, Lemma 2.3]).

Using (4.10), we find that

$$\int_{|\xi'|\le 1} (g_{\sigma-j}(x',\xi',\xi_n,\xi_n) - g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n))(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$= (-\lambda)^{-N} \int_{|\xi'|\le 1} (g_{\sigma-j}(x',\xi',\xi_n,\xi_n) - g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n))\, đ\xi + O\left(\lambda^{-N-\frac{1}{2}+\varepsilon}\right), \tag{5.18}$$

since $(g_{\sigma-j} - g^h_{\sigma-j})\langle\xi_n\rangle^{1-2\varepsilon}$ is integrable on the set where $|\xi'|\le 1$ (cf. (5.6), (5.7)). Here, in view of (3.2), and (4.6) applied in dimension $n-1$,

$$\int_{\xi\in\mathbb{R}^n,|\xi'|\le 1} (g_{\sigma-j}(x',\xi',\xi_n,\xi_n) - g^h_{\sigma-j}(x',\xi',\xi_n,\xi_n))\, đ\xi$$
$$= \int_{\xi'\in\mathbb{R}^{n-1},|\xi'|\le 1} ((\mathrm{tr}_n\, g_{\sigma-j})(x',\xi') - (\mathrm{tr}_n\, g^h_{\sigma-j})(x',\xi'))\, đ\xi'$$
$$= ⨍ (\mathrm{tr}_n\, g_{\sigma-j})(x',\xi')\, đ\xi'; \tag{5.19}$$

this shows (5.14).

For (5.15), we use that $g_{<1-n}$ is integrable in ξ and remains so after multiplication by a small power of $|\xi|$; then (4.10) can be used to show that

$$\int g_{<1-n}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, đ\xi$$
$$= (-\lambda)^{-N}\int g_{<1-n}(x',\xi',\xi_n,\xi_n)\, đ\xi + O\left(\lambda^{-N-\frac{1}{2}+\varepsilon}\right),$$

where

$$\int g_{<1-n}(x',\xi',\xi_n,\xi_n)\, đ\xi = ⨍ (\mathrm{tr}_n\, g_{<1-n})(x',\xi')\, đ\xi',$$

in view of (3.2), and (4.6) in dimension $n-1$. □

If $\sigma\notin\mathbb{Z}$ or $\sigma<1-n$, this ends the analysis, since there is no term in g of order $1-n$. Since

$$⨍ (\mathrm{tr}_n\, g)(x',\xi')\, đ\xi' = \sum_{\sigma-j>1-n} ⨍ (\mathrm{tr}_n\, g_{\sigma-j})(x',\xi')\, đ\xi' + ⨍ (\mathrm{tr}_n\, g_{<1-n})(x',\xi')\, đ\xi'$$

in this case, we thus find:

Theorem 5.3 *When* $\sigma - n + 1 \notin \mathbb{N}$, *the kernel of* $\operatorname{tr}_n(GQ^N_{\lambda,+})$ *on the diagonal* $\{x' = y'\}$ *has for* $N > (\sigma + n)/2$ *an expansion*:

$$K(\operatorname{tr}_n(GQ^N_{\lambda,+}), x', x') = \sum_{0\le j<\sigma+n-1} (-\lambda)^{\frac{\sigma+n-1-j}{2}-N} b_j(x') \\ +(-\lambda)^{-N} ⨍ (\operatorname{tr}_n g)(x', \xi')\, đ\xi + O\left(\lambda^{-N-\frac{1}{2}+\varepsilon}\right) \quad (5.20)$$

any $\varepsilon > 0$, *for* $\lambda \to \infty$ *on rays in* $\mathbb{C}\setminus\mathbb{R}_+$, *where*

$$b_j(x') = \int_{\mathbb{R}^n} g^h_{\sigma-j}(x', \xi', \xi_n, \xi_n)(|\xi|^2 + 1)^{-N}\, đ\xi. \quad (5.21)$$

When σ is an integer $\geq 1 - n$, we must include a study of the contribution from g_{1-n}. Here the Laguerre expansion (5.1) comes into the picture, since we need to treat G_{diag} and G_{off} (cf. (5.5)) by different methods. The symbols of order $1 - n$ are:

$$g_{\text{diag},1-n} = \sum_{l\in\mathbb{N}} g_{ll,1-n}, \quad g_{\text{off},1-n} = \sum_{l,m\in\mathbb{N},l\neq m} g_{lm,1-n}, \text{ where} \\ g_{lm,1-n}(x', \xi', \xi_n, \eta_n) = c_{lm,1-n}(x', \xi')\hat{\varphi}_l([\xi'], \xi_n)\bar{\hat{\varphi}}_m([\xi'], \eta_n). \quad (5.22)$$

We denote $\operatorname{OP}(c_{lm}(x', \xi')) = C_{lm}$.

Proposition 5.4 *The* $g_{ll,1-n}$ *satisfy for* $N \geq 1$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \operatorname{tr} g_{ll,1-n}(x', \xi', \xi_n, \xi_n)(|\xi|^2 - \lambda)^{-N}\, đ\xi \\
&= \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{\mathbb{R}^n} \operatorname{tr} g_{ll,1-n}(x', \xi', \xi_n, \xi_n)(|\xi|^2 - \lambda)^{-1}\, đ\xi \\
&= \frac{\partial_\lambda^{N-1}}{(N-1)!} \Big[\tfrac{1}{2} \operatorname{res}_{x'} C_{ll}(-\lambda)^{-1} \log(-\lambda) + ⨍ \operatorname{tr} c_{ll,1-n}(x', \xi')\, đ\xi'(-\lambda)^{-1} \\
&\quad + \operatorname{res}_{x'} C_{ll}(-\log 2\, (-\lambda)^{-1} + O(\lambda^{-\frac{3}{2}+\varepsilon})) \Big], \\
&= \tfrac{1}{2} \operatorname{res}_{x'} C_{ll}(-\lambda)^{-N} \log(-\lambda) + ⨍ \operatorname{tr} c_{ll,1-n}(x', \xi')\, đ\xi'(-\lambda)^{-N} \\
&\quad + \operatorname{res}_{x'} C_{ll} \left(-(\log 2 + \tfrac{1}{2}\alpha_N)(-\lambda)^{-N} + O(\lambda^{-N-\frac{1}{2}+\varepsilon})\right), \qquad (5.23)
\end{aligned}$$

for $\lambda \to \infty$ *on rays in* $\mathbb{C}\setminus\mathbb{R}_+$.

It follows by summation in l that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \operatorname{tr} g_{\mathrm{diag},1-n}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi \\
&= \tfrac{\partial_\lambda^{N-1}}{(N-1)!}\Big[\tfrac12 \operatorname{res}_{x'}(\operatorname{tr}_n G)(-\lambda)^{-1}\log(-\lambda) + \Big(\fint \operatorname{tr}\operatorname{tr}_n g_{1-n}(x',\xi')\, d\!\!{}^-\!\xi' \\
&\quad - \log 2\, \operatorname{res}_{x'}(\operatorname{tr}_n G)\Big)(-\lambda)^{-1}\Big] + O\Big(\lambda^{-N-\frac12+\varepsilon}\Big), \\
&= \tfrac12 \operatorname{res}_{x'}(\operatorname{tr}_n G)(-\lambda)^{-N}\log(-\lambda) + \Big(\fint \operatorname{tr}\operatorname{tr}_n g_{1-n}(x',\xi')\, d\!\!{}^-\!\xi' \\
&\quad - \big(\log 2 + \tfrac12\alpha_N\big)\operatorname{res}_{x'}(\operatorname{tr}_n G)\Big)(-\lambda)^{-N} + O\Big(\lambda^{-N-\frac12+\varepsilon}\Big).
\end{aligned} \tag{5.24}$$

Proof Since the integrand is in $L_1(\mathbb{R}^n)$ for all $N \geq 1$, we can insert the formula $(|\xi|^2-\lambda)^{-N} = \frac{\partial_\lambda^{N-1}}{(N-1)!}(|\xi|^2-\lambda)^{-1}$ and pull the differential operator $\frac{\partial_\lambda^{N-1}}{(N-1)!}$ outside the integral sign. Then we can do the main work in the case $N = 1$. Write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \operatorname{tr} g_{ll,1-n}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi = I_1 + I_2, \\
&I_1 = \int_{|\xi'|\geq 1} \operatorname{tr} g^h_{ll,1-n}(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi,\ \ I_2 = \int_{|\xi'|\leq 1} \operatorname{tr} g_{ll,1-n}(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi.
\end{aligned} \tag{5.25}$$

The last term is treated as in the preceding proof. Using (4.10), and the orthonormality of the Laguerre functions, we find that

$$\begin{aligned}
I_2 &= \int_{|\xi'|\leq 1} \operatorname{tr} g_{ll,1-n}(x',\xi',\xi_n,\xi_n)(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi \\
&= (-\lambda)^{-N} \int_{|\xi'|\leq 1} \operatorname{tr} g_{ll,1-n}(x',\xi',\xi_n,\xi_n)\, d\!\!{}^-\!\xi + O\Big(\lambda^{-N-\frac12+\varepsilon}\Big) \\
&= (-\lambda)^{-N} \fint \operatorname{tr} c_{ll,1-n}(x',\xi')\, d\!\!{}^-\!\xi' + O(\lambda^{-N-\frac12+\varepsilon}),
\end{aligned} \tag{5.26}$$

in view of (3.2), and (4.6) in dimension $n-1$.

Now consider I_1. It suffices to take $\lambda \in \mathbb{R}_-$; here we write $-\lambda = \mu^2$, $\mu > 0$. The integrand is homogeneous in (ξ',ξ_n) but the integration is over $\{\xi \mid |\xi'| \geq 1\}$; it is here that the Laguerre expansion helps in the calculations. Write in general

$$I_{lm} = \int_{|\xi'|\geq 1} \operatorname{tr} c_{lm,1-n}(x',\xi')\hat{\varphi}_l([\xi'],\xi_n)\bar{\hat{\varphi}}_m([\xi'],\xi_n)(|\xi|^2-\lambda)^{-N}\, d\!\!{}^-\!\xi. \tag{5.27}$$

Denote $r = |\xi'|$, $\kappa = (|\xi'|^2 - \lambda)^{\frac{1}{2}} = (r^2 + \mu^2)^{\frac{1}{2}}$. Then by Lemma A.2,

$$I_1 = I_{ll} = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} \operatorname{res}_{x'} C_{ll} \int_{r \geq 1} \frac{1}{r\kappa(r+\kappa)}\, dr. \tag{5.28}$$

The integral is analyzed by use of Lemma A.3. Here (A.4) implies:

$$\begin{aligned} I_{ll} &= \operatorname{res}_{x'} C_{ll} \tfrac{\partial_\lambda^{N-1}}{(N-1)!} [\mu^{-2}(-\log 2 + \log(\sqrt{1+\mu^2}\, + 1))] \\ &= \operatorname{res}_{x'} C_{ll} \tfrac{\partial_\lambda^{N-1}}{(N-1)!} \Big[(-\lambda)^{-1}\Big(-\log 2 + \log\Big((-\lambda)^{\frac{1}{2}}\Big) + \log\Big(\sqrt{1+(-\lambda)^{-1}}\, + (-\lambda)^{-\frac{1}{2}}\Big)\Big)\Big] \\ &= \operatorname{res}_{x'} C_{ll} \tfrac{\partial_\lambda^{N-1}}{(N-1)!} \Big[\tfrac{1}{2}(-\lambda)^{-1}\log(-\lambda) - \log 2 (-\lambda)^{-1} + O\Big(\lambda^{-\frac{3}{2}}\Big)\Big], \end{aligned} \tag{5.29}$$

for $|\lambda| \to \infty$.

When we add I_1 and I_2, we obtain the statement of (5.23) before the last identity. In the last identity, the term $-\frac{1}{2}\alpha_N \operatorname{res}_{x'}(C_{ll})$ (cf. (2.5)) comes from differentiating the log-term, and it is checked from (5.29) that the error has the asserted order.

Now summation with respect to l gives the second statement in the proposition, in view of (5.4)ff. □

The coefficient $\log 2$ in (5.23), (5.24) may seem a little odd, but fortunately, it will disappear again when the terms are analyzed more and set in relation to compositions with $\log P_1'$. First we make the appropriate analysis of $g_{\mathrm{off},1-n}$. An analysis similar to the above of the I_{lm} with $l \neq m$ can of course be carried out, but as seen from (A.3) and (A.5), this gives coefficients and remainders depending on l and m, leading to a less transparent summation result.

We shall use instead that this part in fact has good enough regularity properties to define a completely local contribution, where the methods of Sect. 3 can be applied. It was observed already in [11] that this part gives a local coefficient and no logarithmic term.

Proposition 5.5 *Define for $N \geq 1$,*

$$\mathcal{S}^{(N)}_{\mathrm{off},\lambda} = \operatorname{tr}_n \left(G_{\mathrm{off}} \tfrac{\partial_\lambda^{N-1}}{(N-1)!} (P_1 - \lambda)_+^{-1} \right), \text{ with symbol } \mathfrak{s}^{(N)}_{\mathrm{off}}; \tag{5.30}$$

it is of order $\sigma - 2N$ and regularity $\sigma - \frac{1}{4}$, and it also equals $(-\lambda)^{-N}$ times a symbol of order σ and regularity $\sigma + \frac{1}{4}$, and satisfies estimates like (4.21). Then we can define the "log-transform" S_{off} with symbol

$$s_{\mathrm{off}}(x', \xi') = \tfrac{i}{2\pi} \int_{\mathcal{C}} \log\lambda\, \mathfrak{s}_{\mathrm{off}}(x', \xi', \lambda)\, d\lambda \tag{5.31}$$

(for $|\xi'| \geq 1$).

The fiber trace of the kernel of $\mathcal{S}^{(N)}_{\mathrm{off},\lambda}$ has an expansion on the diagonal

$$\operatorname{tr} K(\mathcal{S}^{(N)}_{\mathrm{off},\lambda}, x', x') = \sum_{0\le l\le\sigma+n-1} \operatorname{tr} \tilde{\mathfrak{s}}^{(N)}_{\mathrm{off},l}(x')(-\lambda)^{\frac{\sigma+n-1-l}{2}-N} + O\left(\lambda^{-N-\frac{1}{8}}\right), \tag{5.32}$$

where

$$\operatorname{tr} \tilde{\mathfrak{s}}^{(N)}_{\mathrm{off},\sigma+n-1}(x') = -\tfrac{1}{2} \operatorname{res}_{x'} S_{\mathrm{off}}. \tag{5.33}$$

Proof For the symbol $\mathfrak{s}_{\mathrm{off}}(x', \xi', \lambda)$ of $\mathcal{S}_{\mathrm{off},\lambda}$, the immediate rules give that $\mathfrak{s}^{(N)}_{\mathrm{off}}(x', \xi', \lambda) = \frac{\partial_\lambda^{N-1}}{(N-1)!}\mathfrak{s}_{\mathrm{off}}(x', \xi', \lambda)$ is of order $\sigma - 2N$ and regularity $\sigma - \frac{1}{4}$. But it can be rewritten using (4.22) (we let $|\xi'| \ge 1$ and as usual write $|\xi'| = r$ for short, omitting explicit mention of (x', ξ')):

$$\begin{aligned}\mathfrak{s}_{\mathrm{off}}(\lambda) &= \sum_{l\ne m} c_{lm} \operatorname{tr}_n[\hat{\varphi}_l(r, \xi_n)\overline{\hat{\varphi}}_m(r, \eta_n) \circ_n (p_1 - \lambda)^{-1}_+] \\ &= \sum_{l\ne m} c_{lm} \operatorname{tr}_n[\hat{\varphi}_l(r, \xi_n)\overline{\hat{\varphi}}_m(r, \eta_n) \circ_n (-\lambda^{-1} + \lambda^{-1} p_1(p_1 - \lambda)^{-1})_+] \\ &= \sum_{l\ne m} c_{lm} \operatorname{tr}_n[\hat{\varphi}_l(r, \xi_n)\overline{\hat{\varphi}}_m(r, \eta_n) \circ_n (p_1\lambda^{-1}(p_1 - \lambda)^{-1})_+],\end{aligned}$$

where we used that $\operatorname{tr}_n(\hat{\varphi}_l(r, \xi_n)\overline{\hat{\varphi}}_m(r, \eta_n)) = (\hat{\varphi}_l, \hat{\varphi}_m) = 0$ for $l \ne m$. Similarly to Lemma 3.3, we see from this that $\mathcal{S}_{\mathrm{off},\lambda}$ can also be viewed as λ^{-1} times an operator of order σ and regularity $\sigma + \frac{1}{4}$, and likewise $(-\lambda)^N \frac{\partial_\lambda^{N-1}}{(N-1)!}\mathcal{S}_{\mathrm{off},\lambda}$ is of order σ and regularity $\sigma + \frac{1}{4}$; moreover, estimates like (4.21) hold. Then Theorems 3.1 and 3.2 apply, and the result of Theorem 4.5 extends to this case, when we define s_{off} from $\mathfrak{s}_{\mathrm{off}}$ by (5.31). □

These two propositions can be taken together to give a formula for the term $l_0(GQ_{\lambda,+})$. However, to find a more transparent formulation, we shall analyze the ingredients some more.

Insertion of $\mathfrak{s}_{\mathrm{off}}$ in (5.31) gives that (for $|\xi'| \ge 1$)

$$\begin{aligned} &s_{\mathrm{off}}(x', \xi') \\ &\quad = \frac{i}{2\pi} \int_{\mathcal{C}} \log\lambda \sum_{l\ne m} c_{lm}(x', \xi') \operatorname{tr}_n[\hat{\varphi}_l(r, \xi_n)\overline{\hat{\varphi}}_m(r, \eta_n)\circ_n (p_1\lambda^{-1}(p_1 - \lambda)^{-1})_+]\, d\lambda.\end{aligned}$$

One would like to pass the integration in λ inside tr_n, so that $\log p_1$ would appear in the formula, but the complex rule for calculation of $\operatorname{tr}_n(g \circ_n r_+)$ (cf. (5.9), (5.12)) requires a symbol r satisfying the transmission condition, which $\log p_1$ does not (in its principal part). However, inspired from (5.12) we can *define* an extension of the normal trace to non-standard symbols:

Definition 5.6 For functions $r(x',\xi)$ and $g(x',\xi,\eta_n)$, we set

$$\begin{aligned} \operatorname{tr}_n'(g\circ_n r_+) &= \int_{\mathbb{R}} g(x',\xi',\xi_n,\xi_n) r(x',\xi',\xi_n)\, đ\xi_n, \\ \operatorname{tr}_n'(r_+\circ_n g) &= \int_{\mathbb{R}} r(x',\xi',\xi_n) g(x',\xi',\xi_n,\xi_n)\, đ\xi_n, \end{aligned} \tag{5.34}$$

whenever the integral converges. The notation will also be used for the associated operators.

This allows us to write

$$\begin{aligned} s_{\mathrm{off}}(x',\xi') &= \tfrac{i}{2\pi}\int_{\mathcal{C}} \log\lambda \sum_{l\neq m} c_{lm} \operatorname{tr}_n'[\hat\varphi_l(r,\xi_n)\overline{\hat\varphi}_m(r,\eta_n)\circ_n (p_1\lambda^{-1}(p_1-\lambda)^{-1})_+]\,d\lambda \\ &= \tfrac{i}{2\pi}\int_{\mathcal{C}} \log\lambda \sum_{l\neq m} c_{lm} \int_{\mathbb{R}} \hat\varphi_l(r,\xi_n)\overline{\hat\varphi}_m(r,\xi_n) p_1(\xi)\lambda^{-1}(p_1(\xi)-\lambda)^{-1}\, đ\xi_n d\lambda \\ &= \sum_{l\neq m} c_{lm}\int_{\mathbb{R}} \hat\varphi_l(r,\xi_n)\overline{\hat\varphi}_m(r,\xi_n) p_1(\xi)\tfrac{i}{2\pi}\int_{\mathcal{C}} \log\lambda\,\lambda^{-1}(p_1(\xi)-\lambda)^{-1}\,d\lambda đ\xi_n \\ &= \operatorname{tr}_n'(g_{\mathrm{off}}\circ_n (p_1(p_1^{-1}\log p_1))_+) = \operatorname{tr}_n'(g_{\mathrm{off}}\circ_n (\log p_1)_+). \end{aligned} \tag{5.35}$$

In this sense, we have that

$$S_{\mathrm{off}} \sim \operatorname{tr}_n'(G_{\mathrm{off}}(\log P_1')_+) \tag{5.36}$$

(modulo smoothing operators).

We can now extend this point of view to cases where G_{off} is replaced by G_{diag} or G itself. *Define*:

$$\begin{aligned} S_{\mathrm{diag}} &\sim \operatorname{tr}_n'(G_{\mathrm{diag}}(\log P_1')_+), \\ S &\sim \operatorname{tr}_n'(G(\log P_1')_+), \end{aligned} \tag{5.37}$$

with symbols given for $|\xi'|\geq 1$ by:

$$\begin{aligned} s_{\mathrm{diag}}(x',\xi') &= \sum_{l\in\mathbb{N}} c_{ll}(x',\xi')\operatorname{tr}_n'[\hat\varphi_l(r,\xi_n)\overline{\hat\varphi}_l(r,\eta_n)\circ_n(\log p_1)_+], \\ s(x',\xi') &= \sum_{l,m\in\mathbb{N}} c_{lm}(x',\xi')\operatorname{tr}_n'[\hat\varphi_l(r,\xi_n)\overline{\hat\varphi}_m(r,\eta_n)\circ_n(\log p_1)_+] \\ &= \operatorname{tr}_n'[g(x',\xi',\xi_n,\eta_n)\circ_n(\log p_1)_+]. \end{aligned} \tag{5.38}$$

Further calculations show:

Lemma 5.7 *The symbol $s_{\rm diag}$ of $S_{\rm diag}$ satisfies*

$$s_{\rm diag}(x',\xi') \sim (\operatorname{tr}_n g)(x',\xi')(2\log 2 + 2\log[\xi']). \tag{5.39}$$

It is log-polyhomogeneous, and in particular,

$$\operatorname{res}_{x',0}(S_{\rm diag}) = \operatorname{res}_{x',0}(\operatorname{tr}'_n(G_{\rm diag}(\log P'_1)_+)) = 2\log 2\ \operatorname{res}_{x'}(\operatorname{tr}_n G). \tag{5.40}$$

Proof For $r = |\xi'| \ge 1$, write

$$\begin{aligned} s_{\rm diag}(x',\xi') &= \sum_{l\in\mathbb{N}} s_{ll}(x',\xi'), \text{ with} \\ s_{ll}(x',\xi') &= c_{ll}(x',\xi')\operatorname{tr}'_n[\hat\varphi_l(r,\xi_n)\bar{\hat\varphi}_l(r,\eta_n)\circ_n(\log p_1)] \\ &= c_{ll}(x',\xi')\int_{\mathbb{R}} \hat\varphi_l(r,\xi_n)\bar{\hat\varphi}_l(r,\xi_n)\log(r^2+\xi_n^2)\, đ\xi_n. \end{aligned}$$

Here we have

$$\begin{aligned} &\int_{\mathbb{R}} \hat\varphi_l(r,\xi_n)\bar{\hat\varphi}_l(r,\xi_n)\log(r^2+\xi_n^2)\, đ\xi_n \\ &= \int_{\mathbb{R}} 2r\frac{(r-it)^l}{(r+it)^{l+1}}\frac{(r+it)^l}{(r-it)^{l+1}}\log(r^2+t^2)\, đt = \frac{2r}{2\pi}\int_{\mathbb{R}}\frac{\log(r^2+t^2)}{r^2+t^2}\,dt \\ &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{\log(1+s^2)+\log r^2}{1+s^2}\,ds = 2\log 2 + \log r^2 \end{aligned}$$

(see Lemma A.4 in the Appendix). It follows that

$$s_{\rm diag}(x',\xi') = \sum_{l\in\mathbb{N}} c_{ll}(x',\xi')(2\log 2 + 2\log|\xi'|), \tag{5.41}$$

for $|\xi'| \ge 1$, cf. also (5.3). This shows (5.39), and (5.40) follows in view of (5.4). □

It follows from this and (5.35) that $S = \operatorname{tr}'_n(G(\log P'_1)_+)$ is likewise log-polyhomogeneous.

Then we can finally conclude, in the special localized situation:

Theorem 5.8 *The operator family $G(-\Delta-\lambda)_+^{-N}$ on $\mathbb{R}^n_+$ has for $N > (\sigma+n)/2$, expansions as in (2.23), with $\delta = \frac12 - \varepsilon$, for $\lambda \to \infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. Here $a'_0 = \frac12\operatorname{res}(G) = \frac12\int_{\mathbb{R}^{n-1}}\operatorname{res}_{x'}\operatorname{tr}_n G\,dx'$ and*

$$l_0(G(-\Delta-\lambda)_+^{-1}) = \int_{\mathbb{R}^{n-1}} \big(\operatorname{TR}_{x'}\operatorname{tr}_n G - \tfrac12\operatorname{res}_{x',0}\operatorname{tr}'_n(G(\log P'_1)_+)\big)\,dx'. \tag{5.42}$$

Proof Collecting the results of Lemma 5.2, Theorem 5.3 for $\sigma + n - 1 \notin \mathbb{N}$ (where the relevant residues vanish), and Propositions 5.4 and 5.5 for $\sigma + n - 1 \in \mathbb{N}$, we get the expansion with a'_0 as stated, and

$$\begin{aligned} & l_0(G(-\Delta - \lambda)_+^{-1}) \\ &\quad = \int_{\mathbb{R}^{n-1}} \big(\mathrm{TR}_{x'} \operatorname{tr}_n G - \log 2 \operatorname{res}_{x'} \operatorname{tr}_n G - \tfrac{1}{2} \operatorname{res}_{x'} S_{\mathrm{off}}\big)\, dx' \\ &\quad = \int_{\mathbb{R}^{n-1}} \big(\mathrm{TR}_{x'} \operatorname{tr}_n G - \log 2 \operatorname{res}_{x'} \operatorname{tr}_n G - \tfrac{1}{2} \operatorname{res}_{x'} \operatorname{tr}'_n (G_{\mathrm{off}} (\log P'_1)_+)\big)\, dx', \end{aligned} \tag{5.43}$$

in view of (5.36). Now since $G = G_{\mathrm{diag}} + G_{\mathrm{off}}$ with $\operatorname{tr}'_n(G(\log P'_1)_+)$ log-polyhomogeneous,

$$\begin{aligned} \operatorname{res}_{x'} \operatorname{tr}'_n (G_{\mathrm{off}} (\log P'_1)_+) &= \operatorname{res}_{x',0} \operatorname{tr}'_n (G(\log P'_1)_+) - \operatorname{res}_{x',0} \operatorname{tr}'_n (G_{\mathrm{diag}} (\log P'_1)_+) \\ &= \operatorname{res}_{x',0} \operatorname{tr}'_n (G(\log P'_1)_+) - 2 \log 2 \operatorname{res}_{x'} \operatorname{tr}_n G, \end{aligned}$$

by Lemma 5.7. When this is inserted in (5.43), we find (5.42). □

This gives the formula for the contribution to the basic zeta coefficient in a reasonably natural form. Now the global formula with P_1, as well as the formula with P_1 replaced by general P_2, follow as in Sect. 4:

Theorem 5.9 *For a singular Green operator G of order $\sigma \in \mathbb{R}$ and class 0, together with an elliptic differential operator P_2 as in Theorem 4.3, $G(P_2 - \lambda)_+^{-N}$ has expansions as in (2.23) for $N > (\sigma + n)/2$, with $\delta = \frac{1}{2} - \varepsilon$, for $\lambda \to \infty$ on rays in V. Here $a'_0 = \frac{1}{2} \operatorname{res}(G)$ and*

$$l_0(G(P_2 - \lambda)_+^{-1}) = \int_{X'} \big(\mathrm{TR}_{x'} \operatorname{tr}_n G - \tfrac{1}{2} \operatorname{res}_{x',0} \operatorname{tr}'_n (G(\log P_2)_+)\big)\, dx', \tag{5.44}$$

calculated in local coordinates.

It is worth keeping in mind here that $\operatorname{tr}'_n(G(\log P_2)_+))$ in local coordinates is the sum of a usual normal trace, namely that of $G(\log P_2 - \log P_1)_+$, and a generalized normal trace tr'_n in one special case, namely for G composed with the logarithm of the Laplacian.

6 The constants coming from $P_+ G_\lambda$ and $G G_\lambda$

The appropriate expansion of $\operatorname{Tr}(P_+ + G) G_\lambda^{(N)}$ was shown in Sect. 3, and we shall just discuss the constants arising there more in depth. In this analysis m can be any positive integer, and we as usual take $N > (\sigma + n)/m$.

Recall that the operator U introduced in Corollary 3.5 arose from taking the normal trace of $BG_\lambda^{(N)}$ and integrating together with $\log\lambda$; this gave a ψdo on X'. We shall set this in relation to the composition of B with the s.g.o.-like part $G_1^{\log}$ of $\log P_{1,T}$:

$$G_1^{\log} = \lim_{s\searrow 0} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} \log\lambda\, G_\lambda\, d\lambda = \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-1} \log\lambda\, P_1 G_\lambda\, d\lambda \tag{6.1}$$

(cf. (3.13)); it is described in more detail in [10]. The analysis is different for the two terms coming from P_+ resp. G, so let us write

$$U = U_P + U_G, \tag{6.2}$$

where U_P is defined from $B = P_+$ and U_G is defined from $B = G$. We have the trace expansions from Corollary 3.5:

$$\begin{aligned} \operatorname{Tr}(P_+ G_\lambda^{(N)}) &= \sum_{1\le j\le \sigma+n} a_j^{(N)} (-\lambda)^{\frac{\sigma+n-j}{m} - N} + O\left(\lambda^{-N-\frac{1}{4m}}\right) \text{ with} \\ a_{\sigma+n}^{(N)} &= -\tfrac{1}{m} \operatorname{res} U_P \quad (= l_0(P_+ G_\lambda)); \end{aligned} \tag{6.3}$$

$$\begin{aligned} \operatorname{Tr}(G G_\lambda^{(N)}) &= \sum_{1\le j\le \sigma+n} b_j^{(N)} (-\lambda)^{\frac{\sigma+n-j}{m} - N} + O\left(\lambda^{-N-\frac{1}{4m}(+\varepsilon)}\right) \text{ with} \\ b_{\sigma+n}^{(N)} &= -\tfrac{1}{m} \operatorname{res} U_G \quad (= l_0(G G_\lambda)). \end{aligned} \tag{6.4}$$

The natural thing to do would be to interchange the λ-integral and the application of tr_n. This works well for the part with G. Here we can proceed as in the discussion of S in Sect. 4, by appealing to the estimates of the symbol-kernel of G_λ shown by Seeley [23] (and recalled in [10, Theorem 2.4]): The symbol-kernel has in local coordinates an expansion in quasi-homogeneous terms $\tilde g \sim \sum_{j\ge 0} \tilde g_{-m-j}$, satisfying estimates on the rays in V, with $\kappa = |\xi'| + |\lambda|^{\frac{1}{m}}$:

$$|D_{x'}^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'} y_n^l D_{y_n}^{l'} D_\lambda^p \tilde g_{-m-j}| \mathrel{\dot\le} \kappa^{1-m-|\alpha|-k+k'-l+l'-j-mp} e^{-c\kappa(x_n+y_x)} \tag{6.5}$$

for all indices, when $\kappa \ge \varepsilon$. These estimates allow carrying the λ-integration inside tr_n:

Theorem 6.1 *For U_G in (6.2), we have that $U_G = \operatorname{tr}_n(G G_1^{\log})$, and the coefficient of $(-\lambda)^{-N}$ in the trace expansion of $G G_\lambda^{(N)}$ is:*

$$l_0(G G_\lambda) = -\tfrac{1}{m} \operatorname{res} U_G = -\tfrac{1}{m} \operatorname{res} \operatorname{tr}_n(G G_1^{\log}). \tag{6.6}$$

Proof Consider the j'th symbol term $u_{\sigma-j}(x',\xi')$ in U_G, cf. (3.22). It is for $|\xi'|\ge 1$ a linear combination of terms of the form

$$\frac{i}{2\pi}\int_{\mathcal C}\log\lambda\int_{x_n,y_n>0}\partial_{\xi'}^{\alpha}\tilde g_{\sigma-k}(x',x_n,y_n,\xi')\cdot\partial_{x'}^{\alpha}\tilde g_{-l-m}(x',y_n,x_n,\xi',\lambda)\,dx_n dy_n d\lambda$$
$$=\int_{x_n,y_n>0}\partial_{\xi'}^{\alpha}\tilde g_{\sigma-k}(x',x_n,y_n,\xi')\frac{i}{2\pi}\int_{\mathcal C}\log\lambda\cdot\partial_{x'}^{\alpha}\tilde g_{-l-m}(x',y_n,x_n,\xi',\lambda)\,d\lambda dx_n dy_n,$$

for $|\alpha|+k+l=j$ ($\alpha\in\mathbb N^{n-1}$, $k,l\in\mathbb N$); we could interchange the order of integration since the symbol-kernels from G are bounded and those from G_λ have $L_{1,x_n,y_n}(\mathbb R^2_{++})$-norms that are $O(\lambda^{-1-\frac{1}{m}})$. The inner log-integrals define the terms in the symbol-kernel of $G_1^{\log}$ (and derivatives), cf. [10]. We conclude that $\operatorname{tr}_n(GG_1^{\log})$ is a well-defined classical ψdo of order σ which equals U_G modulo a smoothing operator. □

Here it is only a slight extension of the definition of [4] to define

$$\operatorname{res}(GG_1^{\log})=-m\,l_0(GG_\lambda).\tag{6.7}$$

For U_P, it would be nice to be able to write a similar relation of $\operatorname{res}U_P$ to the residue of $\operatorname{tr}_n(P_+G_1^{\log})$. But the latter normal trace rarely exists in the usual sense; it does so when P is of low order, but otherwise needs interpretations involving e.g., the subtraction of a number of symbol terms which do not contribute to the residue. This is apparent already in the case $P=I$, as considered in [10, Sect. 3], where tr_n is defined for $G_1^{\log}$ minus its principal part.

For clarity in the explanation, assume first that in the local coordinates in $\mathbb R^n_+$, where we are considering $P_+G_1^{\log}$, the symbol p is independent of x_n. Then, in view of the transmission condition, it can be decomposed into a differential operator symbol and a symbol of normal order -1:

$$p(x',\xi)=p'(x',\xi)+p''(x',\xi),$$
$$p'(x',\xi)=\sum_{|\alpha|\le\sigma}a_\alpha(x')(\xi')^{\alpha'}\xi_n^{\alpha_n},\quad |p''|\,\dot\le\,\langle\xi_n\rangle^{-1}\langle\xi'\rangle^{\sigma+1}.\tag{6.8}$$

(One can instead let p'' be the part of normal order 0.) For the differential operator part, the composition of $D_{x_n}^{\alpha_n}$ with $G_1^{\log}$ is simply an application of $D_{x_n}^{\alpha_n}$ to the symbol-kernel $\tilde g^{\log}$. This makes the symbol-kernel *more* singular at $x_n+y_n=0$, the larger α_n is, cf. [10, (2.23)–(2.24)]; this is also clear from the example $\tilde g^{\log}=\frac{1}{x_n+y_n}e^{-|\xi'|(x_n+y_n)}$ where $P_1=-\Delta$. However, the term that contributes to the residue escapes this effect since the order is lifted by α_n so that we have to go further down in the series to find the term whose order equals minus the boundary dimension. In short, the relevant term in the $\circ_n$-composition of p' and $\tilde g^{\log}$ is

$$\sum_{|\alpha|\le\sigma,|\alpha'|+\alpha_n-j=1-n}a_\alpha(x')(\xi')^{\alpha'}D_{x_n}^{\alpha_n}\tilde g_{-j}^{\log}(x',x_n,y_n,\xi'),\tag{6.9}$$

which is $O((x_n+y_n)^{-\varepsilon})$ when $n=2$, bounded in x_n+y_n when $n>2$, and integrable for $x_n+y_n\to\infty$, according to [10, (2.26)–(2.29)]. In the full composition there will moreover be terms coming from derivatives in ξ' and x':

$$\sum_{|\alpha|\le\sigma,\beta'\le\alpha',|\alpha'-\beta'|+\alpha_n-j=1-n} c_{\alpha',\beta'}a_\alpha(x')(\xi')^{\alpha'-\beta'}D_{x_n}^{\alpha_n}\partial_{x'}^{\beta'}\tilde g_{-j}^{\log}(x',x_n,y_n,\xi'), \tag{6.10}$$

but we still have estimates as in [10, (2.26)–(2.29)], since $|\alpha'-\beta'|\ge 0$, so tr_n *of the relevant terms* makes sense. In this sense we identify $\operatorname{res} U_{P'}$ with $\operatorname{res}\operatorname{tr}_n(P'_+G_1^{\log})$.

For the part p'' of normal order -1 (or 0) we can use in the $\circ_n$-composition with the symbol-kernel of G_λ that tr_n of it can be simplified, by Lemma 5.1. Now apply the notation tr'_n from Definition 5.6. Considering the terms in the full $\circ$-composition of p'' with $g(x',\xi',\xi_n,\eta_n,\lambda)$, we see that the $\log\lambda$-integration can be interchanged with taking tr'_n, leading to an identification of $U_{P''}$ with $\operatorname{tr}'_n(P''_+G_1^{\log})$ on the symbol level, so that $\operatorname{res} U_{P''}$ identifies with $\operatorname{res}\operatorname{tr}'_n(P''_+G_1^{\log})$.

In the case where the symbol of P depends also on x_n, one applies the preceding considerations to the terms in a Taylor expansion in x_n (a technique used extensively in [5, Chap. 2.7]), up to a term with a so large power of x_n that the order is so low that there is no residue contribution. This just gives some more terms in slightly more complicated formulas; the principle is the same as above.

This is as close as we can get to identifying $\operatorname{res} U_P$ with the residue of a normal trace associated with $P_+G_1^{\log}$. But it gives sufficient justification for *defining*

$$\operatorname{res}_{x'}(P_+G_1^{\log}) = \int_{|\xi'|=1} \operatorname{tr}\,[\operatorname{tr}_n \operatorname{symb}_{1-n}(P'_+G_1^{\log}) + \operatorname{tr}'_n \operatorname{symb}_{1-n}(P''_+G_1^{\log})]\,đS(\xi'), \tag{6.11}$$

consistently with $\operatorname{res}_{x'} U_P$, with reference to the detailed interpretations of the contributions from P' and P'' given above. Then we can write:

Theorem 6.2 *For U_P in (6.2), the coefficient of $(-\lambda)^{-N}$ in the trace expansion of $P_+G_\lambda^{(N)}$ is (cf. (6.11)):*

$$l_0(P_+G_\lambda) = -\tfrac{1}{m}\operatorname{res} U_P = -\tfrac{1}{m}\int_{X'} \operatorname{res}_{x'}(P_+G_1^{\log})\,dx', \tag{6.12}$$

also denoted $-\frac{1}{m}\operatorname{res}(P_+G_1^{\log})$.

7 Consequences

We collect the results of Sects. 4–6 in the following theorem, denoting the general auxiliary operator P_1:

Theorem 7.1 *Let $B = P_+ + G$ on X, of order σ, where P is a ψdo satisfying the transmission condition at X' and G is a singular Green operator of class 0 ($\sigma \in \mathbb{Z}$ if $P \neq 0$, $\sigma \in \mathbb{R}$ otherwise). Moreover, let P_1 be a second-order elliptic differential operator provided with a differential boundary condition $Tu = 0$, such that $P_{1,T}$ has $\mathbb{R}_-$ as a spectral cut, and let V be a sector around $\mathbb{R}_-$ such that the principal symbol and principal boundary symbol realization have no eigenvalues in V. The operator family $B(P_{1,T} - \lambda)^{-N} = (P_+ + G)(Q^N_{\lambda,+} + G^{(N)}_\lambda)$ has for $N > (\sigma + n)/2$ trace expansions as in (2.23), for $\lambda \to \infty$ on rays in V. The basic zeta coefficient $C_0(B, P_{1,T}) = l_0(B(P_{1,T} - \lambda)^{-1})$ is a sum of five invariantly defined terms*:

$$\begin{aligned} C_0(B, P_{1,T}) = l_0((PQ_\lambda)_+) - l_0(L(P, Q_\lambda)) + l_0(GQ_{\lambda,+}) \\ +l_0(P_+G_\lambda) + l_0(GG_\lambda); \end{aligned} \tag{7.1}$$

here

$$\begin{aligned} l_0((PQ_\lambda)_+) &= \int_X [\mathrm{TR}_x\, P - \tfrac{1}{2}\operatorname{res}_{x,0}(P \log P_1)]\, dx, \\ -l_0(L(P, Q_\lambda)) &= \tfrac{1}{2}\operatorname{res} L(P, \log P_1) = \tfrac{1}{2}\int_{X'} \operatorname{res}_{x'} \operatorname{tr}_n L(P, \log P_1))\, dx', \\ l_0(GQ_{\lambda,+}) &= \int_{X'} \left[\mathrm{TR}_{x'} \operatorname{tr}_n G - \tfrac{1}{2}\operatorname{res}_{x',0} \operatorname{tr}'_n(G(\log P_1)_+)\right] dx', \\ l_0(P_+G_\lambda) &= -\tfrac{1}{2}\operatorname{res}(P_+G_1^{\log}) \\ &= -\tfrac{1}{2}\int_{X'}\int_{|\xi'|=1} \operatorname{tr}[\operatorname{tr}_n \operatorname{symb}_{1-n}(P'_+G_1^{\log}) \\ &\quad + \operatorname{tr}'_n \operatorname{symb}_{1-n}(P''_+G_1^{\log})]\, đS(\xi'), \\ l_0(GG_\lambda) &= -\tfrac{1}{2}\operatorname{res}(GG_1^{\log}) = -\tfrac{1}{2}\int_{X'} \operatorname{res}_{x'} \operatorname{tr}_n(GG_1^{\log})\, dx', \end{aligned} \tag{7.2}$$

where the integrals are calculated in local coordinates (P *is decomposed as in* (6.8)ff).

Note that the first and the third terms in (7.1) are global, and closely related to the expression (2.15) found in [22] for the boundaryless case; the other terms are local.

Let us now also show how this can be generalized to auxiliary operators of even order $m = 2k$. The goal is to obtain formulas as in (7.1) and (7.2) with $\frac{1}{2}$ replaced by $\frac{1}{m} = \frac{1}{2k}$.

First choose the second-order operator $P_{1,T}$ as a selfadjoint positive operator (e.g., the Dirichlet realization of $P_0 + P_0^* + a$, where P_0 has principal symbol $|\xi|^2$ and a is sufficiently large). Then $(P_{1,T})^k = (P_1^k)_{T'}$ with trace operator $T' = \{T, TP_1, \dots, TP_1^{k-1}\}$; this is likewise positive selfadjoint, and by definition of C_0,

$$C_0(B, (P_{1,T})^k) = C_0(B, P_{1,T}). \tag{7.3}$$

For the latter, we have the formulas (7.1), (7.2). Since $\log P_1^k = k \log P_1$, $\frac{1}{2}\log P_1$ can be replaced by $\frac{1}{m}\log P_1^k$ in the formulas, so the first and third term, the nonlocal terms, are completely analogous to the case $m = 2$. The last two terms in (7.1) have already been shown to be of the desired form with $\frac{1}{2}$ replaced by $\frac{1}{m}$, since Sects. 3 and 6 treat arbitrary m. For the second term, one can establish an analysis similar to that in Sect. 4, using that the symbol-kernel of $G^-(P_1^k)$ is of the form $-\frac{k}{x_n+y_n}e^{-|\xi'|(x_n+y_n)}$ plus a standard singular Green symbol-kernel (for $|\xi'| \geq 1$), cf. [10, Proposition 2.7].

This shows a generalization of Theorem 7.1 for the special choice $(P_1^k)_{T'}$. The analysis is now extended to more general auxiliary operators $P_{2,T''}$ of order $m = 2k$ using that

$$\begin{aligned} C_0(B, P_{2,T''}) &= C_0(B, P_{2,+}) + l_0(BG_\lambda) \\ &= C_0(B, (P_1^k)_+) + [C_0(B, P_{2,+}) - C_0(B, (P_1^k)_+)] + l_0(BG_\lambda); \end{aligned} \tag{7.4}$$

here Theorem 3.6 shows formula (2.12) for $C_0(B, P_{2,+}) - C_0(B, (P_1^k)_+)$, and the contributions from the singular Green part G_λ of $(P_{2,T''} - \lambda)^{-1}$ are as worked out in Sect. 6. We then obtain:

Corollary 7.2 *Theorem 7.1 holds with P_1 of even order m, when $\frac{1}{2}$ in the formulas is replaced by $\frac{1}{m}$, as written in (2.25).*

As a further consequence of these results, we can formulate some defect formulas (relative formulas) which compare two different choices of auxiliary operator, generalizing (2.12). For one thing, we can supply (2.12) with a formula where the full auxiliary operators, not just their ψdo parts, enter:

Corollary 7.3 *For B together with two auxiliary operators P_{1,T_1} and P_{2,T_2} as in Corollary 7.2, of even orders m_1 resp. m_2, one has*:

$$\begin{aligned} C_0(B, P_{1,T_1}) - C_0(B, P_{2,T_1}) &= -\operatorname{res}\Big(B\Big(\tfrac{1}{m_1}\log P_1 - \tfrac{1}{m_2}\log P_2\Big)_+\Big) \\ &+ \operatorname{res}\Big(L\Big(P, \tfrac{1}{m_1}\log P_1 - \tfrac{1}{m_2}\log P_2\Big)\Big) - \tfrac{1}{m_1}\operatorname{res}(P_+ G_1^{\log}) \\ &+ \tfrac{1}{m_2}\operatorname{res}(P_+ G_2^{\log}) - \tfrac{1}{m_1}\operatorname{res}(G G_1^{\log}) + \tfrac{1}{m_2}\operatorname{res}(G G_2^{\log}), \end{aligned} \tag{7.5}$$

with residues defined as above.

Proof We write $\log P_{i,T_i} = (\log P_i)_+ + G_i^{\log}$ for $i = 1, 2$. Then (7.5) follows simply by applying Corollary 7.2 to the corresponding zeta coefficients and forming the difference. □

The new terms in comparison with (2.12) are local, defined by an extension of the residue concept as explained at the end of Sect. 6.

Another defect that we can consider is the difference between the zeta coefficients arising from different choices of the definition of log. It may happen that the spectrum of $P_{1,T}$ (and that of P_1 on $\widetilde{X}$) divides the complex plane into several sectors with infinitely many eigenvalues, separated by rays without eigenvalues. Consider two such rays $e^{i\theta}\mathbb{R}_+$ and $e^{i\varphi}\mathbb{R}_+$, with $\theta < \varphi < \theta + 2\pi$. We can then define $\log_\theta P_1$ and $\log_\theta P_{1,T}$ by variants of (2.13), where the logarithm is taken with cut at $e^{i\theta}\mathbb{R}_+$ and the integration curve goes around this cut (more details in [10]); the logarithmic term in the symbol of $\log_\theta P_1$ is still $m\log[\xi]$. The difference between the logarithms is closely connected with a spectral projection; in fact,

$$\begin{aligned} \log_\theta P_1 - \log_\varphi P_1 &= \tfrac{2\pi}{i}\Pi_{\theta,\varphi}(P_1) \text{ on } \widetilde{X}, \\ \log_\theta P_{1,T} - \log_\varphi P_{1,T} &= \tfrac{2\pi}{i}\Pi_{\theta,\varphi}(P_{1,T}) \text{ on } X, \end{aligned} \tag{7.6}$$

where $\Pi_{\theta,\varphi}(P_1)$ and $\Pi_{\theta,\varphi}(P_{1,T})$ are the sectorial projections for the sector $\Lambda_{\theta,\varphi}$, essentially projecting onto the generalized eigenspace for eigenvalues of P_1 resp. $P_{1,T}$ in $\Lambda_{\theta,\varphi}$ along the complementing eigenspace (see [10] for the precise description of such sectorial projections). Here the ψdo part of $\Pi_{\theta,\varphi}(P_{1,T})$ is $\Pi_{\theta,\varphi}(P_1)_+$ satisfying the transmission condition since m is even, and the s.g.o.-type part $G_{\theta,\varphi}$ of $\Pi_{\theta,\varphi}(P_{1,T})$ is connected with the s.g.o.-type part of the logs by

$$G_1^{\log_\theta} - G_1^{\log_\varphi} = \tfrac{2\pi}{i} G_{\theta,\varphi}. \tag{7.7}$$

Since the expressions $\mathrm{TR}_x\, P$ and $\mathrm{TR}_{x'}\,\mathrm{tr}_n\, G$ are independent of the auxiliary operator, the terms resulting from these are the same when one calculates the zeta coefficients

$$C_{0,\theta}(B, P_{1,T}) = l_{0,\theta}(BR_\lambda), \text{ resp. } C_{0,\varphi}(B, P_{1,T}) = l_{0,\varphi}(BR_\lambda), \tag{7.8}$$

referring to expansions for $\lambda \to \infty$ along $e^{i\theta}\mathbb{R}_+$ resp. $e^{i\varphi}\mathbb{R}_+$. So they disappear when we take the difference between these zeta coefficients. This implies another local defect formula.

Corollary 7.4 *For the difference between the basic zeta coefficients calculated with respect to two different rays $e^{i\theta}\mathbb{R}_+$ and $e^{i\varphi}\mathbb{R}_+$, for P_1 of even order m, we have*:

$$\begin{aligned} &C_{0,\theta}(B, P_{1,T}) - C_{0,\varphi}(B, P_{1,T}) = -\tfrac{1}{m}\operatorname{res}_+(P(\log_\theta P_1 - \log_\varphi P_1)) \\ &\quad + \tfrac{1}{m}\operatorname{res}(L(P, (\log_\theta P_1 - \log_\varphi P_1))) - \tfrac{1}{m}\operatorname{res}(G(\log_\theta P_1 - \log_\varphi P_1)_+) \\ &\quad - \tfrac{1}{m}\operatorname{res}(P_+(G_1^{\log_\theta} - G_1^{\log_\varphi})) - \tfrac{1}{m}\operatorname{res}(G(G_1^{\log_\theta} - G_1^{\log_\varphi})); \end{aligned}$$

here the first three terms are residues in the sense of [4]*, and the last two are generalizations defined as in* (7.2)*. In view of* (7.6) and (7.7)*, this can also be written*:

$$\begin{aligned} &C_{0,\theta}(B, P_{1,T}) - C_{0,\varphi}(B, P_{1,T}) \\ &\quad = -\tfrac{2\pi}{im}\operatorname{res}_+(P\,\Pi_{\theta,\varphi}(P_1)) + \tfrac{2\pi}{im}\operatorname{res}(L(P, \Pi_{\theta,\varphi}(P_1))) \\ &\qquad - \tfrac{2\pi}{im}\operatorname{res}(G(\Pi_{\theta,\varphi}(P_1))_+) - \tfrac{2\pi}{im}\operatorname{res}((P_+ + G)G_{\theta,\varphi}), \end{aligned} \tag{7.9}$$

in short denoted $-\frac{2\pi}{im}\operatorname{res}(B\,\Pi_{\theta,\varphi}(P_{1,T}))$ (the residues being defined from the homogeneous terms of the relevant dimension as in (7.2)*). When $\Pi_{\theta,\varphi}(P_{1,T})$ belongs to the standard calculus, this is a residue in the sense of* [4].

Finally we have some observations on the traciality of the new residue definitions. Let B be *of order and class* 0; it is a bounded operator in L_2, and its adjoint is of the same kind. Then it can be placed to the right of R_λ^N in all the above calculations, performed in the analogous way. So $\operatorname{Tr}(R_\lambda^N B)$ has an expansion similar to (2.3) (this also follows directly from applying (2.3) to $B^*(P_{1,T}^* - \bar\lambda)^{-N}$ and conjugating). Then since $\operatorname{Tr}(BR_\lambda^N) = \operatorname{Tr}(R_\lambda^N B)$, it follows that

$$\begin{aligned} l_0(BR_\lambda) &= l_0(R_\lambda B), \quad \text{in particular} \\ l_0(BQ_{\lambda,+}) &= l_0(Q_{\lambda,+}B), \quad l_0(BG_\lambda) = l_0(G_\lambda B). \end{aligned} \tag{7.10}$$

This leads to:

Theorem 7.5 *When $B = P_+ + G$ is of order and class* 0 *and $P_{1,T}$ is as in Corollary 7.2, then*

$$\operatorname{res}(BG_1^{\log}) = \operatorname{res}(G_1^{\log}B), \quad \operatorname{res}([B, \log P_{1,T}]) = 0. \tag{7.11}$$

Moreover, in the situation of Corollary 7.4,

$$\begin{aligned} \operatorname{res}(B\, G_{\theta,\varphi}) &= \operatorname{res}(G_{\theta,\varphi}\, B) \\ \operatorname{res}(B\, \Pi_{\theta,\varphi}(P_{1,T})) &= \operatorname{res}(\Pi_{\theta,\varphi}(P_{1,T})\, B). \end{aligned} \tag{7.12}$$

Proof First we note that $\operatorname{Tr}(G_\lambda^{(N)}B)$ has an expansion similar to (3.25), since the normal trace of $G_\lambda^{(N)}B$ allows application of Theorems 3.1 and 3.2 as in the case of $BG_\lambda^{(N)}$. The coefficient of $(-\lambda)^{-N}$ is $l_0(G_\lambda B) = -\frac{1}{m}\operatorname{res} U'$, where the ψdo U' on X' is defined similarly as in Corollary 3.5 from taking the normal trace of $G_\lambda^{(N)}B$ and integrating together with $\log\lambda$. The constant $\operatorname{res} U'$ is interpreted as $\operatorname{res}(G_1^{\log}B)$ in the same way as in Sect. 6; for the part $\operatorname{res} U'_G$ this goes directly by an interchange of the integration in λ and the application of tr_n, and for the part $\operatorname{res} U'_P$ it works as in the consideration of the contribution from P'', where the λ-integration and the application of tr'_n to the relevant terms could be interchanged. The first formula in (7.11) then follows from $l_0(BG_\lambda) = l_0(G_\lambda B)$.

For the second formula in (7.11), we do not have a separate residue definition for $B\log P_{1,T}$ (neither for $\log P_{1,T}B$), since (7.2) contains global terms. However, the TR-components of P and G will be the same in the formulas for $l_0(BR_\lambda)$ and $l_0(R_\lambda B)$, so they cancel out in the difference, leaving a local contribution; it is this that we denote $\operatorname{res}([B, \log P_{1,T}])$. It vanishes because of the first identity in (7.10).

In the situation of Corollary 7.4, the preceding results give the commutativity with $G_1^{\log_\theta}$ or $G_1^{\log_\varphi}$ inserted; this implies the first line in (7.12) in view of (7.7). Now the second line follows since the ψdo part of $\Pi_{\theta,\varphi}(P_{1,T})$ is a standard ψdo in the calculus, for which the property is known from [4]. □

A Appendix

Some lemmas on integrals arising from Laguerre expansions will be included here.

Lemma A.1 *Set*

$$r = |\xi'|, \quad \kappa = (|\xi'|^2 - \lambda)^{\frac{1}{2}} = (r^2 + \mu^2)^{\frac{1}{2}};$$
$$s^{\pm}_{l,m}(\xi', \mu) = \int_{\mathbb{R}} \frac{(r-i\xi_n)^l}{(r+i\xi_n)^{l+1}} \frac{(r+i\xi_n)^m}{(r-i\xi_n)^{m+1}} \frac{1}{\kappa \pm i\xi_n} \, đ\xi_n, \tag{A.1}$$

then one has:

$$\begin{aligned} &\text{For } l > m,\ s^{+}_{l,m} = 0, \quad s^{-}_{l,m} = \frac{(r-\kappa)^{l-m-1}}{(r+\kappa)^{l-m+1}}. \\ &\text{For } l = m,\ s^{+}_{l,l} = \frac{1}{(\kappa+r)2r} = s^{-}_{l,l}. \\ &\text{For } l < m,\ s^{+}_{l,m} = \frac{(r-\kappa)^{m-l-1}}{(r+\kappa)^{m-l+1}}, \quad s^{-}_{l,m} = 0. \end{aligned} \tag{A.2}$$

This was shown in [11, Lemma 5.2]. With the same notation we have:

Lemma A.2 *Let $c_{lm,1-n}(x', \xi')$ be homogeneous of degree $1-n$ in ξ' for $|\xi'| \geq 1$. Then*

$$\begin{aligned} I_{lm} &= \int_{|\xi'|\geq 1} c_{lm,1-n}(x', \xi') 2r \frac{(r-i\xi_n)^l}{(r+i\xi_n)^{l+1}} \frac{(r+i\xi_n)^m}{(r-i\xi_n)^{m+1}} (r^2 + \xi_n^2 - \lambda)^{-N} \, đ\xi \\ &= \int_{r\geq 1} \int_{t\in\mathbb{R}} 2 \frac{(r-it)^{l-m-1}}{(r+it)^{l-m+1}} \frac{\partial_\lambda^{N-1}}{(N-1)!} \left(\frac{1}{2\kappa} \left(\frac{1}{\kappa+it} + \frac{1}{\kappa-it} \right) \right) dr\, đt \int_{|\xi'|=1} c_{lm,1-n}(x', \xi') đS(\xi'), \end{aligned}$$

where

$$\begin{aligned} &\int_{r\geq 1} \int_{t\in\mathbb{R}} \frac{(r-it)^{l-m-1}}{(r+it)^{l-m+1}} \frac{\partial_\lambda^{N-1}}{(N-1)!} \left(\frac{1}{\kappa} \left(\frac{1}{\kappa+it} + \frac{1}{\kappa-it} \right) \right) dr\, đt \\ &= \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{r\geq 1} \int_{t\in\mathbb{R}} \frac{(r-it)^{l-m-1}}{(r+it)^{l-m+1}} \left(\frac{1}{\kappa} \left(\frac{1}{\kappa+it} + \frac{1}{\kappa-it} \right) \right) dr\, đt \\ &= \begin{cases} \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{r\geq 1} \frac{(r-\kappa)^{l-m-1}}{\kappa(r+\kappa)^{l-m+1}} \, dr & \text{if } l > m, \\ \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{r\geq 1} \frac{1}{r\kappa(r+\kappa)} \, dr & \text{if } l = m, \\ \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{r\geq 1} \frac{(r-\kappa)^{m-l-1}}{\kappa(r+\kappa)^{m-l+1}} \, dr & \text{if } l < m. \end{cases} \end{aligned} \tag{A.3}$$

Proof The first calculation is straightforward, using that $c_{lm,1-n}(\xi') = r^{1-n} c_{lm,1-n}(\xi'/|\xi'|)$. The reductions of the integrals over r and t in (A.3) now follow from (A.2). □

The integrals in (A.3) can be calculated by use of the following lemma:

Lemma A.3 *Let $a > 0$. One has that*

$$\int_1^\infty \frac{1}{r\sqrt{r^2+a^2}\,(\sqrt{r^2+a^2}+r)}\,dr = a^{-2}(-\log 2 + \log(\sqrt{1+a^2}+1)), \tag{A.4}$$

and, for $j = 1, 2, \dots,$

$$\int_1^\infty \frac{(\sqrt{r^2+a^2}-r)^{j-1}}{\sqrt{r^2+a^2}\,(\sqrt{r^2+a^2}+r)^{j+1}}\,dr = (2j)^{-1}a^{-2}(\sqrt{1+a^{-2}}-a^{-1})^{2j}. \tag{A.5}$$

Proof Letting $s = r/a$, we have:

$$\begin{aligned}
\int_1^\infty \frac{1}{r\sqrt{r^2+a^2}\,(r+\sqrt{r^2+a^2})}\,dr &= a^{-2}\int_{1/a}^\infty \frac{1}{s\sqrt{s^2+1}\,(\sqrt{s^2+1}+s)}\,ds \\
&= a^{-2}\int_{1/a}^\infty \frac{\sqrt{s^2+1}-s}{s\sqrt{s^2+1}}\,ds = a^{-2}\int_{1/a}^\infty \left(\frac{1}{s}-\frac{1}{\sqrt{s^2+1}}\right)ds \\
&= a^{-2}\Big[\log s - \log(\sqrt{s^2+1}+s)\Big]_{1/a}^\infty = -a^{-2}\Big[\log(\sqrt{1+s^{-2}}+1)\Big]_{1/a}^\infty \\
&= a^{-2}(-\log 2 + \log(\sqrt{1+a^2}+1)),
\end{aligned}$$

showing (A.4). For (A.5), we furthermore set $u = \sqrt{s^2+1}-s$, noting that $du/ds = (s-\sqrt{s^2+1})/\sqrt{s^2+1}$; then

$$\begin{aligned}
\int_1^\infty \frac{(\sqrt{r^2+a^2}-r)^{j-1}}{\sqrt{r^2+a^2}\,(\sqrt{r^2+a^2}+r)^{j+1}}\,dr &= a^{-2}\int_{1/a}^\infty \frac{(\sqrt{s^2+1}-s)^{j-1}}{\sqrt{s^2+1}\,(\sqrt{s^2+1}+s)^{j+1}}\,ds \\
&= a^{-2}\int_{1/a}^\infty \frac{(\sqrt{s^2+1}-s)^{2j}}{\sqrt{s^2+1}}\,ds = -a^{-2}\int_{s=1/a}^{s=\infty} u^{2j-1}du \\
&= -a^{-2}\Big[(2j)^{-1}u^{2j}\Big]_{s=1/a}^{s=\infty} = a^{-2}(2j)^{-1}(\sqrt{a^{-2}+1}-a^{-1})^{2j},
\end{aligned}$$

since

$$\begin{aligned}
\sqrt{s^2+1}-s = s(\sqrt{1+s^{-2}}-1) &= s\left(1+\tfrac{1}{2}s^{-2}+O(s^{-4})-1\right) \\
&= \tfrac{1}{2}s^{-1}+O(s^{-3}) \text{ for } s\to\infty
\end{aligned} \tag{A.6}$$

implies $u \to 0$ for $s \to \infty$. □

We shall also need:

Lemma A.4 $\int_{-\infty}^{\infty} \frac{\log(1+s^2)}{1+s^2}\, ds = 2\pi \log 2.$

Proof Write $\log(1+s^2) = \log(1+is) + \log(1-is)$. Here $\log(1+is)$ extends to a holomorphic function of $s \in \mathbb{C}\setminus\{s = it \mid t \geq 1\}$, and

$$\frac{1}{1+s^2} = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right). \tag{A.7}$$

Since $\log(1+is)/(1+s^2)$ is $O(|s|^{\varepsilon-2})$ for $|s| \to \infty$, we can calculate

$$\int_{-\infty}^{\infty} \frac{\log(1+is)}{1+s^2}\, ds = -\int_{\mathcal{C}_-} \frac{\log(1+is)}{1+s^2}\, ds = \frac{1}{2i}\int_{\mathcal{C}_-} \frac{\log(1+is)}{s+i}\, ds = \pi \log 2,$$

where we replaced the real axis by a closed curve $\mathcal{C}_-$ in the lower halfplane going around the pole $-i$ in the positive direction, and used (A.7).

The integral with $\log(1-is)$ is turned into an integral over a curve around the pole i in the upper halfplane, contributing $\pi \log 2$ in a similar way. □

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